MOORE SPACES AND $w \Delta$-SPACES

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This paper is dedicated to Professor J. H. Roberts on the occasion of his sixty-fifth birthday.

This paper is a study of conditions under which a $w\Omega$-space is a Moore space. In §2 we introduce the notion of a $G^*_\Omega$-diagonal and show that every $w\Omega$-space with a $G^*_\Omega$-diagonal is developable. In §3 we prove that every regular $\theta$-refinable $w\Omega$-space with a point-countable separating open cover is a Moore space. In §4 we introduce the class of $\alpha$-spaces and show that a regular $w\Omega$-space is a Moore space if and only if it is an $\alpha$-space. Finally, in §5 we study a new class of spaces which generalizes both semi-stratifiable and $w\Omega$-spaces.

1. Preliminaries. We begin with some definitions and known results which will be used throughout this paper. Unless otherwise stated no separation axioms are assumed; however regular spaces are always $T_1$ and paracompact spaces are always Hausdorff. The set of natural numbers will be denoted by $N$.

Let $X$ be a set, $\mathcal{G}$ a cover of $X$, $x$ an element of $X$. The star of $x$ with respect to $\mathcal{G}$, denoted $\text{st}(x, \mathcal{G})$, is the union of all elements of $\mathcal{G}$ containing $x$. The order of $x$ with respect to $\mathcal{G}$, denoted $\text{ord}(x, \mathcal{G})$, is the number of elements of $\mathcal{G}$ containing $x$.

A space $X$ is developable if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of open covers of $X$ such that, for each $x$ in $X$, $\{\text{st}(x, \mathcal{G}_n) : n = 1, 2, \cdots\}$ is a fundamental system of neighborhoods of $x$. Such a sequence of open covers is called a development for $X$. A regular developable space is called a Moore space. Bing [1] proved that every paracompact Moore space is metrizable.

According to Borges [3] a space $X$ is a $w\Omega$-space if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of open covers of $X$ such that, for each $x$ in $X$, if $x_n \in \text{st}(x, \mathcal{G}_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a $w\Omega$-sequence for $X$. Clearly every countably compact space is a $w\Omega$-space, and in [3] Borges proved that every developable space and every $M$-space is a $w\Omega$-space. For the relationship between $w\Omega$-spaces, strict $p$-spaces, and $p$-spaces, see [6].

A space $X$ is subparacompact if every open cover of $X$ has a $\sigma$-discrete closed refinement. Every paracompact space is subparacompact [16], and in [8] Creede proved that every semi-stratifiable space is subparacompact. For further properties of subparacompact spaces see [5], [11], and [15].
A space $X$ is \(\theta\)-refinable if for each open cover $\mathcal{V}$ of $X$ there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open refinements of $\mathcal{V}$ such that, for each $x$ in $X$, there is an $n$ in $\mathbb{N}$ such that $\text{ord}(x, \mathcal{G}_n)$ is finite. Such a sequence of open covers is called a $\theta$-refinement of $\mathcal{V}$. In [24] Wicke and Worrell state that every subparacompact space is $\theta$-refinable and that a countably compact $T_1$ space is compact if and only if it is $\theta$-refinable.

2. Spaces with a $G_\gamma$-diagonal. Recall that a space $X$ has a $G_\gamma$-diagonal if its diagonal $\Delta = \{(x, x) : x \in X\}$ is a $G_\gamma$-subset of $X \times X$. The notion of a $G_\gamma$-diagonal plays an important role in metrization theorems; see, for example, [2], [3], [7], [14], and [22]. Every semi-stratifiable Hausdorff space has a $G_\gamma$-diagonal [8]. On the other hand the space $[0, 1] \times \{0, 1\}$ with the lexicographic order is a compact perfectly normal space which fails to have a $G_\gamma$-diagonal [14].

In [7] Ceder obtained this characterization of spaces with a $G_\gamma$-diagonal.

**Proposition 2.1.** (Ceder) A space $X$ has a $G_\gamma$-diagonal if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of $X$ such that, for any two distinct points $x$ and $y$ of $X$, there is an $n$ in $\mathbb{N}$ such that $y \notin \text{st}(x, \mathcal{G}_n)$.

In light of this characterization of a $G_\gamma$-diagonal and Borges' study of spaces with a $\bar{G}_\delta$-diagonal (see [3]), we introduce the following definition.

**Definition 2.2.** A space $X$ has a $G_\delta^\ast$-diagonal if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of $X$ such that, for any two distinct points $x$ and $y$ of $X$, there is an $n$ in $\mathbb{N}$ such that $y \notin \text{st}(x, \mathcal{G}_n)$. Such a sequence of open covers is called a $G_\delta^\ast$-sequence for $X$.

In [13] Kullman proved that every regular $\theta$-refinable space with a $G_\gamma$-diagonal has a $\bar{G}_\delta$-diagonal. Since every space with a $\bar{G}_\delta$-diagonal has a $G_\delta^\ast$-diagonal, we have the following proposition.

**Proposition 2.3.** Every regular $\theta$-refinable space with a $G_\gamma$-diagonal has a $G_\delta^\ast$-diagonal. In particular every regular semi-stratifiable space has a $G_\delta^\ast$-diagonal.

The next result relates the $G_\delta^\ast$-diagonal property to the diagonal $\Delta$.

**Proposition 2.4.** Let $X$ be a space, let $\{V_n : n = 1, 2, \ldots\}$ be a
sequence of open subsets of $X \times X$ containing $\Delta$, and suppose that
$\bigcap_{n=1}^{\infty} V_n = \Delta$. Then $X$ has a $G^*_\delta$-diagonal. In particular, if $X$ is
Hausdorff and $X \times X$ is perfectly normal then $X$ has a $G^*_\delta$-diagonal.

Proof. For $n = 1, 2, \ldots$ let $\mathcal{G}_n = \{G \subseteq X : G \text{ open, } G \times G \subseteq V_n\}$. Since $V_n$ is open and contains $\Delta$, $\mathcal{G}_n$ covers $X$. To show that $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is a $G^*_\delta$-sequence for $X$, let $x$ and $y$ be distinct points of $X$. Choose $n$ in $\mathbb{N}$ such that $(x, y) \notin V_n$, and let $U$ and $W$ be open neighborhoods of $x$ and $y$ respectively such that $(U \times W) \cap V_n = \emptyset$. It follows that $W \cap \text{st}(x, \mathcal{G}_n) = \emptyset$ and so $y \notin \text{st}(x, \mathcal{G}_n)$.

We now prove the main result in this section.

**Theorem 2.5.** Every $w_4$-space with a $G^*_\delta$-diagonal is developable.

Proof. Let $X$ be a space, let $\mathcal{H}_1, \mathcal{H}_2, \ldots$ be a $w_4$-sequence for $X$, and let $\mathcal{G}_1, \mathcal{G}_2, \ldots$ be a $G^*_\delta$-sequence for $X$. For each positive integer $n$ let

$$
\mathcal{G}_n = \left\{ G : G = \left( \bigcap_{i=1}^{n} H_i \right) \cap \left( \bigcap_{i=1}^{n} K_i \right), H_i \in \mathcal{H}_i, K_i \in \mathcal{H}_i, i = 1, \ldots, n \right\}.
$$

It is easy to check that $\mathcal{G}_{n+1}$ is an open refinement of $\mathcal{G}_n$ for all $n$ in $\mathbb{N}$ and that $\mathcal{G}_1, \mathcal{G}_2, \ldots$ in a $w_4$-sequence and a $G^*_\delta$-sequence for $X$.

Suppose that $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is not a development for $X$. Then there is a point $x$, a neighborhood $W$ of $x$, and a sequence $\langle x_n \rangle$ such that for all $n$, $x_n \in \text{st}(x, \mathcal{G}_n)$ and $x_n \notin W$. Since $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is a $w_4$-sequence for $X$, the sequence $\langle x_n \rangle$ has a cluster point $p$. Clearly $p \notin W$ so $p \neq x$. Since $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is a $G^*_\delta$-sequence for $X$, there is a positive integer $k$ and a neighborhood $V$ of $p$ such that $V \cap \text{st}(x, \mathcal{G}_k) = \emptyset$. Now for $n \geq k$, $x_n \in \text{st}(x, \mathcal{G}_n) \subseteq \text{st}(x, \mathcal{G}_k)$ and so $x_n \notin V$. This contradicts the fact that $p$ is a cluster point of $\langle x_n \rangle$. Thus $\mathcal{G}_1, \mathcal{G}_2, \ldots$ is a development for $X$.

**Corollary 2.6.** The following are equivalent for a regular $w_4$-space $X$:

(a) $X$ is a Moore space.
(b) $X$ is semi-stratifiable.
(c) $X$ is $\theta$-refinable and has a $G^*_\delta$-diagonal.
(d) $X$ has a $G^*_\delta$-diagonal.

Proof. The implication (a) $\Rightarrow$ (b) is due to Creede [8]; (b) $\Rightarrow$ (c) follows from results by Creede [8] and Wicke and Worrell [24]; (c) $\Rightarrow$ (d) follows from Proposition 2.3; (d) $\Rightarrow$ (a) follows from Theorem 2.5.
Remark 2.7. The equivalence of (a) and (b) was first proved by Creede in [8], and the equivalence of (a) and (c) is due to Siwiec [23]. It is not known if every regular $\omega I$-space with a $G_\delta$-diagonal is a Moore space. For a study of p-spaces with a $G_\delta$-diagonal, see [13].

Corollary 2.8. The following are equivalent for a regular countably compact space $X$:

(a) $X$ is metrizable.
(b) $X \times X \times X$ is completely normal.
(c) $X \times X$ is perfectly normal.
(d) $X$ has a $G_\delta^*$-diagonal.

Proof. Clearly (a) $\Rightarrow$ (b); (b) $\Rightarrow$ (c) follows from a theorem due to Katetov [12]; (c) $\Rightarrow$ (d) follows from Proposition 2.4. To prove (d) $\Rightarrow$ (a) observe that $X$ is a Moore space (by Corollary 2.6) and recall that every countably compact Moore space is metrizable.

3. Separating covers. In 1938 Filippov [9] proved that every paracompact $M$-space with a point-countable base is metrizable. Filippov's theorem was generalized by Burke and Stoltenberg in [4], and recently Burke [6] obtained another generalization as follows.

Burke's Theorem. Every regular subparacompact $wJ$-space with a point-countable base is a Moore space.

In another direction Nagata [20] proved a metrization theorem which not only generalizes Filippov's theorem but a result by Okuyama as well [22]. In order to state Nagata's theorem succinctly we use the following terminology due to Michael [17]. A cover $\mathcal{V}$ of a set $X$ is said to be separating if given distinct points $x$ and $y$ of $X$, there is a $V$ in $\mathcal{V}$ such that $x \in V, y \notin V$.

Nagata's Theorem. Every paracompact $M$-space with a point-countable separating open cover is metrizable.

In this section we use the techniques developed by Burke, Filippov, Nagata, and Stoltenberg, together with the results in §2, to obtain a generalization of the abovementioned theorems by Burke and Nagata.

In light of the usefulness of the concept of a $\theta$-base in the study of developable spaces (see [24]), we begin with the following definition.
DEFINITION 3.1. A θ-separating cover of a space $X$ is a sequence $\mathcal{G}_1, \mathcal{G}_2, \cdots$ of open collections such that, for any two distinct points $x$ and $y$ in $X$, there is a $n$ in $N$ such that
(a) $\text{ord}(x, \mathcal{G}_n)$ is finite;
(b) there is a $G$ in $\mathcal{G}_n$ such that $x \in G$ and $y \notin G$.

The relationship between a θ-separating cover and a $G_\delta$-diagonal is given by the following two propositions.

PROPOSITION 3.2. Let $X$ be a space with a θ-separating cover. If every closed subset of $X$ is a $G_\delta$ then $X$ has a $G_\delta$-diagonal.

Proof. Let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be a θ-separating cover of $X$. For each pair of positive integers $n$ and $k$ let $\mathcal{H}_{nk} = \{H: H \neq \emptyset, H = \bigcap_{i=1}^{k} G_i, G_1, \cdots, G_k \} \text{ distinct elements of } \mathcal{G}_n$ and let $F_{nk} = X - \bigcup \{H: H \in \mathcal{H}_{nk}\}$. Now $F_{nk}$ is a closed set and so $F_{nk} = \bigcap_{j=1}^{n} W_{nj}$, where each $W_{nj}$ is open. For $j = 1, 2, \cdots$ let $\mathcal{K}_{nkj} = \mathcal{H}_{nk} \cup \{W_{nj}\}$. Then each $\mathcal{K}_{nkj}$ is an open cover of $X$ and the sequence $\{\mathcal{K}_{nkj}: n, k, j \text{ in } N\}$ exhibits the $G_\delta$-diagonal property for $X$.

PROPOSITION 3.3. Every θ-refinable space with a $G_\delta$-diagonal has a θ-separating cover.

Proof. Let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be a θ-separating cover of $X$. For each $n$ in $N$ let $\mathcal{K}_n, \mathcal{K}_{n2}, \cdots$ be a θ-refinement of $\mathcal{G}_n$. Then
\[ \{\mathcal{K}_n: n = 1, 2, \cdots, k = 1, 2, \cdots\} \]

is a θ-separating cover of $X$.

The following lemmas, due to Burke and Miscenko [19], play a key role in the proof of our theorem. For the sake of completeness we sketch the proof of Burke’s result. (See Remark 1.9 in [6]).

LEMMA 3.4. (Burke) Let $X$ be a regular, θ-refinable $\mathcal{W}_\Delta$-space. Then there is a sequence $\mathcal{H}_1, \mathcal{H}_2, \cdots$ of open covers of $X$ such that for each $x$ in $X$,
(a) $C_x = \bigcap_{n=1}^{\infty} \text{st}(x, \mathcal{G}_n)$ is compact;
(b) $\{\text{st}(x, \mathcal{G}_n): n = 1, 2, \cdots\}$ is a base for $C_x$.

Proof. Let $\mathcal{V}_1, \mathcal{V}_2, \cdots$ be a $\mathcal{W}_\Delta$-sequence for $X$. By induction on $n$ construct for each positive integer $n$ a sequence $\mathcal{W}_{n1}, \mathcal{W}_{n2}, \cdots$ of open covers of $X$ such that
\[ \text{ for } k = 1, 2, \cdots, \{\mathcal{W}: W \in \mathcal{W}_{nk}\} \text{ refines } \mathcal{V}_n \text{ and } \mathcal{W}_{nk}, \]
\[ 1 \leq i \leq n - 1, 1 \leq j \leq n - 1; \]
(2) for each \( x \) in \( X \) there is a \( k \) in \( N \) such that \( \text{ord}(x, W_{nk}) \) is finite.

For \( n = 1, 2, \ldots \) let \( \mathcal{G}_n = W_{n1} \). Then the sequence \( \mathcal{G}_1, \mathcal{G}_2, \ldots \) satisfies properties (a) and (b).

**Lemma 3.5.** (Miščenko) Let \( \mathcal{V} \) be a point-countable collection of subsets of a set \( X \) and let \( M \) be a subset of \( X \). Then there are at most countably many finite minimal covers of \( M \) by elements of \( \mathcal{V} \).

We now state and prove the main result in this section.

**Theorem 3.6.** Let \( X \) be a regular, \( \theta \)-refinable \( w\Delta \)-space with a point-countable separating open cover. Then \( X \) is a Moore space.

*Proof.* We are going to show that \( X \) has a \( \theta \)-separating cover and that every closed subset of \( X \) is a \( G_\delta \). It follows by Proposition 3.2 that \( X \) has a \( G_\delta \)-diagonal and hence by Corollary 2.6 \( X \) is a Moore space.

Let \( \mathcal{V} \) be a point-countable separating open cover of \( X \). We assume that \( X \in \mathcal{V} \); and hence for every subset \( M \) of \( X \) there is a finite subcollection of \( \mathcal{V} \) which covers \( M \), namely \( \{X\} \). Let \( \mathcal{G}_1, \mathcal{G}_2, \ldots \) be open covers of \( X \) such that for each \( x \) in \( X \),

- (a) \( C_x = \bigcap_{n=1}^\infty \text{st}(x, \mathcal{G}_n) \) is compact;
- (b) \( \{\text{st}(x, \mathcal{G}_n) : n = 1, 2, \ldots \} \) is a base for \( C_x \).

For each \( n \) in \( N \) let \( H_{nk1}, H_{nk2}, \ldots \) be a \( \theta \)-refinement of \( \mathcal{G}_n \). Recall that

- (c) \( H_{nk} \) refines \( \mathcal{G}_n \), \( k = 1, 2, \ldots \);
- (d) for each \( x \) in \( X \) there is a \( k \) in \( N \) such that \( \text{ord}(x, H_{nk}) \) is finite.

\( X \) has a \( \theta \)-separating cover. For each pair of positive integers \( n \) and \( k \) and for each \( H \) in \( H_{nk} \) let \( H(n, k, 1), H(n, k, 2), \ldots \) be all finite minimal covers of \( H \) by elements of \( \mathcal{V} \); and let

\[ H_{nkJ} = \{H \cap V : H \in H_{nk}, V \in H(n, k, j)\} . \]

To show that \( \{H_{nkJ} : n, k, j \in N\} \) is a \( \theta \)-separating cover of \( X \), let \( x \) and \( y \) be two distinct points of \( X \). Choose \( V_i \) in \( \mathcal{V} \) such that \( x \in V_i \) and \( y \notin V_i \), and let \( \{V_1, \ldots, V_t\} \) be a finite cover of \( C_x \) by elements of \( \mathcal{V} \) such that \( x \in V_i \) for \( i = 2, \ldots, t \). Now \( C_x \subseteq \bigcup_{i=1}^t V_i \) and so by (b) there is a \( n \) in \( N \) such that \( \text{st}(x, \mathcal{G}_n) \subseteq \bigcup_{i=1}^t V_i \).

Choose \( k \) in \( N \) such that \( \text{ord}(x, H_{nk}) \) is finite, and let \( H \) be some element of \( H_{nk} \) such that \( x \in H \). Since \( H_{nk} \) refines \( \mathcal{G}_n \), \( H \subseteq \text{st}(x, \mathcal{G}_n) \).
and so $H \subseteq \bigcup_{i=-1}^{\infty} V_i$. Choose a minimal subcollection of \{V_1, \ldots, V_t\} which covers $H$ and label it $H(n, k, j)$. Note that $V_i \in H(n, k, j)$. Thus $(H \cap V_i) \in \mathcal{H}_{nkj}$, $x \in (H \cap V_i)$, and $y \notin (H \cap V_i)$. Finally, suppose $H_i, \ldots, H_s$ are all elements of $\mathcal{H}_{nk}$ containing $x$. Since $H_i(n, k, j)$ is finite for $i = 1, \ldots, s$ it follows that $\text{ord}(x, \mathcal{H}_{nk})$ is finite. This completes the proof that $X$ has a $\theta$-separating cover.

Every closed subset of $X$ is a $G_\delta$. Let $M$ be a closed subset of $X$. For each pair of positive integers $n$ and $k$, and for each $H$ in $\mathcal{H}_{nk}$ such that $H \cap M \neq \emptyset$, let $H(n, k, j)$, $j = 1, 2, \ldots$ be all finite minimal covers of $H \cap M$ by elements of $\mathcal{V}$. By repeatedly counting a cover if necessary, we may assume that $H(n, k, j)$ exists for all $j$ in $N$. For $j = 1, 2, \ldots$ let $H^*(n, k, j)$ denote the union of all elements of $H(n, k, j)$, and let $W_{nk} = \bigcup \{H \cap (\bigcap_{i=1}^{k} H^*(n, k, i)) : H \in \mathcal{H}_{nk} \}$. Clearly each $W_{nk}$ is open and contains $M$. To complete the proof that $M$ is a $G_\delta$ it suffices to show that if $x \in M$ then there exist $n, k$, and $j$ such that $x \in W_{nkj}$.

First suppose that $C_x \cap M = \emptyset$. Choose $n$ in $N$ such that $\text{st}(x, \mathcal{C}_n) \cap M = \emptyset$, and let $k$ and $j$ be any positive integers. Suppose $x \in W_{nkj}$. Then there is a $H$ in $\mathcal{H}_{nk}$ such that $x \in H$ and $H \cap M \neq \emptyset$. Now $\mathcal{H}_{nk}$ refines $\mathcal{C}_n$ and so $H \subseteq \text{st}(x, \mathcal{C}_n)$. Hence $\text{st}(x, \mathcal{C}_n) \cap M = \emptyset$ and this contradicts the choice of $n$.

Next suppose that $C_x \cap M \neq \emptyset$. Let $\{V_1, \ldots, V_t\}$ be a finite cover of $C_x \cap M$ by elements of $\mathcal{V}$ such that $x \notin V_r$, $r = 1, \ldots, t$. Choose $n$ in $N$ such that $\text{st}(x, \mathcal{C}_n) \subseteq (\bigcup_{r=1}^{t} V_r) \cup (X - M)$. Let $k$ in $N$ be such that $\text{ord}(x, \mathcal{H}_{nk})$ is finite and let $H_1, \ldots, H_s$ be all elements of $\mathcal{H}_{nk}$ which contain $x$ and intersect $M$. For $i = 1, \ldots, s$, $H_i \subseteq \text{st}(x, \mathcal{C}_n)$ and so $H_i \cap M \subseteq \bigcup_{r=1}^{t} V_r$. Select from $\{V_1, \ldots, V_t\}$ a minimal subcollection which covers $H_i \cap M$ and label it $H_i(n, k, j_i)$. Now $x \notin H_i(n, k, j_i)$ and so if we take $j = \max\{j_1, \ldots, j_s\}$ then $x \in W_{nkj}$.

4. $\alpha$-spaces. A space with a $\sigma$-closure preserving separating closed cover is called a $\sigma^\alpha$-space. This definition was introduced by Nagata and Siwiec in [21].

Proposition 4.1. Every subparacompact space with a $G_\delta$-diagonal is a $\sigma^\alpha$-space.

Proof. Let $X$ be a subparacompact space and let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be open covers of $X$ exhibiting the $G_\delta$-diagonal property for $X$. For each $n$ in $N$ let $\mathcal{F}_{n1}, \mathcal{F}_{n2}, \ldots$ be a $\sigma$-discrete closed refinement of $\mathcal{F}_n$. Then $\{\mathcal{F}_{nk} : n = 1, 2, \ldots, k = 1, 2, \ldots\}$ is a $\sigma$-closure preserving
separating closed cover of $X$.

In [6] Burke showed that a regular $w \Delta$-space is a Moore space if and only if it is a $\sigma^*$-space. His method of proof suggests introducing a new class of spaces which we call $\alpha$-spaces. We shall show that $\sigma^*$-spaces are $\alpha$-spaces and that a regular $w \Delta$-space is a Moore space if and only if it is an $\alpha$-space.

**DEFINITION 4.2.** A space $X$ is an $\alpha$-space if there is a function $g$ from $\mathbb{N} \times X$ into the topology of $X$ such that for each $x$ in $X$,

(a) $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}$;

(b) if $y \in g(n, x)$ then $g(n, y) \subseteq g(n, x)$.

Such a function is called an $\alpha$-function for $X$.

**PROPOSITION 4.3.** Every $\sigma^*$-space is an $\alpha$-space.

**Proof.** Let $\mathcal{F}_1, \mathcal{F}_2, \cdots$ be a $\sigma$-closure preserving separating closed cover of a $\sigma^*$-space $X$. For $n$ in $\mathbb{N}$ and $x$ in $X$ let

$$g(n, x) = X - \bigcup \{F \in \mathcal{F}_n : x \notin F\}.$$ 

It is easy to check that the function $g$ is an $\alpha$-function for $X$.

**PROPOSITION 4.4.** Every space with a $\sigma$-point finite separating open cover is an $\alpha$-space. In particular, every $T_1$ space with a $\sigma$-point finite base is an $\alpha$-space.

**Proof.** Let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be a $\sigma$-point finite separating open cover of a space $X$. We may assume that $X \in \mathcal{G}_n$ for all $n$ in $\mathbb{N}$. For $n = 1, 2, \cdots$ and $x$ in $X$ let $g(n, x) = \bigcap \{G \in \mathcal{G}_n : x \in G\}$. Then the function $g$ is an $\alpha$-function for $X$.

The following characterization of semi-stratifiable spaces will be useful in proving the main theorem in this section.

**LEMMA 4.5.** The following are equivalent for a space $X$:

(a) $X$ is semi-stratifiable.

(b) There is a function $g$ from $\mathbb{N} \times X$ into the topology of $X$ such that (1) for each $x$ in $X$, $\bigcap_{n=1}^{\infty} g(n, x) = \{x\}$; (2) if $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then the sequence $\langle x_n \rangle$ converges to $x$.

(c) There is a function $g$ from $\mathbb{N} \times X$ into the topology of $X$ such that (1) for each $x$ in $X$ and $n$ in $\mathbb{N}$, $x \in g(n, x)$; (2) if $x \in g(n, x_n)$ for $n = 1, 2, \cdots$ then $x$ is a cluster point of the sequence $\langle x_n \rangle$.

**Proof.** The equivalence of (a) and (b) is due to Creede [8], and
(b) $\implies$ (c) is obvious. To complete the proof we show that (c) $\implies$ (b). Thus, let $g$ be a function satisfying (c), and assume that $g(n + 1, x) \subseteq g(n, x)$ for all $n$ in $\mathbb{N}$ and $x$ in $X$.

To prove (1) of (b), first let $y \in \bigcap_{n=1}^{\infty} g(n, x)$. Then by (2) of (c), $y$ is a cluster point of the sequence $\{x, x, \ldots\}$ and so $y \in \{x\}^-$. Next let $y \in \{x\}^-$. Then $x \in g(n, y)$ for $n = 1, 2, \ldots$ so by (2) of (c) it follows that $x$ is a cluster point of the sequence $\{y, y, \ldots\}$. Thus $y \in g(n, x)$ for $n = 1, 2, \ldots$ and so $y \in \bigcap_{n=1}^{\infty} g(n, x)$.

To prove (2) of (b), let $x \in g(n, x_n)$, $n = 1, 2, \ldots$ and suppose that the sequence $\langle x_n \rangle$ does not converge to $x$. Then there is a neighborhood $W$ of $x$ and a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $x_{n_k} \notin W$ for all $k$ in $\mathbb{N}$. Now $x \in g(n_x, x_{n_k}) \subseteq g(k, x_{n_k})$ for $k = 1, 2, \ldots$ so by (2) of (c), $x$ is a cluster point of the sequence $\langle x_{n_k} \rangle$. But this is impossible, and so we conclude that $\langle x_n \rangle$ converges to $x$.

**Theorem 4.6.** A regular $\omega$-space is a Moore space if and only if it is an $\alpha$-space.

**Proof.** By Propositions 4.1 and 4.3 every Moore space is an $\alpha$-space. To complete the proof let $X$ be a regular $\omega$-space which is also an $\alpha$-space and let us show that $X$ is a Moore space. By Corollary 2.6 it suffices to show that $X$ is semi-stratifiable.

Let $\mathcal{G}_1, \mathcal{G}_2, \cdots$ be a $\omega$-sequence for $X$, let $g$ be an $\alpha$-function for $X$. We may assume that for $x$ in $X$ and $n$ in $\mathbb{N}$, $g(n + 1, x) \subseteq g(n, x)$. For $x$ in $X$ and $n = 1, 2, \cdots$ let $h(n, x) = g(n, x) \cap \text{st}(x, \mathcal{G}_n)$. We shall show that the function $h$ satisfies (c) of Lemma 4.5.

Clearly (1) of (c) is satisfied. To check (2) let $x \in h(n, x_n)$ for $n = 1, 2, \cdots$. Then for $n = 1, 2, \cdots$, $x \in \text{st}(x_n, \mathcal{G}_n)$ and so $x_n \in \text{st}(x, \mathcal{G}_n)$. Thus the sequence $\langle x_n \rangle$ has a cluster point $y$. Suppose $y \neq x$. Now $\{y\} = \bigcap_{n=1}^{\infty} g(n, y)$ and so there is a $k$ in $\mathbb{N}$ such that $x \in g(k, y)$. Since $y$ is a cluster point of $\langle x_n \rangle$ there is a $m \geq k$ such that $x_m \in g(k, y)$. Since $g$ is an $\alpha$-function for $X$, $x_m \in g(k, y)$ implies $g(k, x_m) \subseteq g(k, y)$. But $x \in h(m, x_m) \subseteq g(m, x_m) \subseteq g(k, x_m)$ and so $x \in g(k, y)$, a contradiction. Thus $x = y$ and $x$ is a cluster point of $\langle x_n \rangle$.

**Corollary 4.7.** Every regular $\omega$-space with a $\sigma$-point finite separating open cover is a Moore space.

**Corollary 4.8.** Every regular countably compact space with a $\sigma$-point finite separating open cover is metrizable.

5. A generalization of semi-stratifiable and $\omega$-spaces. Let $X$ be a space and let $g$ be a function from $\mathbb{N} \times X$ into the topology of
such that for all \( x \in X \) and \( n \in \mathbb{N} \), \( x \in g(n, x) \). Consider the following properties of the function \( g \).

(A) If \( x \in g(n, x_n) \) and \( y_n \in g(n, x_n) \) for \( n = 1, 2, \cdots \) then \( x \) is a cluster point of the sequence \( \langle y_n \rangle \).

(B) If \( x \in g(n, x_n) \) and \( y_n \in g(n, x_n) \) for \( n = 1, 2, \cdots \) then the sequence \( \langle y_n \rangle \) has a cluster point.

(C) If \( x_n \in g(n, x) \) for \( n = 1, 2, \cdots \) then \( x \) is a cluster point of the sequence \( \langle x_n \rangle \).

(D) If \( x_n \in g(n, x) \) for \( n = 1, 2, \cdots \) then the sequence \( \langle x_n \rangle \) has a cluster point.

(E) If \( x \in g(n, x_n) \) for \( n = 1, 2, \cdots \) then \( x \) is a cluster point of the sequence \( \langle x_n \rangle \).

(F) If \( x \in g(n, x_n) \) for \( n = 1, 2, \cdots \) then the sequence \( \langle x_n \rangle \) has a cluster point.

In [10] Heath proved that developable spaces can be characterized in terms of a function \( g \) satisfying (A), and similarly \( wJ \)-spaces can be characterized in terms of a function \( g \) satisfying (B). Clearly 1st countable spaces are characterized by (C), and (D) is precisely the definition of a \( q \)-space [18]. Finally, as proved in §4, semi-stratifiable spaces are characterized by a function \( g \) satisfying (E). These observations suggest introducing a new class of spaces, based on (F), which generalizes semi-stratifiable and \( wJ \)-spaces.

**Definition 5.1.** A space \( X \) is a \( \beta \)-space if there is a function \( g \) from \( \mathbb{N} \times X \) into the topology of \( X \) such that

(a) for all \( x \) in \( X \) and \( n \) in \( \mathbb{N} \), \( x \in g(n, x) \);

(b) if \( x \in g(n, x_n) \) for \( n = 1, 2, \cdots \) then the sequence \( \langle x_n \rangle \) has a cluster point.

Such a function is called a \( \beta \)-function for \( X \).

**Theorem 5.2.** The following are equivalent for a regular space \( X \):

(a) \( X \) is semi-stratifiable.

(b) \( X \) is a \( \beta \)-space with a \( G^*_\beta \)-diagonal.

(c) \( X \) is an \( \alpha \)-space and a \( \beta \)-space.

**Proof.** Clearly (a) \( \Rightarrow \) (b) and (a) \( \Rightarrow \) (c). To prove (b) \( \Rightarrow \) (a) let \( g \) be a \( \beta \)-function for \( X \) and let \( \mathcal{U}, \mathcal{V}, \cdots \) be a \( G^*_\beta \)-sequence for \( X \), where it is assumed that \( \mathcal{V}_{n+1} \) refines \( \mathcal{V}_n \) for all \( n \). For \( x \) in \( X \) and \( n \) in \( \mathbb{N} \) let \( h(n, x) = g(n, x) \cap \mathrm{st}(x, \mathcal{V}_n) \). Then \( h \) satisfies (c) of Lemma 4.5 and so \( X \) is semi-stratifiable.

To prove (c) \( \Rightarrow \) (a) let \( g \) be a \( \beta \)-function for \( X \) and let \( h \) be an \( \alpha \)-function for \( X \), where \( h(n+1, x) \subseteq h(n, x) \) for all \( n \) in \( \mathbb{N} \) and \( x \)
in $X$. For $x$ in $X$ and $n = 1, 2, \ldots$ let $k(n, x) = g(n, x) \cap h(n, x)$. Then $k$ satisfies (c) of Lemma 4.5 and so $X$ is semi-stratifiable.

**Remark 5.3.** The implication $(d) \Rightarrow (a)$ of Corollary 2.6 and Theorem 4.6 can be proved using the above theorem together with Creede’s result that every regular semi-stratifiable $w\mathcal{A}$-space is a Moore space.

6. **Summary.** The relationship between some of the classes of spaces considered in this paper can be summarized in a diagram as follows.

![Diagram](attachment:image.png)

**Fig. 1**

**References**


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DUKE UNIVERSITY
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, *Uniquely representable semigroups on the two-cell* ........................................ 565
Glen Eugene Bredon, *Some examples for the fixed point property* .......... 571
William Lee Bynum, *Characterizations of uniform convexity* .................. 577
Douglas Derry, *The convex hulls of the vertices of a polygon of order n* ...... 583
Edwin Duda and Jack Warren Smith, *Reflexive open mappings* ............... 597
Y. K. Feng and M. V. Subba Rao, *On the density of (k, r) integers* .......... 613
Irving Leonard Glicksberg and Ingemar Wik, *Multipliers of quotients of L₁* ................................................................. 619
John William Green, *Separating certain plane-like spaces by Peano continua* ................................................................. 625
Lawrence Albert Harris, *A continuous form of Schwarz’s lemma in normed linear spaces* .................................................. 635
Richard Earl Hodel, *Moore spaces and w Δ-spaces* .............................. 641
Lawrence Stanislaus Husch, Jr., *Homotopy groups of PL-embedding spaces. II* .......................................................... 653
Yoshinori Isomichi, *New concepts in the theory of topological space—supercondensed set, subcondensed set, and condensed set* ...... 657
J. E. Kerlin, *On algebra actions on a group algebra* ............................... 669
Keizō Kikuchi, *Canonical domains and their geometry in Cⁿ* ................... 681
Ralph David McWilliams, *On iterated w*-sequential closure of cones* ...... 697
C. Robert Miers, *Lie homomorphisms of operator algebras* ................... 717
Louise Elizabeth Moser, *Elementary surgery along a torus knot* .............. 737
Hiroshi Onose, *Oscillatory properties of solutions of even order differential equations* ....................................................... 747
Wellington Ham Ow, *Wiener’s compactification and Φ-bounded harmonic functions in the classification of harmonic spaces* ............... 759
Zalman Rubinstein, *On the multivalence of a class of meromorphic functions* ............................................................... 771
Hans H. Storrer, *Rational extensions of modules* ..................................... 785
Albert Robert Stralka, *The congruence extension property for compact topological lattices* .................................................... 795
Robert Evert Stong, *On the cobordism of pairs* .................................... 803
Albert Leon Whiteman, *An infinite family of skew Hadamard matrices* .... 817
Lynn Roy Williams, *Generalized Hausdorff-Young inequalities and mixed norm spaces* ..................................................... 823