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MOORE SPACES AND $w\Delta$-SPACES

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This paper is dedicated to Professor J. H. Roberts on the occasion of his sixty-fifth birthday.

This paper is a study of conditions under which a $w\Delta$-space is a Moore space. In §2 we introduce the notion of a G_δ^*-diagonal and show that every $w\Delta$-space with a G_δ^*-diagonal is developable. In §3 we prove that every regular θ-refinable $w\Delta$-space with a point-countable separating open cover is a Moore space. In §4 we introduce the class of α-spaces and show that a regular $w\Delta$-space is a Moore space if and only if it is an α-space. Finally, in §5 we study a new class of spaces which generalizes both semi-stratifiable and $w\Delta$-spaces.

1. Preliminaries. We begin with some definitions and known results which will be used throughout this paper. Unless otherwise stated no separation axioms are assumed; however regular spaces are always T_1 and paracompact spaces are always Hausdorff. The set of natural numbers will be denoted by N.

Let X be a set, $\mathscr{G}$ a cover of X, x an element of X. The *star of x with respect to $\mathscr{G}$*, denoted $\mathrm{st}(x, \mathscr{G})$, is the union of all elements of $\mathscr{G}$ containing x. The *order of x with respect to $\mathscr{G}$*, denoted ord $(x, \mathscr{G})$, is the number of elements of $\mathscr{G}$ containing x.

A space X is *developable* if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that, for each x in X, $\{\mathrm{st}(x, \mathscr{G}_n)\colon n = 1, 2, \cdots\}$ is a fundamental system of neighborhoods of x. Such a sequence of open covers is called a *development* for X. A regular developable space is called a *Moore space*. Bing [1] proved that every paracompact Moore space is metrizable.

According to Borges [3] a space X is a *$w\Delta$-space* if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that, for each x in X, if $x_n \in \mathrm{st}(x, \mathscr{G}_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a *$w\Delta$-sequence* for X. Clearly every countably compact space is a $w\Delta$-space, and in [3] Borges proved that every developable space and every M-space is a $w\Delta$-space. For the relationship between $w\Delta$-spaces, strict p-spaces, and p-spaces, see [6].

A space X is *subparacompact* if every open cover of X has a σ-discrete closed refinement. Every paracompact space is subparacompact [16], and in [8] Creede proved that every semi-stratifiable space is subparacompact. For further properties of subparacompact spaces see [5], [11], and [15].

A space X is *θ-refinable* if for each open cover $\mathscr{V}$ of X there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open refinements of $\mathscr{V}$ such that, for each x in X, there is a n in N such that $\text{ord}(x, \mathscr{G}_n)$ is finite. Such a sequence of open covers is called a *θ-refinement* of $\mathscr{V}$. In [24] Wicke and Worrell state that every subparacompact space is θ-refinable and that a countably compact T_1 space is compact if and only if it is θ-refinable.

2. Spaces with a G_δ^*-diagonal. Recall that a space X has a *G_δ-diagonal* if its diagonal $\Delta = \{(x, x): x \text{ in } X\}$ is a G_δ-subset of $X \times X$. The notion of a G_δ-diagonal plays an important role in metrization theorems; see, for example, [2], [3], [7], [14], and [22].

Every semi-stratifiable Hausdorff space has a G_δ-diagonal [8]. On the other hand the space $[0, 1] \times \{0, 1\}$ with the lexicographic order is a compact perfectly normal space which fails to have a G_δ-diagonal [14].

In [7] Ceder obtained this characterization of spaces with a G_δ-diagonal.

PROPOSITION 2.1. (Ceder) *A space X has a G_δ-diagonal if and only if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that, for any two distinct points x and y of X, there is a n in N such that $y \notin \text{st}(x, \mathscr{G}_n)$.*

In light of this characterization of a G_δ-diagonal and Borges' study of spaces with a $\bar{G}_\delta$-diagonal (see [3]), we introduce the following definition.

DEFINITION 2.2. A space X has a *G_δ^*-diagonal* if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that, for any two distinct points x and y of X, there is a n in N such that $y \notin \text{st}(x, \mathscr{G}_n)^-$. Such a sequence of open covers is called a *G_δ^*-sequence* for X.

In [13] Kullman proved that every regular θ-refinable space with a G_δ-diagonal has a $\bar{G}_\delta$-diagonal. Since every space with a $\bar{G}_\delta$-diagonal has a G_δ^*-diagonal, we have the following proposition.

PROPOSITION 2.3. *Every regular θ-refinable space with a G_δ-diagonal has a G_δ^*-diagonal. In particular every regular semi-stratifiable space has a G_δ^*-diagonal.*

The next result relates the G_δ^*-diagonal property to the diagonal Δ.

PROPOSITION 2.4. *Let X be a space, let $\{V_n: n = 1, 2 \cdots\}$ be a* *sequence of open subsets of* $X \times X$ *containing* Δ, *and suppose that* $\bigcap_{n=1}^{\infty} \bar{V}_n = \Delta$. *Then* X *has a* G_δ^**-diagonal. In particular, if* X *is Hausdorff and* $X \times X$ *is perfectly normal then* X *has a* G_δ^**-diagonal.*

Proof. For $n = 1, 2, \cdots$ let $\mathscr{G}_n = \{G \subseteqq X\colon G \text{ open}, G \times G \subseteqq V_n\}$. Since V_n is open and contains Δ, $\mathscr{G}_n$ covers X. To show that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a G_δ^*-sequence for X, let x and y be distinct points of X. Choose n in N such that $(x, y) \notin \bar{V}_n$, and let U and W be open neighborhoods of x and y respectively such that $(U \times W) \cap V_n = \phi$. It follows that $W \cap \operatorname{st}(x, \mathscr{G}_n) = \phi$ and so $y \notin \operatorname{st}(x, \mathscr{G}_n)^-$.

We now prove the main result in this section.

THEOREM 2.5. *Every* $w\Delta$*-space with a* G_δ^**-diagonal is developable.*

Proof. Let X be a space, let $\mathscr{H}_1, \mathscr{H}_2, \cdots$ be a $w\Delta$-sequence for X, and let $\mathscr{K}_1, \mathscr{K}_2, \cdots$ be a G_δ^*-sequence for X. For each positive integer n let

$$\mathscr{G}_n = \left\{G\colon G = \left(\bigcap_{i=1}^{n} H_i\right) \cap \left(\bigcap_{i=1}^{n} K_i\right),\ \mathrm{H}_i \in \mathscr{H}_i,\ K_i \in \mathscr{K}_i,\ i = 1, \cdots, n\right\}.$$

It is easy to check that $\mathscr{G}_{n+1}$ is an open refinement of $\mathscr{G}_n$ for all n in N and that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ in a $w\Delta$-sequence and a G_δ^*-sequence for X.

Suppose that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is not a development for X. Then there is a point x, a neighborhood W of x, and a sequence $\langle x_n \rangle$ such that for all n, $x_n \in \operatorname{st}(x, \mathscr{G}_n)$ and $x_n \notin W$. Since $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a $w\Delta$-sequence for X, the sequence $\langle x_n \rangle$ has a cluster point p. Clearly $p \notin W$ so $p \neq x$. Since $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a G_δ^*-sequence for X, there is a positive integer k and a neighborhood V of p such that $V \cap \operatorname{st}(x, \mathscr{G}_k) = \phi$. Now for $n \geqq k$, $x_n \in \operatorname{st}(x, \mathscr{G}_n) \subseteqq \operatorname{st}(x, \mathscr{G}_k)$ and so $x_n \notin V$. This contradicts the fact that p is a cluster point of $\langle x_n \rangle$. Thus $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a development for X.

COROLLARY 2.6. *The following are equivalent for a regular* $w\Delta$*-space* X:

(a) *X is a Moore space.*
(b) *X is semi-stratifiable.*
(c) *X is θ-refinable and has a G_δ-diagonal.*
(d) *X has a G_δ^*-diagonal.*

Proof. The implication (a) $\Rightarrow$ (b) is due to Creede [8]; (b) $\Rightarrow$ (c) follows from results by Creede [8] and Wicke and Worrell [24]; (c) $\Rightarrow$ (d) follows from Proposition 2.3; (d) $\Rightarrow$ (a) follows from Theorem 2.5.

REMARK 2.7. The equivalence of (a) and (b) was first proved by Creede in [8], and the equivalence of (a) and (c) is due to Siwiec [23]. It is not known if every regular $w\Delta$-space with a G_δ-diagonal is a Moore space. For a study of p-spaces with a G_δ-diagonal, see [13].

COROLLARY 2.8. *The following are equivalent for a regular countably compact space X:*

(a) *X is metrizable.*
(b) *$X \times X \times X$ is completely normal.*
(c) *$X \times X$ is perfectly normal.*
(d) *X has a G_δ^*-diagonal.*

Proof. Clearly (a) $\Rightarrow$ (b); (b) $\Rightarrow$ (c) follows from a theorem due to Katětov [12]; (c) $\Rightarrow$ (d) follows from Proposition 2.4. To prove (d) $\Rightarrow$ (a) observe that X is a Moore space (by Corollary 2.6) and recall that every countably compact Moore space is metrizable.

3. **Separating covers.** In 1968 Filippov [9] proved that every paracompact M-space with a point-countable base is metrizable. Filippov's theorem was generalized by Burke and Stoltenberg in [4], and recently Burke [6] obtained another generalization as follows.

BURKE'S THEOREM. *Every regular subparacompact $w\Delta$-space with a point-countable base is a Moore space.*

In another direction Nagata [20] proved a metrization theorem which not only generalizes Filippov's theorem but a result by Okuyama as well [22]. In order to state Nagata's theorem succinctly we use the following terminology due to Michael [17]. A cover $\mathscr{V}$ of a set X is said to be *separating* if given distinct points x and y of X, there is a V in $\mathscr{V}$ such that $x \in V$, $y \notin V$.

NAGATA'S THEOREM. *Every paracompact M-space with a point-countable separating open cover is metrizable.*

In this section we use the techniques developed by Burke, Filippov, Nagata, and Stoltenberg, together with the results in §2, to obtain a generalization of the abovementioned theorems by Burke and Nagata.

In light of the usefulness of the concept of a θ-base in the study of developable spaces (see [24]), we begin with the following definition.

DEFINITION 3.1. A *θ-separating cover* of a space X is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open collections such that, for any two distinct points x and y in X, there is a n in N such that

(a) $\operatorname{ord}(x, \mathscr{G}_n)$ is finite;

(b) there is a G in $\mathscr{G}_n$ such that $x \in G$ and $y \notin G$.

The relationship between a θ-separating cover and a G_δ-diagonal is given by the following two propositions.

PROPOSITION 3.2. *Let X be a space with a θ-separating cover. If every closed subset of X is a G_δ then X has a G_δ-diagonal.*

Proof. Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a θ-separating cover of X. For each pair of positive integers n and k let $\mathscr{H}_{nk} = \{H\colon H \neq \phi, H = \bigcap_{i=1}^k G_i, G_1, \cdots, G_k$ distinct elements of $\mathscr{G}_n\}$ and let $F_{nk} = X - \cup\{H\colon H \in \mathscr{H}_{nk}\}$. Now F_{nk} is a closed set and so $F_{nk} = \bigcap_{j=1}^\infty W_{nkj}$, where each W_{nkj} is open. For $j = 1, 2, \cdots$ let $\mathscr{K}_{nkj} = \mathscr{H}_{nk} \cup \{W_{nkj}\}$. Then each $\mathscr{K}_{nkj}$ is an open cover of X and the sequence $\{\mathscr{K}_{nkj}\colon n, k, j$ in $N\}$ exhibits the G_δ-diagonal property for X.

PROPOSITION 3.3. *Every θ-refinable space with a G_δ-diagonal has a θ-separating cover.*

Proof. Let X be a θ-refinable space and let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be open covers of X exhibiting the G_δ-diagonal property for X. For each n in N let $\mathscr{H}_{n1}, \mathscr{H}_{n2}, \cdots$ be a θ-refinement of $\mathscr{G}_n$. Then

$$\{\mathscr{H}_{nk}\colon n = 1, 2, \cdots, k = 1, 2, \cdots\}$$

is a θ-separating cover of X.

The following lemmas, due to Burke and Miscenko [19], play a key role in the proof of our theorem. For the sake of completeness we sketch the proof of Burke's result. (See Remark 1.9 in [6]).

LEMMA 3.4. (Burke) *Let X be a regular, θ-refinable $w\Delta$-space. Then there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that for each x in X,*

(a) $C_x = \bigcap_{n=1}^\infty \operatorname{st}(x, \mathscr{G}_n)$ *is compact;*

(b) $\{\operatorname{st}(x, \mathscr{G}_n)\colon n = 1, 2, \cdots\}$ *is a base for C_x.*

Proof. Let $\mathscr{V}_1, \mathscr{V}_2, \cdots$ be a $w\Delta$-sequence for X. By induction on n construct for each positive integer n a sequence $\mathscr{W}_{n1}, \mathscr{W}_{n2}, \cdots$ of open covers of X such that

(1) for $k = 1, 2, \cdots, \{\bar{W}\colon W$ in $\mathscr{W}_{nk}\}$ refines $\mathscr{V}_n$ and $\mathscr{W}_{ij}$, $1 \leqq i \leqq n-1$, $1 \leqq j \leqq n-1$;

(2) for each x in X there is a k in N such that $\mathrm{ord}(x, \mathscr{W}_{nk})$ is finite.

For $n = 1, 2, \cdots$ let $\mathscr{G}_n = \mathscr{W}_{n1}$. Then the sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ satisfies properties (a) and (b).

LEMMA 3.5. (Miščenko) *Let $\mathscr{V}$ be a point-countable collection of subsets of a set X and let M be a subset of X. Then there are at most countably many finite minimal covers of M by elements of $\mathscr{V}$.*

We now state and prove the main result in this section.

THEOREM 3.6. *Let X be a regular, θ-refinable $w\Delta$-space with a point-countable separating open cover. Then X is a Moore space.*

Proof. We are going to show that X has a θ-separating cover and that every closed subset of X is a G_δ. It follows by Proposition 3.2 that X has a G_δ-diagonal and hence by Corollary 2.6 X is a Moore space.

Let $\mathscr{V}$ be a point-countable separating open cover of X. We assume that $X \in \mathscr{V}$, and hence for every subset M of X there is a finite subcollection of $\mathscr{V}$ which covers M, namely $\{X\}$. Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be open covers of X such that for each x in X,

(a) $C_x = \bigcap_{n=1}^{\infty} \mathrm{st}(x, \mathscr{G}_n)$ is compact;

(b) $\{\mathrm{st}(x, \mathscr{G}_n)\colon n = 1, 2, \cdots\}$ is a base for C_x.

For each n in N let $\mathscr{H}_{n1}, \mathscr{H}_{n2}, \cdots$ be a θ-refinement of $\mathscr{G}_n$. Recall that

(c) $\mathscr{H}_{nk}$ refines $\mathscr{G}_n$, $k = 1, 2, \cdots$;

(d) for each x in X there is a k in N such that $\mathrm{ord}(x, \mathscr{H}_{nk})$ is finite.

X has a θ-separating cover. For each pair of positive integers n and k and for each H in $\mathscr{H}_{nk}$ let $H(n, k, 1), H(n, k, 2), \cdots$ be all finite minimal covers of H by elements of $\mathscr{V}$, and let

$$\mathscr{K}_{nkj} = \{H \cap V\colon H \in \mathscr{H}_{nk}, V \in H(n, k, j)\} .$$

To show that $\{\mathscr{K}_{nkj}\colon n, k, j \text{ in } N\}$ is a θ-separating cover of X, let x and y be two distinct points of X. Choose V_1 in $\mathscr{V}$ such that $x \in V_1$ and $y \notin V_1$, and let $\{V_1, \cdots, V_t\}$ be a finite cover of C_x by elements of $\mathscr{V}$ such that $x \notin V_i$ for $i = 2, \cdots, t$. Now $C_x \subseteq \bigcup_{i=1}^t V_i$ and so by (b) there is a n in N such that $\mathrm{st}(x, \mathscr{G}_n) \subseteq \bigcup_{i=1}^t V_i$. Choose k in N such that $\mathrm{ord}(x, \mathscr{H}_{nk})$ is finite, and let H be some element of $\mathscr{H}_{nk}$ such that $x \in H$. Since $\mathscr{H}_{nk}$ refines $\mathscr{G}_n$, $H \subseteq \mathrm{st}(x, \mathscr{G}_n)$

and so $H \subseteqq \bigcup_{i=1}^{t} V_i$. Choose a minimal subcollection of $\{V_1, \cdots, V_t\}$ which covers H and label it $H(n, k, j)$. Note that $V_1 \in H(n, k, j)$. Thus $(H \cap V_1) \in \mathscr{K}_{nkj}$, $x \in (H \cap V_1)$, and $y \notin (H \cap V_1)$. Finally, suppose $H_1, \cdots, H_r$ are all elements of $\mathscr{H}_{nk}$ containing x. Since $H_i(n, k, j)$ is finite for $i = 1, \cdots, r$ it follows that $\mathrm{ord}(x, \mathscr{H}_{nkj})$ is finite. This completes the proof that X has a θ-separating cover.

Every closed subset of X is a G_δ. Let M be a closed subset of X. For each pair of positive integers n and k, and for each H in $\mathscr{H}_{nk}$ such that $H \cap M \neq \varnothing$, let $H(n, k, j), j = 1, 2, \cdots$ be all finite minimal covers of $H \cap M$ by elements of $\mathscr{V}$. By repeatedly counting a cover if necessary, we may assume that $H(n, k, j)$ exists for all j in N. For $j = 1, 2, \cdots$ let $H^*(n, k, j)$ denote the union of all elements of $H(n, k, j)$, and let $W_{nkj} = \cup \{H \cap (\bigcap_{i=1}^{j} H^*(n, k, i)): H \in \mathscr{H}_{nk}, H \cap M \neq \varnothing\}$. Clearly each W_{nkj} is open and contains M. To complete the proof that M is a G_δ it suffices to show that if $x \notin M$ then there exist n, k, and j such that $x \notin W_{nkj}$.

First suppose that $C_x \cap M = \varnothing$. Choose n in N such that $\mathrm{st}(x, \mathscr{G}_n) \cap M = \varnothing$, and let k and j be any positive integers. Suppose $x \in W_{nkj}$. Then there is a H in $\mathscr{H}_{nk}$ such that $x \in H$ and $H \cap M \neq \varnothing$. Now $\mathscr{H}_{nk}$ refines $\mathscr{G}_n$ and so $H \subseteqq \mathrm{st}(x, \mathscr{G}_n)$. Hence $\mathrm{st}(x, \mathscr{G}_n) \cap M \neq \varnothing$ and this contradicts the choice of n.

Next suppose that $C_x \cap M \neq \varnothing$. Let $\{V_1, \cdots, V_t\}$ be a finite cover of $C_x \cap M$ by elements of $\mathscr{V}$ such that $x \notin V_r$, $r = 1, \cdots, t$. Choose n in N such that $\mathrm{st}(x, \mathscr{G}_n) \subseteqq (\bigcup_{r=1}^{t} V_r) \cup (X - M)$. Let k in N be such that $\mathrm{ord}(x, \mathscr{H}_{nk})$ is finite and let $H_1, \cdots, H_s$ be all elements of $\mathscr{H}_{nk}$ which contain x and intersect M. For $i = 1, \cdots, s$, $H_i \subseteqq \mathrm{st}(x, \mathscr{G}_n)$ and so $H_i \cap M \subseteqq \bigcup_{r=1}^{t} V_r$. Select from $\{V_1, \cdots, V_t\}$ a minimal subcollection which covers $H_i \cap M$ and label it $H_i(n, k, j_i)$. Now $x \notin H_i^*(n, k, j_i)$ and so if we take $j = \max\{j_1, \cdots, j_s\}$ then $x \notin W_{nkj}$.

4. α-spaces. A space with a σ-closure preserving separating closed cover is called a *$\sigma^\#$-space.* This definition was introduced by Nagata and Siwiec in [21].

PROPOSITION 4.1. *Every subparacompact space with a G_δ-diagonal is a $\sigma^\#$-space.*

Proof. Let X be a subparacompact space and let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be open covers of X exhibiting the G_δ-diagonal property for X. For each n in N let $\mathscr{F}_{n1}, \mathscr{F}_{n2}, \cdots$ be a σ-discrete closed refinement of $\mathscr{G}_n$. Then $\{\mathscr{F}_{nk}: n = 1, 2, \cdots, k = 1, 2, \cdots\}$ is a σ-closure preserving separating closed cover of X.

In [6] Burke showed that a regular $w\Delta$-space is a Moore space if and only if it is a $\sigma^{\#}$-space. His method of proof suggests introducing a new class of spaces which we call α-spaces. We shall show that $\sigma^{\#}$-spaces are α-spaces and that a regular $w\Delta$-space is a Moore space if and only if it is an α-space.

DEFINITION 4.2. A space X is an *α-space* if there is a function g from $N \times X$ into the topology of X such that for each x in X,

(a) $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}$;

(b) if $y \in g(n, x)$ then $g(n, y) \subseteqq g(n, x)$.

Such a function is called an *α-function* for X.

PROPOSITION 4.3. *Every $\sigma^{\#}$-space is an α-space.*

Proof. Let $\mathscr{F}_1, \mathscr{F}_2, \cdots$ be a σ-closure preserving separating closed cover of a $\sigma^{\#}$-space X. For n in N and x in X let

$$g(n, x) = X - \cup \{F \in \mathscr{F}_n \colon x \notin F\} .$$

It is easy to check that the function g is an α-function for X.

PROPOSITION 4.4. *Every space with a σ-point finite separating open cover is an α-space. In particular, every T_1 space with a σ-point finite base is an α-space.*

Proof. Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a σ-point finite separating open cover of a space X. We may assume that $X \in \mathscr{G}_n$ for all n in N. For $n = 1, 2, \cdots$ and x in X let $g(n, x) = \cap \{G \text{ in } \mathscr{G}_n \colon x \text{ in } G\}$. Then the function g is an α-function for X.

The following characterization of semi-stratifiable spaces will be useful in proving the main theorem in this section.

LEMMA 4.5. *The following are equivalent for a space X:*

(a) *X is semi-stratifiable.*

(b) *There is a function g from $N \times X$ into the topology of X such that* (1) *for each x in X, $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}^-$;* (2) *if $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ converges to x.*

(c) *There is a function g from $N \times X$ into the topology of X such that* (1) *for each x in X and n in N, $x \in g(n, x)$;* (2) *if $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then x is a cluster point of the sequence $\langle x_n \rangle$.*

Proof. The equivalence of (a) and (b) is due to Creede [8], and

(b) $\Rightarrow$ (c) is obvious. To complete the proof we show that (c) $\Rightarrow$ (b). Thus, let g be a function satisfying (c), and assume that $g(n+1, x) \subseteqq g(n, x)$ for all n in N and x in X.

To prove (1) of (b), first let $y \in \bigcap_{n=1}^{\infty} g(n, x)$. Then by (2) of (c), y is a cluster point of the sequence $\{x, x, \cdots\}$ and so $y \in \{x\}^-$. Next let $y \in \{x\}^-$. Then $x \in g(n, y)$ for $n = 1, 2, \cdots$ so by (2) of (c) it follows that x is a cluster point of the sequence $\{y, y, \cdots\}$. Thus $y \in g(n, x)$ for $n = 1, 2, \cdots$ and so $y \in \bigcap_{n=1}^{\infty} g(n, x)$.

To prove (2) of (b), let $x \in g(n, x_n)$, $n = 1, 2, \cdots$ and suppose that the sequence $\langle x_n \rangle$ does not converge to x. Then there is a neighborhood W of x and a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \notin W$ for all k in N. Now $x \in g(n_k, x_{n_k}) \subseteqq g(k, x_{n_k})$ for $k = 1, 2, \cdots$ so by (2) of (c), x is a cluster point of the sequence $\langle x_{n_k} \rangle$. But this is impossible, and so we conclude that $\langle x_n \rangle$ converges to x.

THEOREM 4.6. *A regular $w\Delta$-space is a Moore space if and only if it is an α-space.*

Proof. By Propositions 4.1 and 4.3 every Moore space is an α-space. To complete the proof let X be a regular $w\Delta$-space which is also an α-space and let us show that X is a Moore space. By Corollary 2.6 it suffices to show that X is semi-stratifiable.

Let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a $w\Delta$-sequence for X, let g be an α-function for X. We may assume that for x in X and n in N, $g(n+1, x) \subseteqq g(n, x)$. For x in X and $n = 1, 2, \cdots$ let $h(n, x) = g(n, x) \cap \mathrm{st}(x, \mathscr{G}_n)$. We shall show that the function h satisfies (c) of Lemma 4.5.

Clearly (1) of (c) is satisfied. To check (2) let $x \in h(n, x_n)$ for $n = 1, 2, \cdots$. Then for $n = 1, 2, \cdots$, $x \in \mathrm{st}(x_n, \mathscr{G}_n)$ and so $x_n \in \mathrm{st}(x, \mathscr{G}_n)$. Thus the sequence $\langle x_n \rangle$ has a cluster point y. Suppose $y \neq x$. Now $\{y\} = \bigcap_{n=1}^{\infty} g(n, y)$ and so there is a k in N such that $x \notin g(k, y)$. Since y is a cluster point of $\langle x_n \rangle$ there is a $m \geqq k$ such that $x_m \in g(k, y)$. Since g is an α-function for X, $x_m \in g(k, y)$ implies $g(k, x_m) \subseteqq g(k, y)$. But $x \in h(m, x_m) \subseteqq g(m, x_m) \subseteqq g(k, x_m)$ and so $x \in g(k, y)$, a contradiction. Thus $x = y$ and x is a cluster point of $\langle x_n \rangle$.

COROLLARY 4.7. *Every regular $w\Delta$-space with a σ-point finite separating open cover is a Moore space.*

COROLLARY 4.8. *Every regular countably compact space with a σ-point finite separating open cover is metrizable.*

5. A generalization of semi-stratifiable and $w\Delta$-spaces. Let X be a space and let g be a function from $N \times X$ into the topology of

X such that for all x in X and n in N, $x \in g(n, x)$. Consider the following properties of the function g.

(A) If $x \in g(n, x_n)$ and $y_n \in g(n, x_n)$ for $n = 1, 2, \cdots$ then x is a cluster point of the sequence $\langle y_n \rangle$.

(B) If $x \in g(n, x_n)$ and $y_n \in g(n, x_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle y_n \rangle$ has a cluster point.

(C) If $x_n \in g(n, x)$ for $n = 1, 2, \cdots$ then x is a cluster point of the sequence $\langle x_n \rangle$.

(D) If $x_n \in g(n, x)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point.

(E) If $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then x is a cluster point of the sequence $\langle x_n \rangle$.

(F) If $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point.

In [10] Heath proved that developable spaces can be characterized in terms of a function g satisfying (A), and similarly $w\Delta$-spaces can be characterized in terms of a function g satisfying (B). Clearly 1^{st} countable spaces are characterized by (C), and (D) is precisely the definition of a q-space [18]. Finally, as proved in §4, semi-stratifiable spaces are characterized by a function g satisfying (E). These observations suggest introducing a new class of spaces, based on (F), which generalizes semi-stratifiable and $w\Delta$-spaces.

DEFINITION 5.1. A space X is a β-*space* if there is a function g from $N \times X$ into the topology of X such that

(a) for all x in X and n in N, $x \in g(n, x)$;

(b) if $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point.

Such a function is called a β-*function* for X.

THEOREM 5.2. *The following are equivalent for a regular space* X:

(a) *X is semi-stratifiable.*

(b) *X is a β-space with a G_δ^*-diagonal.*

(c) *X is an α-space and a β-space.*

Proof. Clearly (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c). To prove (b) $\Rightarrow$ (a) let g be a β-function for X and let $\mathscr{G}_1, \mathscr{G}_2, \cdots$ be a G_δ^*-sequence for X, where it is assumed that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$ for all n. For x in X and n in N let $h(n, x) = g(n, x) \cap \mathrm{st}(x, \mathscr{G}_n)$. Then h satisfies (c) of Lemma 4.5 and so X is semi-stratifiable.

To prove (c) $\Rightarrow$ (a) let g be a β-function for X and let h be an α-function for X, where $h(n+1, x) \subseteq h(n, x)$ for all n in N and x

in X. For x in X and $n = 1, 2, \cdots$ let $k(n, x) = g(n, x) \cap h(n, x)$. Then k satifies (c) of Lemma 4.5 and so X is semi-stratifiable.

REMARK 5.3. The implication (d) $\Rightarrow$ (a) of Corollary 2.6 and Theorem 4.6 can be proved using the above theorem together with Creede's result that every regular semi-stratifiable $w\Delta$-space is a Moore space.

6. Summary. The relationship between some of the classes of spaces considered in this paper can be summarized in a diagram as follows.

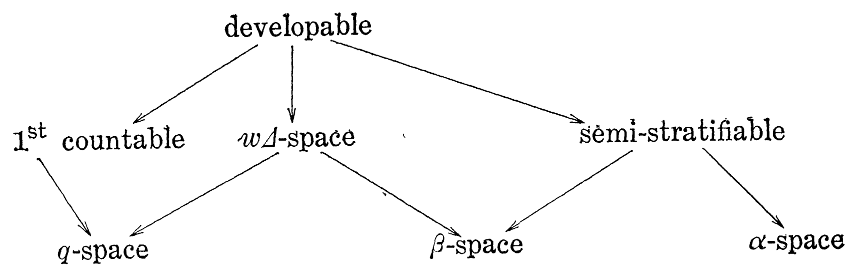

Fig. 1

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