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HOMOTOPY GROUPS OF PL-EMBEDDING SPACES. II

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THEOREM. For $i \leq m - 2$ and $n \leq m - 3$, $\pi_i PL(S^n, S^m)$ is isomorphic to $\pi_i V_{m,n}^{PL}$, the homotopy groups of the PL -Stiefel manifold of n -planes in Euclidean m -space.

E. C. Zeeman [10] conjectured that the homotopy groups, $\pi_i PL(S^n, S^m)$, $m \geq n + i + 3$, of the space of PL -embeddings of the n -sphere into the m -sphere were trivial. As indicated in [4], results of M. C. Irwin [5] and C. Morlet [7] can be used to verify this conjecture. In the theorem above, we generalize this result.

In particular, we have the following [2].

COROLLARY. $\pi_i PL(S^n, S^m) = 0$ for $i < m - n$.

The author expresses his gratitude to N. Max for correcting errors in an earlier version of this note.

We shall assume familiarity with the Δ -set theory of C. P. Rourke and B. J. Sanderson [9] (or equivalently, the quasisimplicial theory of C. Morlet [8]). Let Δ^i be the standard i -simplex and let $\partial_k: \Delta^i \rightarrow \Delta^{i-1}$ be the k th face map. We shall consider the following Δ -sets which are easily seen to be Kan Δ -sets. We indicate an i -simplex from each. All maps commute with the projection along the second factor and $\partial_k f$ is defined to be the restriction to the product of the appropriate set and $\partial_k \Delta^i$.

$PL(S^n, S^m)$	$f: S^n \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL -embedding.
$PL(S^n, S^m \text{ mod } X)$	$f: S^n \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL -embedding such that $f X \times \Delta^i$ is the identity, $X \subseteq S^n$.
$\text{Aut}(S^m)$	$f: S^m \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL automorphism.
$\text{Aut}(S^m \text{ mod } X)$	$f: S^m \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL -automorphism such that $f X \times \Delta^i$ is the identity, $X \subseteq S^m$.
PL_m	Germ of a PL -automorphism $f: R^m \times \Delta^i \rightarrow R^m \times \Delta^i$ such that $f 0 \times \Delta^i$ is the identity; R^m is Euclidean m -space and 0 is the origin.
$PL_{m,n}$	Germ of a PL -automorphism $f: R^m \times \Delta^i \rightarrow R^m \times \Delta^i$ such that $f R^n \times \Delta^i$ is the identity; $R^n = R^n \times 0 \subseteq R^m \times R^{m-n} = R^m$.

The quotient complex $PL_m/PL_{m,n} = V_{m,n}^{PL}$ is the PL -Stiefel manifold introduced by A. Heffliger and V. Poenaru [1].

PROPOSITION 1. $PL_{m,n} \subseteq PL_m \xrightarrow{p} V_{m,n}^{PL}$ is a Kan fibration where p is the natural projection.

Let $S^n \subseteq S^m$ be the standard inclusion and define $r: \text{Aut}(S^m) \rightarrow PL(S^n, S^m)$ by $r(f) = f|_{S^n} \times \Delta^i$ where f is an i -simplex of $\text{Aut}(S^m)$. The following was proved by C. Morlet [8].

PROPOSITION 2. $\text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m) \xrightarrow{r} PL(S^n, S^m)$ is a Kan fibration.

Let x and y be distinct points of S^n and define similar to r the map $r': \text{Aut}(S^m \text{ mod } x, y) \rightarrow PL(S^n, S^m \text{ mod } x, y)$. One can similarly prove the following.

PROPOSITION 3. $\text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m \text{ mod } x, y) \xrightarrow{r'} PL(S^n, S^m \text{ mod } x, y)$ is a Kan fibration.

Let $h: S^m - x \rightarrow R^m$ be a PL-homeomorphism such that h is onto, $h(S^n - x) = R^n$ and $h(y) = 0$. Define $q: \text{Aut}(S^m \text{ mod } x, y) \rightarrow PL_m$ by $q(f) = \text{germ of } (h \times id.)f(h \times id.)^{-1}$. Note that $q(\text{Aut}(S^m \text{ mod } S^n)) \subseteq PL_{m,n}$. Let $q' = q|_{\text{Aut}(S^m \text{ mod } S^n)}: \text{Aut}(S^m \text{ mod } S^n) \rightarrow PL_{m,n}$.

PROPOSITION 4. q and q' are homotopy equivalences.

The first part was proved by N. H. Kuiper and R. K. Lashof [6] and the second part can be proved similarly, also, from [6] we have the following.

PROPOSITION 5. The inclusion $\text{Aut}(S^m \text{ mod } x, y) \subseteq \text{Aut}(S^m)$ induces isomorphisms $\pi_i \text{Aut}(S^m \text{ mod } x, y) \rightarrow \pi_i \text{Aut}(S^m)$ for $i \leq m - 2$.

Let f be an i -simplex in $PL(S^n, S^m \text{ mod } x, y)$. By J. F. P. Hudson [3], there exists an i -simplex f' in $\text{Aut}(S^m \text{ mod } x, y)$ such that $r'(f') = f$. Define $q'': PL(S^n, S^m \text{ mod } x, y) \rightarrow V_{m,n}^{PL}$ by $q''(f) = pq(f')$.

PROPOSITION 6. q'' is a well defined Δ -map such that the following diagram is commutative.

$$\begin{array}{ccccc}
 \text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m \text{ mod } x, y) & \xrightarrow{r'} & PL(S^n, S^m \text{ mod } x, y) & & \\
 \downarrow q' & & \downarrow q & & \downarrow q'' \\
 PL_{m,n} & \subseteq & PL_m & \xrightarrow{p} & V_{m,n}^{PL}
 \end{array}$$

Proof. Suppose $F'' \in \text{Aut}(S^m \text{ mod } x, y)$ such that $r'(F'') = f$. Hence there exists $g \in \text{Aut}(S^m \text{ mod } S^n)$ such that $F'' = gf'$. Therefore, $q(F'') = q(gf') = q(g)q(f')$ and $pq(F'') = pq(f')$ since $q(g)$ is in $PL_{m,n}$.

Proof of Theorem. It follows from the above propositions that q'' induces isomorphisms $\pi_i PL(S^n, S^m \text{ mod } x, y) \rightarrow \pi_i V_{m,n}^{PL}$ for all i . Note that the following diagram is commutative.

$$\begin{array}{ccc} \text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m \text{ mod } x, y) & \xrightarrow{r'} & PL(S^n, S^m \text{ mod } x, y) \\ \parallel & & \parallel \\ \text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m) & \xrightarrow{r} & PL(S^n, S^m). \end{array}$$

Hence, from the above propositions, the inclusion induces isomorphisms $\pi_i PL(S^n, S^m \text{ mod } x, y) \rightarrow \pi_i PL(S^n, S^m)$ for $i \leq m - 2$, from which the theorem follows.

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