

# Pacific Journal of Mathematics

**A CONTINUOUS FORM OF SCHWARZ'S LEMMA IN NORMED  
LINEAR SPACES**

LAWRENCE ALBERT HARRIS

## A CONTINUOUS FORM OF SCHWARZ'S LEMMA IN NORMED LINEAR SPACES

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**Our main result is an inequality which shows that a holomorphic function mapping the open unit ball of one normed linear space into the closed unit ball of another is close to being a linear map when the Fréchet derivative of the function at 0 is close to being a surjective isometry. We deduce this result as a corollary of a kind of uniform rotundity at the identity of the sup norm on bounded holomorphic functions mapping the open unit ball of a normed linear space into the same space.**

Let  $\Delta$  be the open unit disc of the complex plane, and let  $f: \Delta \rightarrow \bar{\Delta}$  be a holomorphic function with  $f(0) = 0$ . It is easy to show that the inequality

$$(1) \quad |f(z) - f'(0)z| \leq \frac{2|z|^2}{1-|z|} (1 - |f'(0)|)$$

holds for all  $z \in \Delta$ . (For example, apply the lemma given in [5] to the function  $z^{-1}f(z)$ . See also [3, §292].) Qualitatively, inequality (1) means that if  $f'(0)$  is close to the unit circle then  $f(z)$  is close to being a linear function of  $z$  as long as  $z$  remains a fixed positive distance away from the exterior of the unit disc. Our purpose is to prove a version of (1) which applies to vector-valued holomorphic functions of vectors. We deduce this result from an extremal inequality for holomorphic functions, which reduces to a theorem of G. Lumer in the linear case. It should be pointed out that the inequalities we obtain cannot be proved simply by composing with linear functionals and applying the 1-dimensional case, as for instance the generalized Cauchy inequalities can.

**1. Main results.** In the following, a function  $h$  defined on an open subset of a complex normed linear space with range in another is called *holomorphic* if the Fréchet derivative of  $h$  at  $x$  (denoted by  $Dh(x)$ ) exists as a bounded complex-linear map for each  $x$  in the domain of definition of  $h$ . (See [7, Def. 3.16.4].) Denote the open (resp., closed) unit ball of a normed linear space  $X$  by  $X_0$  (resp.,  $X_1$ ). Throughout,  $X$  and  $Y$  denote arbitrary complex normed linear spaces. Our main result is

**THEOREM 1.** *Let  $h: X_0 \rightarrow Y_1$  be a holomorphic function with*

$h(0) = 0$ . Put  $L = Dh(0)$  and let  $\mathcal{U}$  be the set of all linear isometries of  $X$  onto  $Y$ . Suppose  $\mathcal{U}$  is nonempty and let  $d(L, \mathcal{U})$  denote the distance of  $L$  from  $\mathcal{U}$  in the operator norm.

Then

$$\|h(x) - L(x)\| \leq \frac{8\|x\|^2}{(1 - \|x\|)^2} d(L, \mathcal{U}), \quad (x \in X_0).$$

Clearly Theorem 1 contains the main result of [5], i.e.,  $h = L$  when  $L$  is in  $\mathcal{U}$ . In fact, it is a consequence of Theorem 1 that any sequence of holomorphic functions  $h_n: X_0 \rightarrow Y_1$  converges uniformly to a linear map  $L$  in  $\mathcal{U}$  on closed subballs of  $X_0$  whenever the sequence of derivatives  $Dh_n(0)$  converges to  $L$  in the operator norm. This may be proved by showing as in [5] that  $h_n(0) \rightarrow 0$ , and then applying Theorem 1 to the function  $(1 + \|h_n(0)\|)^{-1} [h_n(x) - h_n(0)]$ .

Let  $I$  be the identity map on  $X$  and let the symbol  $\|\cdot\|$ , when applied to functions, denote the supremum over  $X_0$ . We deduce Theorem 1 from

**THEOREM 2.** *Let  $\delta \geq 0$  and suppose  $h: X_0 \rightarrow X$  is a holomorphic function satisfying*

$$(2) \quad \|I + \lambda h\| \leq 1 + \delta$$

for all  $\lambda \in \bar{A}$ . Let  $P_m$  be the  $m$ th term of the Taylor series expansion for  $h$  about 0. Then

$$(3) \quad \|P_m\| \leq K_m \delta,$$

where  $K_0 = 1$ ,  $K_1 = e$  and  $K_m = m^{m/(m-1)}$ ,  $m \geq 2$ . If inequality (2) holds when the values of  $\lambda$  are restricted to  $\pm 1$ , then (3) still holds but with  $\delta$  replaced by  $\sqrt{\delta(2 + \delta)}$ .

Recall that by definition

$$(4) \quad P_m(x) = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} h(\lambda x) \right]_{\lambda=0}, \quad P_0(x) = h(0).$$

Hence  $P_1 = Dh(0)$ . Moreover [7, Th. 26.3.6],  $P_m$  is of the form  $P_m(x) = F_m(x, \dots, x)$ , where  $F_m$  is a continuous symmetric  $m$ -linear map. It should be noted that in general  $P_m$  is a mapping of  $X$  into the completion of  $X$ .

**2. Proof of Theorem 1 assuming Theorem 2.** Let  $h: X_0 \rightarrow X_1$  be a holomorphic function with  $h(0) = 0$  and put  $L = Dh(0)$ . It suffices to prove that  $h$  satisfies the inequality

$$(5) \quad \|h(x) - L(x)\| \leq \frac{8\|x\|^2}{(1 - \|x\|)^2} \|I - L\|, \quad (x \in X_0);$$

for Theorem 1 can then be deduced by composing the given function with inverses of linear maps in  $\mathcal{U}$  and applying (5). Thus to prove (5), let

$$h(x) = P_1(x) + P_2(x) + \dots, (P_1 = L),$$

be the Taylor series expansion for  $h$  about 0. This series converges to  $h(x)$  for every  $x$  in  $X_0$ . (See [7, pp. 109-113].) Let  $x \in X_1$  and let  $\ell$  be a linear functional on the completion of  $X$  with  $\|\ell\| \leq 1$ . Define  $f(\lambda) = \ell(\lambda^{-1}h(\lambda x))$ . Then  $f: \Delta \rightarrow \bar{\Delta}$  is holomorphic and

$$f(\lambda) = \sum_{m=0}^{\infty} a_m \lambda^m, a_m = \ell(P_{m+1}(x)).$$

By [9, p. 172], we have  $|a_{m-1}| \leq 1 - |a_0|^2 \leq 2(1 - |a_0|)$  for  $m \geq 2$ , and hence

$$\left| \ell \left( L(x) + \frac{1}{2} \lambda P_m(x) \right) \right| \leq 1$$

for all  $\lambda \in \Delta$ . It follows from the Hahn-Banach Theorem that  $\|L + 1/2 \lambda P_m\| \leq 1$ , and therefore

$$\left\| I + \frac{1}{2} \lambda P_m \right\| \leq 1 + \delta, \delta = \|I - L\|,$$

for all  $\lambda \in \Delta$ . Since  $P_m$  extends to the completion of  $X$ , Theorem 2 applies to show that

$$\|P_m\| \leq 2K_m \delta \leq 8(m - 1)\delta,$$

where the last inequality follows from the inequalities  $m/(m - 1) \leq 2$  and  $m \leq 2^{m-1}$ . Hence if  $x \in X_0$ ,

$$\|h(x) - L(x)\| \leq \sum_{m=2}^{\infty} \|P_m(x)\| \leq \frac{8 \|x\|^2 \delta}{(1 - \|x\|)^2},$$

which is (5).

3. *Proof of Theorem 2.* Our proof is an elaboration of an iteration argument due to H. Cartan. (See [1, pp. 13-14].) Clearly we may suppose that  $\delta > 0$  and that inequality (2) is strict. Let  $N$  be any positive integer satisfying  $N \geq 1/\delta$  and put  $r = 1/(N\delta)$ . Then by the triangle inequality,

$$(6) \quad \|I + \lambda r h\| = \|(1 - r)I + r(I + \lambda h)\| < 1 + 1/N$$

for all  $\lambda \in \Delta$ . Take  $\alpha = (1 + 1/N)^{-1}$ . Our strategy is to compute the derivatives with respect to  $\lambda$  of the  $n$ th iterate of the function  $\alpha I + \lambda \alpha r h$  and then apply the generalized Cauchy inequalities [7, p. 97]. The number  $n$  of iterations we take will depend on  $N$ .

Let  $x \in X_0$  and define

$$f_n(\lambda) = (\alpha I + \lambda \alpha r h)^n(x) .$$

By (6),  $f_n: \Delta \rightarrow X$  is a well-defined holomorphic function satisfying

$$(7) \quad \|f_n(\lambda)\| < 1, (\lambda \in \Delta) .$$

Clearly  $f'_1(0) = \alpha r h(x)$ , and differentiating the identity

$$f_{n+1}(\lambda) = \alpha f_n(\lambda) + \lambda \alpha r h(f_n(\lambda)) ,$$

we have

$$f'_{n+1}(0) = \alpha f'_n(0) + \alpha r h(\alpha^n x) .$$

Therefore, by induction

$$(8) \quad f'_n(0) = \sum_{k=0}^{n-1} \alpha^{n-k} r h(\alpha^k x) .$$

By (7) and Cauchy's inequality,

$$(9) \quad \|f'_n(0)\| \leq 1 .$$

Let  $\Phi_n(x)$  be the right hand side of (8). Clearly each  $\Phi_n$  is holomorphic in  $X_0$  and by (9),  $\|\Phi_n\| \leq 1$ . Applying the Cauchy inequalities, we have

$$\left\| \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \Phi_n(\lambda x) \right]_{\lambda=0} \right\| \leq 1, \quad (x \in X_0) .$$

Hence by (4),

$$(10) \quad \left\| \sum_{k=0}^{n-1} \alpha^{n+(m-1)k} r P_m(x) \right\| \leq 1, \quad (x \in X_0) ,$$

so

$$\|P_m\| \leq \frac{1 - \alpha^{m-1}}{r \alpha^n [1 - \alpha^{n(m-1)}]} ,$$

assuming  $m \geq 2$ . Since  $1 - \alpha^{m-1} \leq (m - 1)(1 - \alpha)$ ,  $1/r = N\delta$  and  $N(1/\alpha - 1) = 1$ , it follows that

$$(11) \quad \|P_m\| \leq \frac{(m - 1)\delta}{\alpha^{n-1} [1 - \alpha^{n(m-1)}]} .$$

Finally, letting  $n$  be the greatest integer in  $N(m - 1)^{-1} \log m$  and taking the limit in (11) as  $N \rightarrow \infty$ , we obtain inequality (3) for  $m \geq 2$ . When  $m = 1$ , inequality (3) follows from (10) with  $n = N$ . When  $m = 0$ , we may obtain (3) from (9) by letting  $x = 0$  and taking the limit as  $n \rightarrow \infty$ .

The proof of the second part of Theorem 2 follows from quite

general considerations. Suppose  $\|I \pm h\| \leq 1 + \delta$ . By the first part of Theorem 2, it suffices to prove that the inequality

$$(12) \quad \|I + \lambda th\| \leq 1 + t^2, \quad t = \sqrt{\delta(2 + \delta)},$$

holds for all  $\lambda \in \mathcal{A}$ . To do this, let  $x \in X_0$  and  $\varphi \in (X^*)_1$  be given. Then  $|\varphi(x) \pm \varphi(h(x))| \leq 1 + \delta$ , and consequently  $|\varphi(x)|^2 + |\varphi(h(x))|^2 \leq (1 + \delta)^2$ . Hence if  $\lambda \in \mathcal{A}$ ,  $|\varphi(x + \lambda th(x))| \leq |\varphi(x)| + t |\varphi(h(x))|$

$$\leq (1 + t^2)^{1/2}(1 + \delta) = 1 + t^2,$$

where the last inequality follows from the Cauchy-Schwarz inequality. This in conjunction with the Hahn-Banach Theorem proves (12).

4. **Further remarks.** Note that by Theorem 2 (or by [2, §§2, 3]) if  $\delta \geq 0$  and  $L: X \rightarrow X$  is a linear map satisfying  $\|I \pm L\| \leq 1 + \delta$ , then  $\|L\| \leq e\sqrt{\delta(2 + \delta)}$ . This readily implies Theorem 18 of [8]. Note also that in the case  $\delta = 0$ , Theorem 2 shows that  $I$  is an extreme point of  $H^\infty(X_0, X)_1$ , where  $H^\infty(X_0, X)$  denotes the space of all bounded holomorphic functions  $h: X_0 \rightarrow X$  with the sup norm. A simpler proof of this fact has already been given in [6]. It would be interesting to know whether or not  $K_m$  is the best possible constant in (3) which is independent of  $\delta$  and  $h$ . See [4] for a related result.

*Note added in proof.* The author has recently shown that the answer to the above is affirmative.

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Received October 16, 1970.



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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.



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