ON ITERATED $w^*$-SEQUENTIAL CLOSURE OF CONES

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In this paper it is proved that for each countable ordinal number $\alpha \geq 2$ there exists a separable Banach space $X$ containing a cone $P$ such that, if $J_X$ is the canonical map of $X$ into its bidual $X^{**}$, then the $\alpha$th iterated $w^*$-sequential closure $K_\alpha(J_XP)$ of $J_XP$ fails to be norm-closed in $X^{**}$. From such spaces there is constructed a separable space $W$ containing a cone $P$ such that if $2 \leq \beta \leq \alpha$, then $K_\beta(J_WP)$ fails to be norm-closed in $W^{**}$. Further, there is constructed a (non-separable) space $Z$ containing a cone $P$ such that if $2 \leq \beta < \Omega$, then $K_\beta(J_ZP)$ fails to be norm-closed in $Z^{**}$.

1. If $X$ is a real Banach space and $Y$ a subset of $X^{**}$, let $K(Y)$ be the set of elements of $X^{**}$ which are $w^*$-limits of sequences in $Y$. Let $K_\alpha(Y) = Y$ and inductively let $K_\alpha(Y) = K(\bigcup_{\beta < \alpha} K_\beta(Y))$ for $0 < \alpha \leq \Omega$, where $\Omega$ is the first uncountable ordinal. A cone in $X$ is a subset of $X$ which is closed under addition and under multiplication by nonnegative scalars. Our main theorem extends the result of [6] that if $P$ is a cone in $X$, then $K_1(J_XP)$ must be norm-closed but $K_2(J_XP)$ can fail to be norm-closed in $X^{**}$. By contrast it is noted that if $S$ is a compact Hausdorff space and $X = C(S)$ and $\alpha < \Omega$, then $K_\alpha(J_XS)$ is norm-closed, even though for example if $S$ is compact, metric, and uncountable, then $K_\alpha(J_XS)$ is not $w^*$-sequentially closed. It is obvious that for each Banach space $X$ and each subset $Y$ of $X^{**}$, $K_\alpha(Y)$ is $w^*$-sequentially closed and hence norm-closed.

In [7] a Banach space $X$ was exhibited such that $K_2(J_XS)$ is not norm-closed. Whether $K_\alpha(J_XS)$ can fail to be norm-closed for $2 < \alpha < \Omega$ is not known to the author. However, in the present paper it will be convenient to use constructions involving spaces studied in [7].

Section 2 is devoted to a useful relationship between $w^*$-sequential convergence and pointwise convergence of bounded sequences of functions, § 3 to further study of a space constructed in [7], and §§ 4 and 5 to preparation for and proof of the main theorems.

2. Let $S$ be a compact Hausdorff space, $B(S)$ the Banach space of bounded real functions on $S$ with the supremum norm, and $C(S)$ the closed subspace of $B(S)$ consisting of the continuous real functions on $S$. If $A$ is a subset of $B(S)$, let $L(A)$ be the set of all pointwise limits of bounded sequences in $A$, and let $L_\alpha(A)$ be defined inductively by $L_0(A) = A$ and $L_\alpha(A) = L(\bigcup_{\beta < \alpha} L_\beta(A))$ for each ordinal $\alpha$ such that $0 < \alpha \leq \Omega$.

If $X$ is a norm-closed subspace of $C(S)$ and $z \in L_\alpha(X)$, then $z$ is
bounded and Borel measurable and hence is integrable with respect to each finite regular Borel signed measure $\mu$ on $S$. For each $f \in X^*$ there exists a finite regular Borel signed measure $\mu_f$ on $S$ such that $f(x) = \int_S x \, d\mu_f$ for each $x \in X$ [3, p. 265], and by the Hahn-Banach theorem $\mu_f$ can be chosen so that $\|\mu_f\| = \|f\|$. If $\nu_f$ is another finite regular Borel signed measure on $S$ such that $f(x) = \int_S x \, d\nu_f$ for each $x \in X$ then also $\int_S zd\mu_f = \int_S zd\nu_f$ for each $z \in L_0(X)$, by virtue of the bounded convergence theorem and transfinite induction. Hence a mapping $T$ is unambiguously defined from $L_0(X)$ into the space of real functions on $X^*$ by

$$(Tz)(f) = \int_S zd\mu_f \quad (z \in L_0(X), \ f \in X^*).$$

**Theorem 2.1.** If $S$ is a compact Hausdorff space and $X$ a norm-closed subspace of $C(S)$, then $T$ is an isometric isomorphism from $L_0(X)$ onto $K_0(J_xX)$, and $T$ maps $L_0(A)$ onto $K_0(J_xA)$ for each subset $A$ of $X$ and each $\alpha \leq \Omega$.

**Proof.** For each $z \in L_0(X)$ it is trivial that $Tz$ is linear on $X^*$ and that $| (Tz)(f) | \leq ||z|| \|f\|$ for every $f \in X^*$, so that $Tz \in X^{**}$ and $\|Tz\| \leq ||z||$. For each $t \in S$ let $f_t(x) = x(t)$ for all $x \in X$; then clearly $f_t \in X^*$ with $\|f_t\| \leq 1$, and it is easily seen that $(Tz)(f_t) = \int_S zd\mu_f = z(t)$, so that $|z(t)| \leq \|Tz\| \|f_t\| \leq \|Tz\|$ and hence $||z|| \leq \|Tz\|$. Since $T$ is obviously linear, it follows that $T$ is an isometric isomorphism from $L_0(X)$ into $X^{**}$.

Now let $A$ be a subset of $X$. Since the restriction of $T$ to $X$ is $J_x$, it follows that $T[L_0(A)] = TA = J_xA = K_0(J_xA)$. If $0 < \alpha \leq \Omega$ and it is assumed that $T[L_0(A)] = K_0(J_xA)$ for each $\beta < \alpha$, then for each $z \in L_0(A)$ there exists a bounded sequence $\{z_n\}$ in $\bigcup_{\beta < \alpha} L_0(J_xA)$ which converges pointwise to $z$. By the bounded convergence theorem $(Tz)(f) = \lim_n (Tz_n)(f)$ for each $f \in X^*$. Since by assumption $(Tz_n) \subset \bigcup_{\beta < \alpha} K_0(J_xA)$, it follows that $Tz \in K_0(J_xA)$. Conversely, if $F \in K_0(J_xA)$ there exists a sequence $\{F_n\} \subset \bigcup_{\beta < \alpha} K_0(J_xA)$ such that $F_n \xrightarrow{w^*} F$; the sequence $\{F_n\}$ must be bounded [3, p. 60], and by assumption there exists a sequence $\{z_n\} \subset \bigcup_{\beta < \alpha} L_0(J_xA)$ such that $Tz_n = F_n$ for each $n$. Now $\{z_n\}$ is bounded, and if $z(t)$ is defined to be $F(f_t)$ for each $t \in S$ it follows that $\{z_n\}$ converges pointwise to $z$ so that $z \in L_0(A)$. For every $f \in X^*$, $(Tz)(f) = \lim_n (Tz_n)(f)$ by the bounded convergence theorem. Thus $F = Tz \in T[L_0(A)]$, completing the proof that $T[L_0(A)] = K_0(J_xA)$.

By transfinite induction the theorem follows.

**Remark.** If $S$ is a compact Hausdorff space and $X$ is the Banach...
space $C(S)$, then for each $\alpha \leq \Omega$, $L_\alpha(X)$ is the space of bounded Baire functions on $S$ of order $\leq \alpha$ and, just as in the special case of a metric space $S$ [8, p. 132], $L_\alpha(X)$ is norm-closed in $B(S)$ and hence also $K_\alpha(J_X X)$ is norm-closed in $X^{**}$. If $S$ is a compact metric space with uncountably many elements then $S$ has a nonempty dense-in-itself kernel [1, Ch. 9, p. 34]. Hence for each countable $\alpha$ there is a subset $T$ of $S$ of Borel order exactly $\alpha$ [4, p. 207], but then it follows that $L_\alpha(X) \neq L_{\alpha+1}(X)$ [5, p. 299] and hence that $K_\alpha(J_X X) \neq K_{\alpha+1}(J_X X)$ for each countable $\alpha$.

3. The reader is now referred to the proof of Theorem 1 of [7] for the construction, for each real $c \geq 1$, of a Banach space $X \subset C([0; 3])$ having the property that there exists an $x^h \in L_\alpha(X)$ such that $\|x^h\| = 1$ but if $\{y_k\}$ is a bounded sequence in $L_\alpha(X)$ which converges pointwise to $x^h$, then $\lim \inf_k \|y_k\| \geq c$. The remainder of the present paper depends heavily on properties of the space $X$, and the reader will occasionally need to refer to [7]. In particular, note that $X$ is generated by a set $\{x_{pq}: p, q \in \omega\}$ of piecewise linear nonnegative functions of norm $c$ on $[0; 3]$ and that $x^p$ is the pointwise limit of the sequence $\{x^p\} \subset L_\alpha(X)$, where $x^p$ is the pointwise limit of $\{x_{pq}\} \subset X$ and $\|x^p\| = c$ for each $p$. Each $x_{pq}$ has truncated peaks centered at certain of the points $s_{ui}, t_{vj}, 2 + s_{ui}$ where $s_{ui} = 2^{-u}i$ and $t_{vj} = 2 - 2^{-v}(1 + 2^{-j})$ for $u, i, v, j \in \omega$ and $i < 2^u$. Specifically, $x_{pq}(s_{ui}) = x_{pq}(2 + s_{ui}) = 1$ if $p \geq u$, and $x_{pq}(s_{ui}) = 1$ if and only if $p \geq u$. Further, $x_{pq}(t_{vj}) = c$ if $v \leq p \leq j < p + q$ and 0 otherwise. If $\chi(S)$ denotes the characteristic function of the subset $S$ of $[0; 3]$, it turns out that

$$x^p = \chi([s_{pi}: i < 2^p] \cup [2 + s_{pi}: i < 2^p]) + c\chi([t_{vj}: v \leq p \leq j])$$

and that

$$x^0 = \chi([s_{pi}: p \in \omega, i < 2^p] \cup [2 + s_{pi}: p \in \omega, i < 2^p]).$$

**Lemma 3.1.** Let $Q$ be the norm-closed cone in $X$ generated by $\{x_{pq}: p, q \in \omega\}$. Then $Q$ coincides with

$$Q_0 = \{\Sigma_p \Sigma_q a_{pq} x_{pq}: a_{pq} \geq 0, \Sigma_p \Sigma_q a_{pq} < \infty\},$$

where the indicated summations are over the set $\omega$ of all positive integers.

**Proof.** It is clear that $Q_0$ is a cone containing $\{x_{pq}: p, q \in \omega\}$ and contained in $Q$. If $\{z_n\}$ is a sequence in $Q_0$ which converges in norm to some $x \in X$, then each $z_n$ has the form $z_n = \Sigma_p \Sigma_q a_{npq} x_{pq}$ with $a_{npq} \geq 0$ and $\Sigma_p \Sigma_q a_{npq} < \infty$. As noted in [7] the limit $\lim_n a_{npq} = a_{pq}$ exists for all $p, q$; indeed, in the notation of [7],
\[ a_{pq} = e^\alpha (x(t_{pp} - 2^{-2p-q}) - x(t_{pp} - 2^{-2p-q-1})). \]

Clearly each \( a_{pq} \geq 0 \), and if \( r, s \in \omega \) then
\[
\sum_{p \leq s} \sum_{q \leq t} a_{pq} = \lim_{n \to \infty} \sum_{p \leq s} \sum_{q \leq t} a_{npq} \leq \lim_{n} z_{n}(s_{11}) = x(s_{11});
\]

hence \( \sum_{p \leq s} \sum_{q \leq t} a_{pq} \leq x(s_{11}) \) and \( z = \sum_{p \leq s} \sum_{q \leq t} a_{pq} x_{pq} \in Q_{0} \).

Let \( \varepsilon > 0 \) be given. It follows from \([7, p. 1196]\) that each \( x_{pq} \) is continuous and vanishes at 0 and at \( 2 - 2^{-1} \) and hence that each element of \( X \) shares these properties. Since \( s_{p_{1}} \to 0 \), there exists \( t_{i} \in \omega \) such that \( z(s') < \varepsilon \) and \( x(t') < \varepsilon \) for \( s' = s_{p_{1} + 1} \). Since \( \|z_{n} - x\| \to 0 \), there exists \( n' \) such that \( z_{n}(s') < \varepsilon \) for all \( n > n' \). Thus, by \([7]\),
\[
\sum_{p > r} \sum_{q > s} a_{pq} = z(s') < \varepsilon \quad \text{and} \quad \sum_{p \geq r} \sum_{q \leq s} a_{npq} = z(s') < \varepsilon \quad \text{for} \quad n > n'.
\]

Further, since \( t_{i} \to 2 - 2^{-1} \), there exists by continuity \( q_{i} \geq p_{i} \) such that \( z(t_{i}, q_{i}) \) is such that \( z_{n}(t_{i}, q_{i}) < \varepsilon \) for all \( n > n' \). It follows from \([7]\) that
\[
\sum_{p \leq s} \sum_{q \geq t} a_{pq} \leq \sum_{p \leq q_{i}} \sum_{q \geq t_{i} - p} a_{pq} = e^{-\alpha} z(t_{i}, q_{i}) < \varepsilon
\]
and similarly \( \sum_{p \geq r} \sum_{q > s} a_{npq} \leq e^{-\alpha} z(t_{i}, q_{i}) < \varepsilon \) for all \( n > n'' \). Moreover, since \( a_{npq} \to a_{pq} \), there exists \( n_{i} \geq n'' \) such that \( \sum_{p \leq q_{i}} \sum_{q \leq t_{i}} a_{pq} - a_{npq} < \varepsilon \) for all \( n > n_{i} \). Hence for \( n > n_{i} \), the triangle inequality implies that
\[
\|z - z_{n}\| \leq \|\sum_{p > r} \sum_{q > s} a_{pq} x_{pq}\| + \|\sum_{p \geq r} \sum_{q \leq s} a_{npq} x_{pq}\| + \|\sum_{p \leq q_{i}} \sum_{q \geq t_{i} - p} a_{pq} - a_{npq} x_{pq}\| + \|\sum_{p \geq r} \sum_{q > s} a_{npq} x_{pq}\| < 5\varepsilon,
\]

since \( \|x_{pq}\| = c \) for all \( p, q \). Thus \( \|z - z_{n}\| \to 0 \) and therefore \( x = z \in Q_{0} \), proving that \( Q_{0} \) is norm-closed.

**Lemma 3.2.** Let \( Q_{1} = \{\sum_{p \geq 0} b_{p} x^{p} : b_{p} \geq 0, \sum_{p} b_{p} < \infty\} \). Then \( L_{1}(Q) = Q + Q_{1} \).

**Proof.** Since \( L_{1}(Q) \) is a norm-closed cone in \( B([0; 3]) \) by \([6, Theorem 1, p. 192]\) and \( \text{Theorem 2.1, and since } [x^{p}]_{p} \subset L_{1}(Q) \), it is clear that \( Q + Q_{1} \subset L_{1}(Q) \). If \( \{z_{n}\} \) is a bounded sequence in \( Q \) which is pointwise convergent to some \( z \in L_{1}(Q) \), each \( z_{n} \) has the form \( z_{n} = \sum_{p \geq 0} \sum_{q \geq 0} a_{npq} x_{pq} \) with \( a_{npq} \geq 0 \) and \( \sum_{p \geq 0} \sum_{q \geq 0} a_{npq} < \infty \). As in the proof of Lemma 3.1, for all \( p, q \in \omega \) the limit \( a_{pq} = \lim_{n} a_{npq} \) exists. For all \( a_{pq} \),
\[
\sum_{q \geq t} a_{pq} = \lim_{n} \sum_{q \geq t} a_{npq} \leq \lim_{n} e^{-\alpha} z_{n}(t_{pp}) = e^{-\alpha} z(t_{pp});
\]
hence \( \sum_{q \geq t} a_{pq} \leq e^{-\alpha} z(t_{pp}) \) for each \( p \in \omega \). Let \( b_{p} = e^{-\alpha} z(t_{pp}) - \sum_{q \geq t} a_{pq} \) for each \( p \), and note that all the numbers \( a_{pq} \) and \( b_{p} \) are nonnegative.

For \( n, p \in \omega \) let \( u_{np} = \sum_{q \geq t} a_{npq} x_{pq} \) and \( u_{p} = \sum_{q \geq t} a_{pq} x_{pq} + b_{p} x^{p} \). For each \( p \), if \( t \in [0; 3] \) and \( t \) is not of the form \( s_{pi}, 2 + s_{pi}, \) or \( t_{ij} \) with \( v \leq p \)
\( \leq j \), in the notation of \([7, p. 1196]\), \( x_{pq}(t) = 0 \) for all sufficiently large \( q \) and hence \( x^p(t) = 0 \), so that \( u_{np}(t) \rightarrow u_p(t), \) if \( t = s_{p\ell} \) or \( t = 2 + s_{p\ell} \), then
\[
u_{np}(t) = \sum_{q \leq r} a_{npq} = c^{-1}z_n(t_{pp}) \rightarrow c^{-1}z(t_{pp}) = u_p(t).
\]
Finally, if \( v \leq p \leq j \), then
\[
u_{np}(t) = c\sum_{q \leq r} a_{npq} \rightarrow z(t_{pp}) - c\sum_{q < r} a_{pq} = c[b_p + \sum_{q < r} a_{pq}] = u_p(t_{s,i}),
\]
proving that \( \{u_{np}\} \) converges pointwise to \( u_p \) on \([0; 3]\),
For each \( r \in \omega \),
\[
\sum_{r \leq r} \sum_{q \leq r} a_{ppq} + b_p = c^{-1}\sum_{q \leq r} z(t_{pp}) = c^{-1}\lim_n \sum_{r \leq r} z_n(t_{pp}) = \lim_n \sum_{r \leq r} \sum_{q \leq r} a_{npq} \\
\leq \lim_n z_n(s_{1i}) = z(s_{1i}),
\]
Hence \( \sum_{r \leq r} u_p \in Q + Q_1 \). Let \( w = z - \sum_{r \leq r} u_p \); then \( w \) is easily seen to be a Baire function of the first class on \([0; 3] \) and hence by \([8, p. 143]\) \( w \) must have a point \( t \) of continuity in \([2; 3]\).
At each point of the form \( t = 2 + s_{r_i} \) with \( i \) odd, \( u_p(t) = u_p(s_{1i}) \) for each \( p \geq r \) and hence
\[
w(t) = \lim_n (\sum_{q < r} u_{np}(t) + \sum_{r \leq r} \sum_{q \leq r} a_{npq}) - \sum_{r \leq r} u_p(t) \\
= \lim_n (z_n(s_{1i}) - \sum_{r \leq r} u_{np}(s_{1i})) - \sum_{r \leq r} u_p(t) \\
= z(s_{1i}) - \sum_{r \leq r} u_p(s_{1i}) = w(s_{1i}).
\]
Since the set of such points \( t \) is dense in \([2; 3] \), \( w(t_i) = w(s_{1i}) \). On the other hand, it follows from \([7]\) that for each point of the form \( s = 2 + s_{r_i} \pm 2c_{r_i} \) with \( i \) odd, \( x_{pq}(s) = 0 \) whenever \( p \geq r \), and hence
\[
w(s) = \lim_n \sum_{q < r} u_{np}(s) - \sum_{q < r} u_p(s) = 0.
\]
Since the set of such points \( s \) is also dense in \([2; 3]\), it follows that \( w(t_0) = 0 \) and hence that \( w(s_{1i}) = 0 \).
For each \( r \in \omega \) let \( w_r = z - \sum_{q < r} u_p \). Then \( w_r \rightarrow w \) in the norm topology, and \( w_r \) is the pointwise limit of \( \{\sum_{q \leq r} u_{np}\} \). Hence
\[
\|w_r\| \leq \lim sup_n \|\sum_{q \leq r} u_{np}\| \leq c \lim_n \sum_{q \leq r} u_{np}(s_{1i}) = cw_r(s_{1i})
\]
and consequently
\[
\|w\| = \lim_r \|w_r\| \leq c \lim_r w_r(s_{1i}) = cw(s_{1i}) = 0.
\]
Therefore \( w = 0 \) and \( z = \sum_{r \leq r} u_p \in Q + Q_1 \), completing the proof of the lemma.

Note. The last paragraph of the previous proof shows that if
\{z_n\} is a bounded pointwise convergent sequence in \(Q\), then in the notation of that proof for each \(\varepsilon > 0\) there exist \(p_i, n_i \in \omega\) such that \(\Sigma_{p \geq p_i} \Sigma_{q \geq q_i} c_{aq} < \varepsilon\) for all \(n \geq n_i\). Indeed, given \(\varepsilon > 0\) there exists \(n_i\) such that \(\Sigma_{p \geq p_i} \Sigma_{q \geq q_i} c_{aq} < \varepsilon\). Since \(\limsup_n \|\Sigma_{p \geq p_i} u_{aq}\| \leq c w_{p_i} (s_{n_i})\), there exists \(n_i\) such that
\[
\Sigma_{p \geq p_i} \Sigma_{q \geq q_i} c_{aq} = (\Sigma_{p \geq p_i} u_{aq})(s_{n_i}) \leq \|\Sigma_{p \geq p_i} u_{aq}\| < \varepsilon.
\]

**Lemma 3.3.** Let \(Q_2 = \{c_0x^0: c_0 \geq 0\}\). Then \(L_2(Q) = L_2(Q) = Q + Q_1 + Q_2\).

**Proof.** Clearly \(Q + Q_1 + Q_2\) is a cone containing \(L_1(Q)\) and contained in \(L_2(Q)\). To prove the lemma it suffices to show that \(L(Q + Q_1 + Q_2) \subseteq Q + Q_1 + Q_2\). If \(\{z_n\}\) is a bounded sequence in \(Q + Q_1 + Q_2\) which is pointwise convergent to a function \(z\), then each \(z_n\) has the form
\[
z_n = y_n + \Sigma p b_{np} x^p + c_n x^0
\]
where \(y_n \in Q, b_{np} \geq 0, c_n \geq 0, \) and \(\Sigma p b_{np} < \infty\). Since \(\{z_n\}\) is bounded, the diagonal process yields a subsequence \(\{z_{n_i}\}\) of \(z_n\) such that \(c_0 = \lim_i c_{n_i}\) and \(b = \lim_i \Sigma p b_{np}\) exist and \(b = \lim_i \Sigma p b_{np}\) exists for each \(p \in \omega\). It is easily seen from [7, p. 1196] that these limits are finite and nonnegative, that \(\Sigma p b_p \leq b\), and that the sequence \(\{\Sigma p b_{np} x^p + c_n x^0\}\) is pointwise convergent to \(\Sigma p b_p x^p + (c_0 + b - \Sigma p b_p) x^0\). Hence also \(\{y_{n_i}\}\) is pointwise convergent, and by Lemma 3.2 its pointwise limit is in \(Q + Q_1 + Q_2\). Since \(z\) is the pointwise limit of \(\{z_{n_i}\}\), it follows that \(z \in Q + Q_1 + Q_2\).

**Remark.** It is clear from [7] that the representation of each \(z \in L_2(Q)\) in the form \(\Sigma p \Sigma q a_{pq} x^q + \Sigma p b_p x^p + c_0 x^0\) is unique.

4. Given an arbitrary countable ordinal \(\alpha \geq 2\) and a number \(c \geq 1\), we now construct a separable Banach space \(X_\alpha\) containing a cone \(P_\alpha\) for which there exists \(z_\alpha \in L_\alpha(P_\alpha)\) such that \(\|z_\alpha\| = 1\) but such that if \(\{w_n\}\) is a bounded sequence in \(\bigcup_{\beta \leq \alpha} L_\beta(P_\beta)\) converging pointwise to \(z_\alpha\), then \(\lim_n \|w_n\| \geq c\).

Let \(B_\alpha\) be the countable set \(\{(2, 1)\} \cup \{(\beta, \gamma): \alpha \geq \beta > \gamma \geq 2\}\). Then there exists a one-to-one mapping \(\nu_\alpha\) from \(D_\alpha\) onto \(B_\alpha\), where \(D_\alpha = \{1, \cdots, 2^{-3}(\alpha^2 - 3\alpha + 4)\}\) if \(\alpha < \omega\) and \(D_\alpha = \omega\) if \(\alpha \geq \omega\), such that \(\nu_\alpha(1) = (2, 1)\). Let \(U = \{0\} \cup \{n^-: n \in D_\alpha\}\) and let \(S_\alpha\) be the compact subset \([0; 6] \times U\) of \(E^2\). For each real function \(z\) defined on \(S_\alpha\) and each \(u \in U\), let
\[
z^{1,u}(t) = z(t, u), \quad z^{2,u}(t) = z(t + 3, u).
\]
for $t \in [0; 3]$. Further, let $\mathcal{S}_\alpha$ be the set of all type $- \alpha$ generalized sequences $s = (s_\beta; 1 \leq \beta \leq \alpha)$ of positive integers.

Letting $x_{pq}$ be as in § 3 and noting by [7] that $x_{pq}(0) = x_{pq}(3) = 0$ for $p, q \in \omega$, we easily verify that for each $s \in \mathcal{S}_\alpha$ the function $x_s$ defined by

$$
x_s^t = \begin{cases} x_{s(\beta)}^t & \text{if } u > 0, u^{-1} \leq s_1, \nu_\alpha(u^{-1}) = (\beta, \gamma) \\ 0 & \text{if } u > 0, u^{-1} > s_1 \\ 0 & \text{if } u = 0 \\ nx_{s(\gamma)}^t & \text{if } u > 0, \nu_\alpha(u^{-1}) = (\beta, \gamma) \\ 0 & \text{if } u = 0 
\end{cases}
$$

is an element of $C(S_\alpha)$. Let $X_\alpha$ be the norm-closed subspace and $P_\alpha$ the norm-closed cone in $C(S_\alpha)$ generated by $\{x_s; s \in \mathcal{S}_\alpha\}$. Since $S_\alpha$ is compact metric, $C(S_\alpha)$ is separable [3, p. 340] and hence also $X_\alpha$ is separable. Note that $\|x_s\| = c$ for each $s \in \mathcal{S}_\alpha$.

For $1 \leq \delta \leq \alpha$ and $s \in \mathcal{S}_\alpha$ let $z_s$ be defined on $S_\alpha$ by

$$
z_s^t = u^{-1}z_s^{t-1} \begin{cases} x_{s+1}^t & \text{if } u > 0, \nu_\alpha(u^{-1}) = (\beta, \gamma), \beta > \gamma > \delta \\ x_s^t & \text{if } u > 0, \nu_\alpha(u^{-1}) = (\beta, \gamma), \beta > \delta \geq \gamma \\ x_0^t & \text{if } u > 0, \nu_\alpha(u^{-1}) = (\beta, \gamma), \delta \geq \beta > \gamma 
\end{cases}
$$

Thus $\|z_s\| = c$ if $1 \leq \delta < \alpha$, but $\|z_s\| = 1$ for each $s \in \mathcal{S}_\alpha$. In fact, $z_s$ is independent of $s \in \mathcal{S}_\alpha$ and we simply write $z_s$ instead of $z_{s,a}$.

**Lemma 4.1.** For each $s \in \mathcal{S}_\alpha$ and $1 \leq \delta \leq \alpha$, $z_s, z_\delta \in L_\alpha(P_\alpha)$.

*Proof.* If $\delta = 1$ and $s \in \mathcal{S}_\alpha$, then for each $q \in \omega$ let $s^q \in \mathcal{S}_\alpha$ be defined by

$$
s^q_\beta = \begin{cases} q & \text{if } \beta = 1 \\ s_\beta & \text{if } 1 < \beta \leq \alpha. 
\end{cases}
$$

It is easy to verify that $\{x_{s^q}\}_{q=1}^\infty$ is a bounded sequence in $P_\alpha$ converging pointwise to $z_s$, so that $z_s, z_\delta \in L_\alpha(P_\alpha)$.

Proceeding by transfinite induction, assume that $1 < \delta \leq \alpha$ and that $z_s, z_\delta \in L_\alpha(P_\alpha)$ for each $s \in \mathcal{S}_\alpha$ and $1 \leq \varepsilon < \delta$. Let $s \in \mathcal{S}_\alpha$ be given, and let $t^q \in \mathcal{S}_\alpha$ be defined for each $q \in \omega$ by

$$
t^q_\beta = \begin{cases} s_\beta & \text{if } \delta \neq \beta \leq \alpha \\ q & \text{if } \beta = \delta. 
\end{cases}
$$

If $\delta$ is not a limiting ordinal, then $\delta$ has an immediate predecessor $\delta - 1$, and it is straightforward to show that the bounded sequence
\{z_{s_i, \delta-1}\}_{s=1}^{\alpha} in L_{\delta-1}(P_s) converges pointwise to \(z_{s, \delta}\) on \(S_a\). On the other hand, if the countable ordinal \(\delta\) is limiting, there exists an increasing sequence \(\{\varepsilon_j\}_{j=1}^{\infty}\) of ordinals whose limit is \(\delta\), and it can be verified that the bounded sequence \(\{z_{s, \varepsilon_j}\}_{j=1}^{\infty}\) in \(U_{s < \alpha} L_{\delta}(P_s)\) is pointwise convergent to \(z_{s, \delta}\). Thus the lemma is proved inductively. In particular, our proof has shown that \(z_{s, \delta}\), whose norm is 1, is the pointwise limit of a sequence of elements of norm \(c\) in \(U_{s < \alpha} L_{\delta}(P_s)\).

Note that if \(1 \leq \delta \leq \Omega, z \in L_{\delta}(P_a), \ i \in \{1, 2\}\), and \(u \in U\), then \(z^{i, u} \in L_{\delta}(Q) \subseteq L_{\theta}(Q) = Q + Q_1 + Q_2\) by Lemma 3.3, and trivially \(z^{i, 0} = 0\).

**Lemma 4.2.** Let \(1 \leq \delta \leq \Omega\) and \(z \in L_{\delta}(P_a)\) with

\[
z^{*, i} = \sum p \sum q \sum a_{pq}x_{pq} + \sum p \sum b_{pq}x_{pq} + c_{pq}x_{pq},
\]

Then also \(y \in L_{\delta}(P_a)\), where

\[
y^{*, i} = y^{i, *} = \sum p \sum q \sum a_{pq}x_{pq} + \sum p \sum b_{pq}x_{pq} + c_{pq}x_{pq},
\]

\(y^{i, 0} = 0\), and \(uy^{i, u} = y^{i, u} = z^{i, u}\) for each \(u \in U \setminus \{0, 1\}\).

**Proof.** The proof will be by induction on \(\delta\). If \(\delta = 1\), then \(z^{i, 1} \in L_{\delta}(Q) = Q + Q_1\) and hence \(c = 0\). There exists a bounded sequence \(\{w_n\}\) in \(P_s\) which converges pointwise to \(z\) on \(S_a\). Since the finite linear combinations with nonnegative coefficients of elements in \(\{x_s: s \in \mathcal{S}_a\}\) are norm-dense in \(P_s\), each \(w_n\) can be assumed to have the form \(w_n = \sum_{i \in \omega} r_{ni}x_{i_n}\), where each \(s^{i}_n \in \mathcal{S}_a\) each \(r_{ni} \geq 0\), and for each \(n\) there exist only finitely many \(i\) such that \(r_{ni} > 0\). If \(t^{s_i} \in \mathcal{S}_a\) is defined for all \(n, i \in \omega\) by \((t^{s_i})_s = (s^{i}_n)_s\) for \(2 \leq \beta \leq \alpha\) and \((t^{s_i})_1 = n\), then the sequence \(\{w'_n\}\), where \(w'_n = \sum_{i \in \omega} r_{ni}x_{i_n}\), is clearly a bounded sequence in \(P_s\). It will now be shown that \(\{w'_n\}\) converges pointwise to \(y\).

For each \(u \in U \setminus \{0, 1\}\), \(v_u(w^{i, u}) = (\beta, \gamma)\) for some \(\beta, \gamma\) such that \(\beta > \gamma \geq 2\), and hence for each \(n \geq u^{-1}\),

\[
w^{i, u}_n = u^{-1}w^{i, u}_n = \sum_{i \in \omega} r_{ni}x_{i_n},
\]

therefore, \(w^{i, u}_n(t) \rightarrow u^{-1}z^{i, u}(t) = y^{i, u}(t)\) and \(w^{i, u}_n(t) \rightarrow z^{i, u}(t) = y^{i, u}(t)\) for all \(t \in [0; 3]\).

Since the situation for \(u = 0\) is trivial, it remains only to consider the case in which \(u = 1\). Given \(n, p, q \in \omega\) let

\[
a_{pq} = \sum \{r_{ni}: (s^{i}_n)_s = p, (s^{i}_n)_1 = q\}.
\]

Thus each \(a_{pq} \geq 0\), and for each \(n\) there are only finitely many pairs \((p, q)\) for which \(a_{pq} > 0\). Since \(w^{i, u}_n = \sum p \sum q a_{pq}x_{pq}\) for each \(n\), it follows from the proof of Lemma 3.2 and the note following that proof that
for each \( p, q \); that

\[
\lim \sum \alpha_{pq} = a_{pq}
\]

for each \( p \); and that \( \lim \sup \sum \alpha_{pq} \to 0 \) as \( r \to \infty \). Thus given \( \varepsilon > 0 \), there exist \( r \) and \( n_1 \) such that \( \sum_{p \geq r} (\alpha_{pq} + b_p) < \varepsilon/3c \) and \( \sum_{p \geq r} \sum \alpha_{npq} < \varepsilon/3c \) for all \( n > n_1 \). Now \( w^{i_1}_n = \sum_{p} (\sum \alpha_{npq})x_{pq} \), and for each \( t \in [0; 3] \) there exists \( n_2(t) > n_1 \) such that

\[
| (\sum \alpha_{npq})x_{pq}(t) - (\sum \alpha_{pq} + b_p)x_p(t) | < \frac{\varepsilon}{3r}
\]

for each \( n > n_2(t) \) and \( p < r \). It follows easily by the triangle inequality that

\[
| w^{i_1}_n(t) - \sum \alpha_{pq} x_p(t) | < \varepsilon
\]

for each \( n > n_2(t) \). Thus

\[
w^{i_1}_n(t) = w^{i_1}_n(t) \longrightarrow y^{i_1}(t) = y^{i_1}(t)
\]

for all \( t \), completing the proof for \( \delta = 1 \).

Now let \( \delta > 1 \) and assume that the statement of the lemma is true for each ordinal \( \varepsilon \) such that \( 1 \leq \varepsilon < \delta \). If \( z \in L_\varepsilon(P_\alpha) \), there exists a bounded sequence \( \{w_n\} \subset \bigcup_{\varepsilon < \delta} L_\varepsilon(P_\alpha) \) which converges pointwise to \( z \). By the induction hypothesis the sequence \( \{y_n\} \) is contained in \( \bigcup_{\varepsilon < \delta} L_\varepsilon(P_\alpha) \), where, if

\[
w^{i_1}_n = \sum_{p} \alpha_{npq} x_{pq} + \sum_{p} b_{np} x_{p} + c_n x_o,
\]

then

\[
y^{i_1}_n = y^{i_1}_n = \sum_{p} (b_{np} + \sum \alpha_{npq}) x_{p} + c_n x_o,
\]

and \( y^{i_0}_n = y^{i_0}_n = 0 \) and \( uy^{i_0}_n = y^{i_0}_n = w^{i_0}_n \) for \( u \neq 0, 1 \). An easy induction argument shows that \( ||f^*_u|| \leq uc f^{i_0}_u(s_{i_1}) \) for each \( u \in U \) and \( f \in L_\varepsilon(P_\alpha) \), and from this result it follows that the sequence \( \{y_n\} \) is bounded. To see that \( \{y_n\} \) converges pointwise to \( y \), note first that \( y^{i_0}_n = y^{i_0}_n = 0 = y^{i_0}_n = y^{i_0}_n \) for each \( n \). Next, if \( u \neq 0, 1 \) and \( t \in [0; 3] \), then

\[
uy^{i_0}_n(t) = y^{i_0}_n(t) = w^{i_0}_n(t) \longrightarrow z^{i_0}(t) = uy^{i_0}_n(t) = y^{i_0}_n(t).
\]

For \( u = 1 \), since \( y^{i_1}_n = y^{i_1}_n \) and \( y^{i_1}_n = y^{i_1}_n \), it remains only to show that \( y^{i_1}_n(t) \to y^{i_1}(t) \) for each \( t \in [0; 3] \). If \( t \) is not of the form \( s_{pi} \), \( 2 + s_{pi} \), or \( t_{ij} \) with \( v \leq j \), then \( y^{i_1}_n(t) = 0 = y^{i_1}_n(t) \). If \( t = s_{pi} \) or \( 2 + s_{pi} \) with \( i \) odd, then

\[
y^{i_1}_n(t) = w^{i_1}_n(t) - \sum_{p < pi} \sum \alpha_{npq} x_{pq}(t)
\]

and
\[ y^{i,i}(t) = z^{i,i}(t) - \sum_{p < p_i} \sum_{q} a_{pq} x_{pq}(t); \]

since \( w^{i,i}_n(t) \rightarrow z^{i,i}(t) \) and \( a_{npq} \rightarrow a_{pq} \) (as noted in the proof of Lemma 3.1), and since there exists \( q_i \) such that \( x_{pq}(t) = 0 \) whenever \( p < p_i, q > q_i \), it follows that \( y^{i,i}_n(t) \rightarrow y^{i,i}(t) \). Finally, if \( t = t_{ij} \) with \( 1 \leq v \leq j \), then

\[ y^{i,i}_n(t) = w^{i,i}_n(t) + c \sum_{p = p_i} \sum_{q} a_{npq} \rightarrow z^{i,i}(t) + c \sum_{p = p_i} \sum_{q} a_{pq} = y^{i,i}(t). \]

This completes the induction step and hence the proof of the lemma.

**Lemma 4.3.** Let \( 0 \leq \delta \leq \Omega \) and \( z \in L_0(P_\delta) \). Then \( z^{i,u} \leq u^{-1}z^{i,u} \) for each \( u \in U \setminus \{0\} \). If

\[ z^{i,i} = \sum_p \sum_q a_{pq} x_{pq} + \sum_p b_p x^p + c_0 x^0 \]

and if \( q_i \in \omega \), then

\[ z^{i,u} \leq u^{-1}z^{i,u} - c \sum_p \sum_{q < q_i} a_{pq} \]

for each \( u \geq q_i^{-1} \).

**Proof.** The first assertion is immediate by induction on \( \delta \). For the second assertion suppose first that \( z \) has the form \( z = \sum_{s \in \sigma} d_s x_s \) where \( \sigma \) is a finite subset of \( \mathcal{S}_\delta \) and \( d_s \geq 0 \) for each \( s \). Then \( z^{i,i} = \sum_p \sum_q a_{pq} x_{pq} \), where

\[ a_{pq} = \sum_{s \in \sigma : s \neq p, s_i = q}. \]

Thus \( \sum_p \sum_{q < q_i} a_{pq} = \sigma \{s \in \sigma : s_i < q_i\} \) and hence if \( u \geq q_i^{-1} \) and \( \nu_a(u^{-i}) = (\beta, \gamma) \), then

\[ z^{i,u} = u \sum_{s \in \sigma} d_s x_{s^\beta} = u z^{i,u} + u \sum_{s \in \sigma} d_s x_{s^\beta} \leq u(z^{i,u} + \sum_{s \in \sigma} d_s x_{s^\beta}) = u(z^{i,u} + c \sum_p x_{pq}), \]

as desired.

Next, suppose \( z \) is the pointwise limit of a bounded sequence \( \{w_n\}_{n \in \mathbb{N}} \) in \( L_0(P_\delta) \) such that each \( w_n \) has the desired property; i.e., for each \( u \geq q_i^{-1} \),

\[ w^{i,u}_n \geq u^{-1}w^{i,u}_n - c \sum_p \sum_{q < q_i} a_{pq} \]

where

\[ w^{i,i}_n = \sum_p \sum_q a_{pq} x_{pq} + \sum_p b_{np} x^p + c_0 x^0. \]

By the proof of Lemma 3.3 there is a subsequence \( \{w_n\} \) of \( \{w_n\} \) such that \( \{\sum_p \sum_q a_{npq} x_{pq}\} \) is pointwise convergent, and by the note following
Lemma 3.2 for each \( \zeta > 0 \) there exist \( p_i \) and \( i_i \) such that for each \( i > i_i \),
\[
\Sigma_{p \geq p_i} \Sigma_{q < q_i} a_{pq} < \epsilon^*.
\]
Since \( a_{pq} \to a_{pq} \) for each \( p \) and \( q \), there exists \( i_2 > i_1 \) such that for each \( i > i_2 \),
\[
\Sigma_{p < p_i} \Sigma_{q < q_i} a_{pq} < \Sigma_{p < p_i} \Sigma_{q < q_i} a_{pq} + \zeta.
\]
Hence, for each \( i > i_2 \),
\[
\Sigma_{p \geq p_i} \Sigma_{q < q_i} a_{pq} < \Sigma_{p < p_i} \Sigma_{q < q_i} a_{pq} + (1 + c)\zeta.
\]
For each \( t \in [0; 3] \) and \( u \geq q_i^{-1} \),
\[
z^{1,w}(t) = \lim_i w_{i_1}^{w_i}(t) \geq \lim_i (u^{-1}w_{i_1}^{w_i}(t) - c\Sigma_{p \geq p_i} \Sigma_{q < q_i} a_{pq})
\geq u^{-1}z^{1,w}(t) - c[\Sigma_{p \geq p_i} \Sigma_{q < q_i} a_{pq} + (1 + c)\zeta].
\]
Since \( \zeta \) can be arbitrarily small,
\[
z^{1,w} \geq u^{-1}z^{1,w} - c\Sigma_{p \geq p_i} \Sigma_{q < q_i} a_{pq}
\]
for each \( u \geq q_i^{-1} \), as desired.

The preceding paragraphs provide both the base step and the inductive step for the proof of the second assertion of the lemma.

**Lemma 4.4.** Let \( G \) be the set of all \( z \in L_\omega(P_\alpha) \) such that \( z^{1,w} \in Q_1 + Q_2 \). If \( z \in G \), then \( z^{1,w} = u^{-1}z^{1,w} \) for each \( u \in U \setminus \{0\} \).

**Proof.** In the notation of Lemma 4.3, \( a_{pq} = 0 \) for all \( p, q \) and hence \( \Sigma_{p \geq p_i} \Sigma_{q < q_i} a_{pq} = 0 \). The present result now follows immediately from Lemma 4.3.

**Lemma 4.5.** \( L_\delta(P_\alpha) \cap G = \begin{cases} L_{\delta-1}(L_1(P_\alpha) \cap G) & \text{if } 1 \leq \delta < \omega \\ L_\delta(L_1(P_\alpha) \cap G) & \text{if } \omega \leq \delta \leq \Omega. \end{cases} \)

**Proof.** The result is trivial for \( \delta = 1 \). Let \( 1 < \delta < \omega \) and assume the result is true for all \( \varepsilon < \delta \). Then for each \( z \in L_\varepsilon(P_\alpha) \cap G \) it follows from Lemma 4.4 that \( z^{1,w} = u^{-1}z^{1,w} \) for each \( u \neq 0 \). Since \( z \in G \), it follows that \( z \) is identical with the \( y \) occurring in the statement of Lemma 4.2 and hence is the pointwise limit of the bounded sequence \( \{y_n\} \subset G \cap \bigcup_{1 \leq \varepsilon < \delta} L_\varepsilon(P_\alpha) \) which appears in the inductive step of the proof of Lemma 4.2. By the inductive hypothesis
\[
\{y_n\} \subset \bigcup_{1 \leq \varepsilon < \delta} L_{\varepsilon-1}(L_1(P_\alpha) \cap G) = L_{\varepsilon-2}(L_1(P_\delta) \cap G)
\]
and hence $z \in L_{\delta - \varepsilon}(L(P_a) \cap G)$. Conversely, if $z \in L_{\delta - \varepsilon}(L(P_a) \cap G)$, then $z$ is the pointwise limit of a bounded sequence \( \{w_n\} \subset L_{\delta - \varepsilon}(L(P_a) \cap G) \). By the inductive hypothesis $L_{\delta - \varepsilon}(L(P_a) \cap G) = L_{\delta - \varepsilon}(P_a) \cap G$. Hence clearly $z \in L(P_a)$, and also $z \in G$ by the proof of Lemma 3.3. Thus the proof is complete for $\delta < \omega$.

Now let $\omega \leq \delta \leq \Omega$ and assume the result is true for all $\varepsilon < \delta$.

As in the previous case each $z \in L_{\delta}(L(P_a) \cap G)$ is the pointwise limit of a bounded sequence \( \{y_n\} \subset L_{\varepsilon}(L(P_a) \cap G) \). By the inductive hypothesis \( \{y_n\} \subset \bigcup_{\varepsilon < \delta} L_{\varepsilon}(L(P_a) \cap G) \), and hence $z \in L_{\varepsilon}(L(P_a) \cap G)$. Conversely, if $z \in L_{\varepsilon}(L(P_a) \cap G)$, then $z$ is the pointwise limit of a bounded sequence \( \{w_n\} \subset \bigcup_{\varepsilon < \delta} L_{\varepsilon}(L(P_a) \cap G) \). By the inductive hypothesis \( \{w_n\} \subset G \cap \bigcup_{\varepsilon < \delta} L_{\varepsilon}(L(P_a)) \) and hence $z \in G \cap L_{\delta}(L(P_a))$, completing the proof of the lemma.

**Lemma 4.6.** Let $\{w_n\}$ be a bounded sequence in $\bigcup_{\varepsilon < \delta} L_{\varepsilon}(L(P_a))$ which converges pointwise on $S_a$ to the function $z_a$ defined earlier in the present section. If

$$ w_n^{i^1} = \sum_p \sum_q a_{npq} x_{pq} + \sum_p b_n x^p + c_n x^0 $$

for each $n \in \omega$, then $\lim_{n} \sum_p \sum_q a_{npq} = 0$.

**Proof.** If the conclusion is not true, then as in the proof of Lemma 3.3 a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ exists such that $\inf_n \sum_p \sum_q a_{npq} > 0$ and such that the limits $c_0 = \lim c_{n_i}$, $b = \lim b_{n_i}$, $b_p = \lim b_{n_i, p}$, and $a_p = \lim \sum_q a_{npq}$ all exist $(p \in \omega)$. Since $z_{\omega}^{i^1} = x^0$ by definition of $z_a$, the coefficient of each $x_{pq}$ in the unique expansion of $z_{\omega}^{i^1}$ must vanish and it is easily verified that $\{\sum_p b_{n_i, p} x^p + c_{n_i} x^0\}$ and $\sum_p \sum_q a_{npq} x_{pq}$ converge pointwise to $\sum_p b_p x^p + (c_0 + b - \sum_p b_p) x^0$ and $\sum_p a_{npq} x^p$ respectively, as in the proofs of Lemmas 3.3 and 3.2 (note that the symbol $b_p$ is used differently in those two proofs). Hence

$$ z_{\omega}^{i^1} = \sum_p (a_p + b_p) x^p + (c_0 + b - \sum_p b_p) x^0. $$

Now the uniqueness of the expansion of $z_{\omega}^{i^1}$ shows that $a_p + b_p = 0$ for each $p$ and $c_0 + b - \sum_p b_p = 1$. Since $a_p$ and $b_p$ are nonnegative, they must both vanish for each $p$ and hence $c_0 + b = 1$. Now

$$ 1 = z_{\omega}^{i^1}(s_{11}) = \lim_{n} (\sum_p \sum_q a_{npq} + \sum_p b_{n_i, p} + c_{n_i}) $$

$$ = \lim_{n} \sum_p \sum_q a_{npq} + b + c_0 $$

and hence $\lim_{n} \sum_p \sum_q a_{npq} = 0$, contradicting our assumption and thus proving the lemma.

**Theorem 4.1.** If $\{w_n\}$ is a bounded sequence in $\bigcup_{\varepsilon < \delta} L_{\varepsilon}(P_a)$ which converges pointwise to $z_a$, then there exists a sequence
\{y_n\} \subset G \cap \bigcup_{i \leq n} L_i(P_a) \text{ such that } \|y_n - w_n\| \to 0.

**Proof.** Each \(w_n^{1:1}\) has the form
\[
w_n^{1:1} = \sum_p \sum_q a_{npq}x_{pq} + \sum_p b_{np}x^p + c_nx^0.
\]
By Lemma 4.2 there exists a sequence \(\{y_n\} \subset \bigcup_{i \leq n} L_i(P_a)\) such that
\[
y_n^{1:1} = \sum_p \sum_q a_{npq}x_{pq} + \sum_p b_{np}x^p + c_nx^0,
\]
and \(y_n^{0,0} = y_n^{1,0} = 0\) and \(w_n^{1:1} = w_n^{0,0}\) for each \(u \neq 0,1\). Since obviously \(\{y_n\} \subset G\), if remains only to show that \(\lim_n \|y_n - w_n\| = 0\).

First note that \((y_n - w_n)^{1:1} = 0\) and \((y_n - w_n)^{0,0} = 0\) for all \(u \neq 1\).
For each real \(r > 0\) there exists by Lemma 4.6 an \(n_r \in \omega\) such that \(\Sigma_p \Sigma_q a_{npq} < r\) for all \(n > n_r\). For each \(u \neq 0\) there exists \(q_u \in \omega\) such that \(u \geq q_u^{1:1}\) and hence by Lemma 4.3,
\[
u^{-1}w_n^{1:1} - cr < \nu^{-1}w_n^{0:0} - c\Sigma_p \Sigma_q a_{npq} 
\leq w_n^{0:0} \leq \nu^{-1}w_n^{1:1}
\]
for each \(n > n_r\). Since \(y_n^{1:1} = w_n^{1:1}\) for each \(u \neq 1\),
\[
\| (y_n - w_n)^{1:1} \| = \| \nu^{-1}y_n^{1:1} - w_n^{1:1} \| = \| \nu^{-1}w_n^{1:1} - w_n^{1:1} \| \leq cr
\]
for each \(n > n_r\) and \(u \neq 0,1\).
Finally, since \(x^{1:1} = x^{1:1}\) for each \(x \in L_\nu(P_a)\),
\[
\| (y_n - w_n)^{1:1} \| = \| (y_n - w_n)^{1:1} \| = \| \Sigma_p (\Sigma_q a_{npq} x_{pq} - \Sigma_q a_{npq} x_{pq}) \| 
< 2cr
\]
for each \(n > n_r\).

We have now shown that \(\|y_n - w_n\| < 2cr\) for each \(n > n_r\), completing the proof of the theorem.

**Lemma 4.7.** Let \(\zeta\) be a countable ordinal, and let \(y \in L_\zeta(L_i(P_a) \cap G)\).
Let \(\zeta' = \zeta + 1\) if \(\zeta < \omega\) and \(\zeta' = \zeta + 1\) if \(\zeta \geq \omega\). If \(u \in U \setminus \{0\}\) and \(\nu_u(w^{-i}) = (\beta, \gamma)\) with \(\beta > \gamma > \zeta'\), then \(y^{1:1}\) is continuous and hence has the form \(y^{1:1} = \Sigma_p \Sigma_q a_{npq} x_{pq}\). If also \(v \in U \setminus \{0\}\) and \(\nu_u(v^{-i}) = (\gamma, \delta)\) with \(\beta > \gamma > \zeta'\), then for each \(r \in \omega\), \(\Sigma_p a_{nrp} = \Sigma_q a_{nqs}\).

**Proof.** The proof will be by induction on \(\zeta\). If \(y \in L_\zeta(L_i(P_a) \cap G) = L_i(P_a) \cap G\), there is a bounded sequence \(\{w_n\} \subset P_a\) which converges pointwise to \(y\). The sequence \(\{w_n\}\) can be chosen so that each \(w_n\) is a finite linear combination of elements of \(\{x_s : s \in S\}\), and hence there exists a countable subset \(\sigma\) of \(S\) such that each \(w_n\) has the form \(w_n = \Sigma_{s \in \sigma} b_{ns} x_s\), where each \(b_{ns}\) is nonnegative and for each \(n\) only a finite number of the \(b_{ns}\) are nonzero. If \(u \neq 0\) and \(\nu_u(w^{-i}) = (\beta, \gamma)\), then
\[ w^u_n = u\sum_{x \in p} b_{nx} x_{p^q} = u\sum_p \sum_q a^n_{pq} x_{pq}, \]

where

\[ a^n_{pq} = \{ b_{nx} : s_x = p, s_y = q \}. \]

Now \( y^{i,u} = u^{-1} y^z_n \) by Lemma 4.4 since \( y \in G \); hence \( y^{i,u} \) is the pointwise limit of the bounded sequence \( \{ \Sigma_p \Sigma_q a^n_{pq} x_{pq} \} \). The function \( y^{i,u} \) is in \( L(Q) \) and hence has the form

\[ y^{i,u} = \Sigma_p \Sigma_q a^n_{pq} x_{pq} + \Sigma_p b^n_{p} x^n; \]

by the proof of Lemma 3.2, \( a^n_{pq} = \lim_n a^n_{pq} \) for all \( p, q \) and

\[ b^n_{p} = c^{-1} y^{i,u}(t_{pp}) - \Sigma_q a^n_{pq} = \lim_n \Sigma_q a^n_{pq} - \Sigma_q a^n_{pq} \]

for all \( p \).

Now assume further that \( \nu_u(w^{-}) = (\beta, \gamma) \) with \( \gamma > 1 \), and let \( \lambda = 2 \) if \( \gamma > 2 \) and \( \lambda = 1 \) if \( \gamma = 2 \). Then \((\gamma, \lambda) \in B_{n} \) so there exists \( v_i \in U \setminus \{0\} \) such that \( \nu_u(w^{-}) = (\gamma, \lambda) \). Since \( \{ \Sigma_p \Sigma_q a^n_{pq} x_{pq} \} \) and \( \{ \Sigma_p \Sigma_q a^n_{pq} x_{pq} \} \) are bounded pointwise convergent sequences in \( Q \), it follows from the note following Lemma 3.2 that for each real \( \epsilon > 0 \) there exist integers \( p_i \) and \( n_i \) such that \( \Sigma_{p > p_i} \Sigma_q a^n_{pq} x_{pq} < \epsilon \) and \( \Sigma_{p > p_i} \Sigma_q a^n_{pq} x_{pq} < \epsilon \) for all \( n \geq n_i \). Since

\[ \Sigma_{p > p_i} \Sigma_q a^n_{pq} x_{pq} = \Sigma \{ b_{nx} : s_x > p_i \} = \Sigma_{p > p_i} \Sigma_q a^n_{pq} x_{pq} < \epsilon \]

for each \( n \geq n_i \), it follows that if \( f_n = \Sigma_{p \leq p_i} \Sigma_{q \leq p_i} a^n_{pq} x_{pq} \),

\[ ||w^{-} w^n_{u} - f_n|| \leq \epsilon \Sigma \{ a^n_{pq} : p > p_i \} \text{ or } q < p_i \] \[ > 2e\epsilon \]

for each \( n \geq n_i \). Since \( ||f_n|| \leq ||w^{-} w^n_{u}|| \leq \sup_n ||w_n|| \) for each \( n \), it follows that for each \( n \geq n_i \), \( f_n \) belongs to the compact subset

\[ \mathcal{C}_{n, p} = \{ \Sigma_{p \leq p_i} \Sigma_{q \leq p_i} k_{pq} x_{pq} : k_{pq} \geq 0, \Sigma_{p \leq p_i} \Sigma_{q \leq p_i} k_{pq} \leq w^{-} \sup_n ||w_n|| \} \]

of \( C[0; 3] \). By compactness some subsequence \( \{ f_{n_i} \} \) of \( \{ f_n \} \) must converge to an element \( f \) of \( \mathcal{C}_{n, p} \), and since \( \{ w^{-} w^n_{u} \} \) converges pointwise to \( y^{i,u} \), it follows that \( ||y^{i,u} - f|| \leq 2e\epsilon \). Thus, for each \( \epsilon > 0 \) there exists an \( f \in C[0; 3] \), depending on \( \epsilon \), such that \( ||y^{i,u} - f|| \leq 2e\epsilon \). Since \( C[0; 3] \) is complete in norm, \( y^{i,u} \in C[0; 3] \) and must therefore be equal to \( \Sigma_p \Sigma_q a^n_{pq} x_{pq} \).

Now if \( 0 \neq v \in U \) and \( \nu_u(w^{-}) = (\gamma, \delta) \) with \( \gamma > \delta > 1 \), then for all \( n \) and \( r \),

\[ \Sigma_q a^n_{npr} = \Sigma \{ b_{nx} : s_x = r \} = \Sigma_q a^n_{nrr} \]

Since \( y^{i,u} = \Sigma_p \Sigma_q a^n_{pq} x_{pq} \), it follows that
\[ \Sigma_q a_n^\sigma = e^{-1} y^{1,\nu}(t_{rr}) = \lim_n e^{-1} v^{-1} w_n^{1,\nu}(t_{rr}) = \lim_n \Sigma_q a_n^{u_q} = \lim_n \Sigma_q a_n^{u_{q+r}}. \]

On the other hand the bounded sequence \{\Sigma_p \Sigma_q a_n^{x_{pq}}\} converges pointwise to \(y^{1,\nu} = \Sigma_p \Sigma_q a_n^{x_{pq}}\). By the note following Lemma 3.2, for each \(\varepsilon > 0\) there exist \(p_1, n_1\) such that \(\Sigma_{p > p_1} \Sigma_q a_{nq} < \varepsilon\) for all \(n \geq n_1\) and also \(\Sigma_{p > p_1} \Sigma_q a_{nq}^{u_q} < \varepsilon\). Hence
\[
|\Sigma_p a_{pr}^{u_q} - \lim_n \Sigma_p a_{npr}^{u_q}| < 2\varepsilon + |\Sigma_{p \leq p_1} a_{pr}^{u_q} - \lim_n \Sigma_{p \leq p_1} a_{npr}^{u_q}| = 2\varepsilon.
\]

Since \(\varepsilon\) is an arbitrary positive number,
\[ \Sigma_p a_{pr}^{u_q} = \lim_n \Sigma_p a_{npr}^{u_q} = \Sigma_q a_{r_q}^{u_q}. \]

This completes the proof of the lemma for \(\zeta = 0\).

For the induction step let \(0 < \zeta < \Omega\), assume the desired result holds for each \(\eta < \zeta\), and let \(y, \zeta', u, \beta, \) and \(\gamma\) be as in the statement of the lemma. Then there exists a bounded sequence \(\{y_n\}\) in \(\bigcup_{\gamma \leq \alpha} L_\eta(L_\gamma(P_\alpha) \cap G)\) which converges pointwise to \(y\). Since \(1 < \zeta' < \gamma \leq \alpha\), there exists \(v_i \in U \setminus \{0\}\) such that \(v_i(v_i^{-1}) = (\gamma, \zeta')\). For each \(n\) there exists \(\eta_n < \zeta\) such that \(y_n \in L_{v_i}(L_\eta(P_\alpha) \cap G)\), and it follows that \(\beta > \gamma > \zeta' > \eta_n^*\) for each \(n\), where \(\eta_n^*\) is defined in terms of \(\eta_n\) as \(\zeta'\) was defined in terms of \(\zeta\). By the induction assumption \(y_n^{1,\nu}\) and \(y_n^{1,\nu}\) are continuous and have the form \(y_n^{1,\nu} = \Sigma_p \Sigma_q a_n^{x_{pq}}x_{pq}\) and \(y_n^{1,\nu} = \Sigma_p \Sigma_q a_n^{x_{pq}}x_{pq}\), and \(\Sigma_p a_{npr}^{u_q} = \Sigma_q a_{r_q}^{u_q}\) for all \(n\) and \(r\).

As in the proof for \(\zeta = 0\), for each \(\varepsilon > 0\) there exist \(n_1, p_1\) such that \(\Sigma_{p > p_1} a_{nq}^{u_q} < \varepsilon\) and \(\Sigma_{p > p_1} \Sigma_q a_{nq}^{\nu_q} < \varepsilon\) for all \(n \geq n_1\). Hence, since \(\Sigma_p a_{npr}^{u_q} = \Sigma_q a_{npr}^{u_q}\) for all \(n\) and \(r\), it follows that for \(n \geq n_1\), the distance between \(y_n^{1,\nu}\) and each \(\gamma_{n^*}\) is less than \(2\varepsilon\). Since \(\{y_n^{1,\nu}\}\) converges pointwise to \(y^{1,\nu}\), the compactness of \(D_{p_1}\) implies that \(|y^{1,\nu} - w| \leq 2\varepsilon\) for some continuous \(w\) depending on \(\varepsilon\). Then the completeness of \(C[0, 3]\) implies that \(y^{1,\nu} \in C[0, 3]\) and therefore, since also \(y^{1,\nu} \in L_\eta(Q)\), that \(y^{1,\nu}\) has the form \(\Sigma_p \Sigma_q a_{nq}^{x_{pq}}x_{pq}\).

If also \(0 \neq v \in U\) and \(\nu_n(v^{-1}) = (\gamma, \delta)\) with \(\beta > \gamma > \delta > \zeta'\), then \(y^{1,\nu}\) and each \(y_n^{1,\nu}\) are continuous and have form corresponding to \(y^{1,\nu}\) and \(y_n^{1,\nu}\) respectively. Further, by the induction assumption, \(\Sigma_p a_{npr}^{u_q} = \Sigma_q a_{npr}^{u_q}\) for all \(n\) and \(r\). Hence
\[ \Sigma_p a_{pr}^{u_q} = e^{-1} y^{1,\nu}(t_{rr}) = \lim_n e^{-1} y_n^{1,\nu}(t_{rr}) = \lim_n \Sigma_q a_{npr}^{u_q}, \]

Exactly as in the last part of the proof for \(\zeta = 0\) it is seen that
\[ \Sigma \alpha_{n}^* = \lim \Sigma \alpha_{n+p}^* \text{.} \] This completes the proof of the induction step and hence of the lemma.

**Lemma 4.8.** If \( y \in L_{\xi}(L_{\alpha}(P_{\alpha}) \cap G) \) for some countable \( \xi \) and if \( u, v \in U \setminus \{0\} \) with \( \nu_{u}(u^{-1}) = (\beta, \gamma) \) and \( \nu_{v}(v^{-1}) = (\beta, \delta) \) for certain ordinals \( \beta, \gamma, \delta \) then in the expression

\[ y^{i,u} = \Sigma \alpha_{u}^* x_{pq} + \Sigma b_{p}^* x_{p} + c_{v}^* \]

and the corresponding expression for \( y^{i,v} \) it must be true that \( y^{i,u}(2^{-i}) = y^{i,v}(2^{-i}) \), \( c_{v} = c_{u} \), and \( b_{p}^* + \Sigma \alpha_{pq}^* = b_{p}^* + \Sigma \alpha_{pq}^* \) for each \( p \).

**Proof.** By Lemma 4.5, \( y \in G \). Hence, by Lemma 4.4, \( y^{i,u} = u^{-y^{i,v}} \) and \( y^{i,v} = v^{-y^{i,v}} \).

If \( \xi = 0 \), then \( y \) is the pointwise limit of a bounded sequence \( \{y_{n}\} \) of functions of the form \( y_{n} = \Sigma_{s \in S_{n}} a_{n} s_{n}, \) where \( a_{n} \) is a finite subset of \( S_{a} \) and each \( b_{n} \) is nonnegative. For each \( p \) and \( n \),

\[ u^{-y_{n}}(t_{pp}) = c^{\{b_{n}: s_{p} = p\}} = v^{-y_{n}}(t_{pp}). \]

Since \( \{y_{n}\} \) converges pointwise to \( y^{i,v} \),

\[ y^{i,u}(t_{pp}) = u^{-y^{i,v}}(t_{pp}) = v^{-y^{i,v}}(t_{pp}) = y^{i,v}(t_{pp}) \]

for each \( p \), and hence it follows immediately that

\[ b_{p}^* + \Sigma \alpha_{pq}^* = c^{y^{i,u}}(t_{pp}) = c^{y^{i,v}}(t_{pp}) \]

\[ = b_{p}^* + \Sigma \alpha_{pq}^* \]

for each \( p \). Since \( y^{i,u} \) and \( y^{i,v} \) are Baire functions of the first class, \( c_{u} = 0 = c_{v} \). Hence

\[ y^{i,u}(2^{-i}) = \Sigma (b_{p}^* + \Sigma \alpha_{pq}^*) = y^{i,v}(2^{-i}). \]

For the induction step let \( \xi > 0 \) and assume the statement of the lemma holds for each \( \eta < \xi \). By hypothesis there exists a bounded sequence \( \{y_{\eta}\} \) in \( \bigcup_{\eta < \xi} L_{\xi}(L_{\alpha}(P_{\alpha}) \cap G) \) which converges pointwise to \( y \). Under the usual notation the relations

\[ b_{p}^* + \Sigma \alpha_{pq}^* = b_{p}^* + \Sigma \alpha_{pq}^* \]

\( c_{n}^* = c_{n}^* \), and \( y_{n}^{i,u}(2^{-i}) = y_{n}^{i,v}(2^{-i}) \) must hold for all \( n \) and \( p \). It is seen immediately that \( y^{i,u}(2^{-i}) = y^{i,v}(2^{-i}) \) and \( y^{i,u}(t_{pp}) = y^{i,v}(t_{pp}) \) for all \( p \), from which the remaining desired relations for \( y^{i,u} \) and \( y^{i,v} \) follow. The proof is thus complete.

**Theorem 4.2.** Let \( \xi \) be a countable ordinal, and let \( \xi' \) be defined as in Lemma 4.7. If \( y \in L_{\xi}(L_{\alpha}(P_{\alpha}) \cap G) \) and \( 0 \neq u \in U \) with \( \nu_{u}(u^{-1}) = (\beta, \gamma) \)
and $\beta > \zeta'$, then $y^{i,*} \in Q + Q_1$.

Proof. If $\zeta = 0$, then $y \in L(P_a)$ and hence trivially $y^{i,*} \in L(P_a)$, which is equal to $Q + Q_1$ by Lemma 3.2.

If $\zeta > 0$ and the desired result is true for each $\eta < \zeta$, then $2^\alpha \zeta' < \beta \alpha$ and hence there exists $v \in U(0)$ such that $v(u)^* = (\beta, \zeta')$. There exists a bounded sequence $\{y_n\}$ in $\bigcup_{\eta < \zeta} L_{\eta}(P_a) \cap G$ which converges pointwise to $y$. Since $\beta > \zeta' > \gamma'$ for each $\eta < \zeta$ it follows from Lemma 4.7 that each $y^{i,v}$ is continuous and hence belongs to $Q$. Hence $y^{i,v} \in L_a(Q) = Q + Q_1$. Thus in the usual notation for $y^{i,u}$ and $y^{i,v}$ it follows that $v^* = 0$, but then also $c^* = 0$ by Lemma 4.8, hence $y^{i,v} \in Q + Q_1$, and the proof is complete.

The following theorem justifies the claim made at the beginning of the present section.

**Theorem 4.3.** The element $z_a \in L_a(P_a)$ has the property that $\|z_a\| = 1$ but that if $\{w_n\}$ is a bounded sequence in $\bigcup_{\beta < \alpha} L_{\beta}(P_a)$ converging pointwise to $z_a$, then $\lim_n \|w_n\| \geq c$.

Proof. By Lemma 4.1 and the remarks preceding it we know that $z_a \in L_a(P_a)$ and $\|z_a\| = 1$. If $\{w_n\}$ is a bounded sequence in $\bigcup_{\beta < \alpha} L_{\beta}(P_a)$ converging pointwise to $z_a$, then by Theorem 4.1 there exists a sequence $\{y_n\}$ in $G \cap \bigcup_{\beta < \alpha} L_{\beta}(P_a)$ such that $\|y_n - w_n\| \to 0$. Clearly $\lim_n \|w_n\| = \lim_n \|y_n\|$. Now by Lemma 4.5,

$$\{y_n\} \subset \begin{cases} L_{\alpha-2}(L_1(P_a) \cap G) & \text{if } 2 \leq \alpha < \omega \\ \bigcup_{\beta < \alpha} L_{\beta}(L_1(P_a) \cap G) & \text{if } \omega \leq \alpha < \Omega. \end{cases}$$

Defining $\zeta'$ as in Lemma 4.7, one sees easily that each $y_n \in L_{\alpha}(L_1(P_a) \cap G)$ for some $\zeta_n$ such that $\alpha > \zeta_n$. Now there exists $u_1 \in U(0)$ such that $v(u)^* = (\alpha, \gamma)$ for some $\gamma < \alpha$; for example, take $\gamma = 1$ if $\alpha = 2$ and $\gamma = 2$ if $\alpha > 2$. Then by Theorem 4.2, $y^{i,v_n} \in Q + Q_1 = L_1(Q)$ for each $n$. Now $z^{i,v_n} = x^a$ by definition, and hence $\lim_n \|y^{i,v_n}\| \geq c$ by Theorem 1 of [7]. It follows that

$$\lim_n \|w_n\| = \lim_n \|y_n\| \geq \lim_n \|y^{i,v_n}\| \geq c.$$

**Corollary 4.1.** Let $T$ be the mapping of Theorem 2.1 for the space $X_a$, and let $G_a = Tz_a$. Then $G_a \in K_a(J_{x,a}P_a)$ and $\|G_a\| = 1$, but if $\{F_n\}$ is a sequence in $\bigcup_{\beta < \alpha} K_{\beta}(J_{x,a}P_a)$ such that $F_n \overset{w*}{\to} G_a$, then $\lim_n \|F_n\| \geq c$.

Proof. It is immediate from Theorem 2.1 that $G_a \in K_a(J_{x,a}P_a)$ and $\|G_a\| = 1$. If $\{F_n\} \subset \bigcup_{\beta < \alpha} K_{\beta}(J_{x,a}P_a)$ and $F_n \overset{w*}{\to} G_a$, then by Theorem 2.1 the sequence $\{T^{-1}F_n\}$ is in $\bigcup_{\beta < \alpha} L_{\beta}(P_a)$ and $\|T^{-1}F_n\| = \|F_n\|$ for each
\( n. \) Now \( \sup_n \| T^{-1} F_n \| = \sup_n \| F_n \| < \infty \) since \( \{ F_n \} \) is \( w^* \)-convergent. For each \( t \in S_a \) let \( f_t \in X_a^* \) be defined as in the proof of Theorem 2.1. Then

\[
(T^{-1} F_n)(t) = F_n(f_t) \longrightarrow G_n(f_t) = z_a(t)
\]

for each \( t \), and hence

\[
\lim_n \| F_n \| = \lim_n \| T^{-1} F_n \| \geq c.
\]

5. Our main theorems will now be proved through consideration of product spaces, as defined in [2, p. 31], of spaces of the type \( X_a \). Since \( X_a \), \( P_a \), and \( G_a \) depend on the given number \( c \geq 1 \) as well as on \( \alpha \), the objects mentioned will henceforth be indicated with double subscripts as \( X_{c, \alpha} \), \( P_{c, \alpha} \), and \( G_{c, \alpha} \) respectively. Recall that if \( I \) is a set and \( X_\alpha \) is a Banach space for each \( \alpha \in I \), then the product spaces \( \Pi_{\alpha \in I} X_\alpha \) and \( \Pi_{\alpha \in I} X_\alpha^{**} \) are respectively the dual and bidual of the Banach space \( \Pi_{\alpha \in I} X_\alpha \) under the natural identifications.

**Theorem 5.1.** For each countable ordinal \( \alpha \geq 2 \) let \( Y_\alpha \) be the Banach space \( \Pi_{\mu \in \omega} X_{\mu, \alpha}^* \) and let

\[
Q_\alpha = \bigcap_{\alpha \in \omega} \{ y \in Y_\alpha : y(n) \in P_{n, \alpha} \}.
\]

Then \( Y_\alpha \) is separable, and \( Q_\alpha \) is a norm-closed cone in \( Y_\alpha \) such that \( K_\alpha(J_{Y_\alpha} Q_\alpha) \) is not norm-closed in \( Y_\alpha^{**} \).

**Proof.** It is evident that \( Y_\alpha \) is separable and \( Q_\alpha \) is a closed cone in \( Y_\alpha \). An easy transfinite induction argument shows that for each \( n \) the functional \( F_n \) belongs to \( K_\alpha(J_{Y_\alpha} Q_\alpha) \), where \( F_n(n) = G_{n, \alpha} \) and \( F_n(i) = 0 \) for all \( i \neq n \). Hence \( \sum_{n=1}^{m} n^{-1} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha) \) for each positive integer \( m \), and therefore \( \sum_{n \in \omega} n^{-1} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha) \). If \( \{ H_k \} \) were a sequence in \( \bigcup_{\beta < \alpha} K_\beta(J_{Y_\alpha} Q_\alpha) \) such that \( H_k \longrightarrow \sum_{n} n^{-1} F_n \), then for each \( i \in \omega \) it would follow that

\[
\{ H_k(i) \}_k \subset \bigcup_{\beta < \alpha} K_{\beta}(J_{X_{\beta, \alpha}^*} P_{\beta, \alpha})
\]

and

\[
H_k(i) \longrightarrow \sum_{n} n^{-1} F_n(i) = i^{-1} G_{i, \alpha}.
\]

It would then result by Corollary 4.1 that

\[
\lim_k \| H_k \| \geq \lim_k \| H_k(i) \| \geq i,
\]

but then since \( i \) is arbitrary the sequence \( \{ H_k \} \) would be unbounded in norm, contradicting the fact that a \( w^* \)-convergent sequence in \( Y_\alpha^{**} \) must be bounded [3, p. 60]. Hence \( \sum_{n} n^{-1} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha) \), and the proof
is complete.

**Theorem 5.2.** For each countable ordinal $\alpha \geq 2$ there exists a separable Banach space $W_\alpha$ containing a norm-closed cone $R_\alpha$ such that if $2 \leq \beta \leq \alpha$, then $K_\beta(J_{W_\alpha}R_\alpha)$ is not norm-closed in $W_\alpha^{**}$.

**Proof.** Let $A_\alpha = \{\beta: 2 \leq \beta \leq \alpha\}$ and for each $\beta \in A_\alpha$ let $Y_\beta$ and $Q_\beta$ be as defined in Theorem 5.1. Let $W_\alpha = \Pi_{\beta \in A_\alpha} Y_\beta$ and $R_\alpha = \bigcap_{\beta \in A_\alpha} \{w \in W_\alpha: w(\beta) \in Q_\beta\}$. Then the Banach space $W_\alpha$ is separable since $A_\alpha$ is countable, and $R_\alpha$ is clearly a norm-closed cone in $W_\alpha$. For each $\beta \in A_\alpha$ there exists by Theorem 5.1 a sequence $\{\phi_{\beta,n}\}$ in $K_\beta(J_{Y_\beta}Q_\beta)$ which converges in norm to an element $\phi_{\beta,0} \in Y_\beta^{**}$ not in $K_\beta(J_{Y_\beta}Q_\beta)$. If $\psi_{\beta,n}$ is defined for each integer $n \geq 0$ by $\psi_{\beta,n}(\gamma) = 0$ for $\gamma \neq \beta$ and $\psi_{\beta,n}(\beta) = \phi_{\beta,n}$, it is easily shown that $\{\psi_{\beta,n}\}_{n \in \mathbb{N}} \subset K_\beta(J_{W_\alpha}R_\alpha)$ and $\{\psi_{\beta,n}\}$ converges in norm to $\psi_{\beta,0}$, but that $\psi_{\beta,0} \in K_\beta(J_{W_\alpha}R_\alpha)$. Hence for each $\beta \in A_\alpha$, $K_\beta(J_{W_\alpha}R_\alpha)$ fails to be norm-closed in $W_\alpha^{**}$.

**Theorem 5.3.** There exists a Banach space $Z$ containing a norm-closed cone $P$ such that if $\beta$ is a countable ordinal $\geq 2$, then $K_\beta(J_ZP)$ fails to be norm-closed in $Z^{**}$.

**Proof.** The proof is almost identical with that of Theorem 5.2. Let $A = \{\beta: 2 \leq \beta < \Omega\}$, $Z = \Pi_{\beta \in A} Y_\beta$, and $P = \bigcap_{\beta \in A} \{z \in Z: z(\beta) \in Q_\beta\}$. Since $A$ is uncountable, the Banach space $Z$ is nonseparable. It is clear that $P$ is a closed cone in $Z$. The proof that $K_\beta(J_ZP)$ fails to be norm-closed in $Z^{**}$ for each $\beta \in A$ is identical with the corresponding part of the proof of Theorem 5.2, in which it was shown that $K_\beta(J_{W_\alpha}R_\alpha)$ fails to be norm-closed in $W_\alpha^{**}$ for each $\beta \in A_\alpha$.

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