

# Pacific Journal of Mathematics

## **HOMOTOPY GROUPS OF PL-EMBEDDING SPACES. II**

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# HOMOTOPY GROUPS OF PL-EMBEDDING SPACES, II

L. S. HUSCH

**THEOREM.** For  $i \leq m - 2$  and  $n \leq m - 3$ ,  $\pi_i PL(S^n, S^m)$  is isomorphic to  $\pi_i V_{m,n}^{PL}$ , the homotopy groups of the PL-Stiefel manifold of  $n$ -planes in Euclidean  $m$ -space.

E. C. Zeeman [10] conjectured that the homotopy groups,  $\pi_i PL(S^n, S^m)$ ,  $m \geq n + i + 3$ , of the space of PL-embeddings of the  $n$ -sphere into the  $m$ -sphere were trivial. As indicated in [4], results of M. C. Irwin [5] and C. Morlet [7] can be used to verify this conjecture. In the theorem above, we generalize this result.

In particular, we have the following [2].

**COROLLARY.**  $\pi_i PL(S^n, S^m) = 0$  for  $i < m - n$ .

The author expresses his gratitude to N. Max for correcting errors in an earlier version of this note.

We shall assume familiarity with the  $\Delta$ -set theory of C. P. Rourke and B. J. Sanderson [9] (or equivalently, the quasisimplicial theory of C. Morlet [8]). Let  $\Delta^i$  be the standard  $i$ -simplex and let  $\partial_k: \Delta^i \rightarrow \Delta^{i-1}$  be the  $k$ th face map. We shall consider the following  $\Delta$ -sets which are easily seen to be Kan  $\Delta$ -sets. We indicate an  $i$ -simplex from each. All maps commute with the projection along the second factor and  $\partial_k f$  is defined to be the restriction to the product of the appropriate set and  $\partial_k \Delta^i$ .

$PL(S^n, S^m)$	$f: S^n \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL-embedding.
$PL(S^n, S^m \text{ mod } X)$	$f: S^n \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL-embedding such that $f X \times \Delta^i$ is the identity, $X \subseteq S^n$ .
$\text{Aut}(S^m)$	$f: S^m \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL automorphism.
$\text{Aut}(S^m \text{ mod } X)$	$f: S^m \times \Delta^i \rightarrow S^m \times \Delta^i$ is a PL-automorphism such that $f X \times \Delta^i$ is the identity, $X \subseteq S^m$ .
$PL_m$	Germ of a PL-automorphism $f: R^m \times \Delta^i \rightarrow R^m \times \Delta^i$ such that $f 0 \times \Delta^i$ is the identity; $R^m$ is Euclidean $m$ -space and 0 is the origin.
$PL_{m,n}$	Germ of a PL-automorphism $f: R^m \times \Delta^i \rightarrow R^m \times \Delta^i$ such that $f R^n \times \Delta^i$ is the identity; $R^n = R^n \times 0 \subseteq R^n \times R^{m-n} = R^m$ .

The quotient complex  $PL_m/PL_{m,n} = V_{m,n}^{PL}$  is the PL-Stiefel manifold introduced by A. Heafiger and V. Poenaru [1].

**PROPOSITION 1.**  $PL_{m,n} \subseteq PL_m \xrightarrow{p} V_{m,n}^{PL}$  is a Kan fibration where  $p$  is the natural projection.

Let  $S^n \subseteq S^m$  be the standard inclusion and define  $r: \text{Aut}(S^m) \rightarrow PL(S^n, S^m)$  by  $r(f) = f|S^n \times \Delta^i$  where  $f$  is an  $i$ -simplex of  $\text{Aut}(S^m)$ . The following was proved by C. Morlet [8].

PROPOSITION 2.  $\text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m) \xrightarrow{r} PL(S^n, S^m)$  is a Kan fibration.

Let  $x$  and  $y$  be distinct points of  $S^n$  and define similar to  $r$  the map  $r': \text{Aut}(S^m \text{ mod } x, y) \rightarrow PL(S^n, S^m \text{ mod } x, y)$ . One can similarly prove the following.

PROPOSITION 3.  $\text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m \text{ mod } x, y) \xrightarrow{r'} PL(S^n, S^m \text{ mod } x, y)$  is a Kan fibration.

Let  $h: S^m - x \rightarrow R^m$  be a  $PL$ -homeomorphism such that  $h$  is onto,  $h(S^n - x) = R^n$  and  $h(y) = 0$ . Define  $q: \text{Aut}(S^m \text{ mod } x, y) \rightarrow PL_m$  by  $q(f) = \text{germ of } (h \times id.)f(h \times id.)^{-1}$ . Note that  $q(\text{Aut}(S^m \text{ mod } S^n)) \subseteq PL_{m,n}$ . Let  $q' = q | \text{Aut}(S^m \text{ mod } S^n): \text{Aut}(S^m \text{ mod } S^n) \rightarrow PL_{m,n}$ .

PROPOSITION 4.  $q$  and  $q'$  are homotopy equivalences.

The first part was proved by N. H. Kuiper and R. K. Lashof [6] and the second part can be proved similarly, also, from [6] we have the following.

PROPOSITION 5. The inclusion  $\text{Aut}(S^m \text{ mod } x, y) \subseteq \text{Aut}(S^m)$  induces isomorphisms  $\pi_i \text{Aut}(S^m \text{ mod } x, y) \rightarrow \pi_i \text{Aut}(S^m)$  for  $i \leq m - 2$ .

Let  $f$  be an  $i$ -simplex in  $PL(S^n, S^m \text{ mod } x, y)$ . By J. F. P. Hudson [3], there exists an  $i$ -simplex  $f'$  in  $\text{Aut}(S^m \text{ mod } x, y)$  such that  $r'(f') = f$ . Define  $q'': PL(S^n, S^m \text{ mod } x, y) \rightarrow V_{m,n}^{PL}$  by  $q''(f) = pq(f')$ .

PROPOSITION 6.  $q''$  is a well defined  $\Delta$ -map such that the following diagram is commutative.

$$\begin{array}{ccccc}
 \text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m \text{ mod } x, y) & \xrightarrow{r'} & PL(S^n, S^m \text{ mod } x, y) & & \\
 \downarrow q' & & \downarrow q & & \downarrow q'' \\
 PL_{m,n} & \subseteq & PL_m & \xrightarrow{p} & V_{m,n}^{PL}
 \end{array}$$

*Proof.* Suppose  $F'' \in \text{Aut}(S^m \text{ mod } x, y)$  such that  $r'(F'') = f$ . Hence there exists  $g \in \text{Aut}(S^m \text{ mod } S^n)$  such that  $F'' = gf'$ . Therefore,  $q(F'') = q(gf') = q(g)q(f')$  and  $pq(F'') = pq(f')$  since  $q(g)$  is in  $PL_{m,n}$ .

*Proof of Theorem.* It follows from the above propositions that  $q''$  induces isomorphisms  $\pi_i PL(S^n, S^m \text{ mod } x, y) \rightarrow \pi_i V_{m,n}^{PL}$  for all  $i$ . Note that the following diagram is commutative.

$$\begin{array}{ccc} \text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m \text{ mod } x, y) & \xrightarrow{r'} & PL(S^n, S^m \text{ mod } x, y) \\ \parallel & & \parallel \\ \text{Aut}(S^m \text{ mod } S^n) \subseteq \text{Aut}(S^m) & \xrightarrow{r} & PL(S^n, S^m). \end{array}$$

Hence, from the above propositions, the inclusion induces isomorphisms  $\pi_i PL(S^n, S^m \text{ mod } x, y) \rightarrow \pi_i PL(S^n, S^m)$  for  $i \leq m - 2$ , from which the theorem follows.

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