ON ITERATED $w^*$-SEQUENTIAL CLOSURE OF CONES

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In this paper it is proved that for each countable ordinal number \( \alpha \geq 2 \) there exists a separable Banach space \( X \) containing a cone \( P \) such that, if \( J_X \) is the canonical map of \( X \) into its bidual \( X^{**} \), then the \( \alpha \)th iterated \textit{w*-sequential closure} \( K_\alpha(J_X P) \) of \( J_X P \) fails to be norm-closed in \( X^{**} \). From such spaces there is constructed a separable space \( W \) containing a cone \( P \) such that if \( 2 \leq \beta < \alpha \), then \( K_\beta(J_w P) \) fails to be norm-closed in \( W^{**} \). Further, there is constructed a (non-separable) space \( Z \) containing a cone \( P \) such that if \( 2 \leq \beta < \Omega \), then \( K_\beta(J_z P) \) fails to be norm-closed in \( Z^{**} \).

1. If \( X \) is a real Banach space and \( Y \) a subset of \( X^{**} \), let \( K(Y) \) be the set of elements of \( X^{**} \) which are \textit{w*-limits} of sequences in \( Y \). Let \( K_0(Y) = Y \) and inductively let \( K_\alpha(Y) = K(\bigcup_{\beta < \alpha} K_\beta(Y)) \) for \( 0 < \alpha \leq \Omega \), where \( \Omega \) is the first uncountable ordinal. A cone in \( X \) is a subset of \( X \) which is closed under addition and under multiplication by nonnegative scalars. Our main theorem extends the result of [6] that if \( P \) is a cone in \( X \), then \( K_1(J_X P) \) must be norm-closed but \( K_2(J_X P) \) can fail to be norm-closed in \( X^{**} \). By contrast it is noted that if \( S \) is a compact Hausdorff space and \( X = C(S) \) and \( \alpha < \Omega \), then \( K_\alpha(J_X X) \) is norm-closed, even though for example if \( S \) is compact, metric, and uncountable, then \( K_\alpha(J_X X) \) is not \textit{w*-sequentially} closed. It is obvious that for each Banach space \( X \) and each subset \( Y \) of \( X^{**} \), \( K_\Omega(Y) \) is \textit{w*-sequentially} closed and hence norm-closed.

In [7] a Banach space \( X \) was exhibited such that \( K_\alpha(J_X X) \) is not norm-closed. Whether \( K_\alpha(J_X X) \) can fail to be norm-closed for \( 2 < \alpha < \Omega \) is not known to the author. However, in the present paper it will be convenient to use constructions involving spaces studied in [7].

Section 2 is devoted to a useful relationship between \textit{w*-sequential convergence} and pointwise convergence of bounded sequences of functions, § 3 to further study of a space constructed in [7], and §§ 4 and 5 to preparation for and proof of the main theorems.

2. Let \( S \) be a compact Hausdorff space, \( B(S) \) the Banach space of bounded real functions on \( S \) with the supremum norm, and \( C(S) \) the closed subspace of \( B(S) \) consisting of the continuous real functions on \( S \). If \( A \) is a subset of \( B(S) \), let \( L(A) \) be the set of all pointwise limits of bounded sequences in \( A \), and let \( L_\alpha(A) \) be defined inductively by \( L_0(A) = A \) and \( L_\alpha(A) = L(\bigcup_{\beta < \alpha} L_\beta(A)) \) for each ordinal \( \alpha \) such that \( 0 < \alpha \leq \Omega \).

If \( X \) is a norm-closed subspace of \( C(S) \) and \( z \in L_\alpha(X) \), then \( z \) is
bounded and Borel measurable and hence is integrable with respect to each finite regular Borel signed measure \( \mu \) on \( S \). For each \( f \in X^* \) there exists a finite regular Borel signed measure \( \mu_f \) on \( S \) such that \( f(x) = \int_s x \, d\mu_f \) for each \( x \in X \) [3, p. 265], and by the Hahn-Banach theorem \( \mu_f \) can be chosen so that \( \| \mu_f \| = \| f \| \). If \( \nu_f \) is another finite regular Borel signed measure on \( S \) such that \( f(x) = \int_s x \, d\nu_f \) for each \( x \in X \) then also \( \int_s zd\mu_f = \int_s zd\nu_f \) for each \( z \in L_a(X) \), by virtue of the bounded convergence theorem and transfinite induction. Hence a mapping \( T \) is unambiguously defined from \( L_a(X) \) into the space of real functions on \( X^* \) by

\[
(Tz)(f) = \int_s zd\mu_f \quad (z \in L_a(X), \ f \in X^*).
\]

**Theorem 2.1.** If \( S \) is a compact Hausdorff space and \( X \) a norm-closed subspace of \( C(S) \), then \( T \) is an isometric isomorphism from \( L_a(X) \) onto \( K_a(J_x X) \), and \( T \) maps \( L_a(A) \) onto \( K_a(J_x A) \) for each subset \( A \) of \( X \) and each \( \alpha \leq \Omega \).

**Proof.** For each \( z \in L_a(X) \) it is trivial that \( Tz \) is linear on \( X^* \) and that \( |(Tz)(f)| \leq \| z \| \| f \| \) for every \( f \in X^* \), so that \( Tz \in X^{**} \) and \( \| Tz \| \leq \| z \| \). For each \( t \in S \) let \( f_t(x) = x(t) \) for all \( x \in X \); then clearly \( f_t \in X^* \) with \( \| f_t \| \leq 1 \), and it is easily seen that \( (Tz)(f_t) = \int_s zd\mu_{f_t} = z(t) \), so that \( |z(t)| \leq \| Tz \| \| f_t \| \leq \| Tz \| \) and hence \( \| z \| \leq \| Tz \| \). Since \( T \) is obviously linear, it follows that \( T \) is an isometric isomorphism from \( L_a(X) \) into \( X^{**} \).

Now let \( A \) be a subset of \( X \). Since the restriction of \( T \) to \( X \) is \( J_x \), it follows that \( T[L_a(A)] = TA = J_x A = K_a(J_x A) \). If \( 0 < \alpha \leq \Omega \) and it is assumed that \( T[L_\beta(A)] = K_\beta(J_x A) \) for each \( \beta < \alpha \), then for each \( z \in L_\alpha(A) \) there exists a bounded sequence \( \{ z_n \} \) in \( \bigcup_{\beta < \alpha} L_\beta(A) \) which converges pointwise to \( z \). By the bounded convergence theorem \((Tz)(f) = \lim_n (Tz_n)(f)\) for each \( f \in X^* \). Since by assumption \( \{ Tz_n \} \subset \bigcup_{\beta < \alpha} K_\beta(J_x A) \), it follows that \( Tz \in K_\alpha(J_x A) \). Conversely, if \( F \in K_\alpha(J_x A) \) there exists a sequence \( \{ F_n \} \subset \bigcup_{\beta < \alpha} K_\beta(J_x A) \) such that \( F_n \xrightarrow{w^*} F \); the sequence \( \{ F_n \} \) must be bounded [3, p. 60], and by assumption there exists a sequence \( \{ z_n \} \subset \bigcup_{\beta < \alpha} L_\beta(A) \) such that \( Tz_n = F_n \) for each \( n \). Now \( \{ z_n \} \) is bounded, and if \( (z(t)) \) is defined to be \( F(f_t) \) for each \( t \in S \) it follows that \( \{ z_n \} \) converges pointwise to \( z \) so that \( z \in L_\alpha(A) \). For every \( f \in X^* \), \((Tz)(f) = \lim_n (Tz_n)(f)\) by the bounded convergence theorem. Thus \( F = Tz \in T[L_\alpha(A)] \), completing the proof that \( T[L_\alpha(A)] = K_\alpha(J_x A) \). By transfinite induction the theorem follows.

**Remark.** If \( S \) is a compact Hausdorff space and \( X \) is the Banach
space $C(S)$, then for each $\alpha \leq \Omega$, $L_\alpha(X)$ is the space of bounded Baire functions on $S$ of order $\leq \alpha$ and, just as in the special case of a metric space $S$ [8, p. 132], $L_\alpha(X)$ is norm-closed in $B(S)$ and hence also $K_\alpha(J_X X)$ is norm-closed in $X^{**}$. If $S$ is a compact metric space with uncountably many elements then $S$ has a nonempty dense-in-itself kernel [1, Ch. 9, p. 34]. Hence for each countable $\alpha$ there is a subset $T$ of $S$ of Borel order exactly $\alpha$ [4, p. 207], but then it follows that $L_\alpha(X) \neq L_{\alpha+1}(X)$ [5, p. 299] and hence that $K_\alpha(J_X X) \neq K_{\alpha+1}(J_X X)$ for each countable $\alpha$.

3. The reader is now referred to the proof of Theorem 1 of [7] for the construction, for each real $c \geq 1$, of a Banach space $X \subset C([0; 3])$ having the property that there exists an $x^0 \in L_2(X)$ such that $\|x^0\| = 1$ but if $\{y^k\}$ is a bounded sequence in $L_1(X)$ which converges pointwise to $x^0$, then $\liminf_k \|y^k\| \geq c$. The remainder of the present paper depends heavily on properties of the space $X$, and the reader will occasionally need to refer to [7]. In particular, note that $X$ is generated by a set $\{x_{pq}: p, q \in \omega\}$ of piecewise linear nonnegative functions of norm $c$ on $[0; 3]$ and that $x^0$ is the pointwise limit of the sequence $\{x^p\} \subset L_1(X)$, where $x^p$ is the pointwise limit of $\{x_{pq}\}_{q \in \omega}$ and $\|x^p\| = c$ for each $p$. Each $x_{pq}$ has truncated peaks centered at certain points $s_{ui}, t_{vj}, 2 + s_{ui}$ where $s_{ui} = 2^{-u}i$ and $t_{vj} = 2 - 2^{-v}(1 + 2^{-j})$ for $u, i, v, j \in \omega$ and $i < 2^u$. Specifically, $x_{pq}(s_{ui}) = x_{pq}(2 + s_{ui}) = 1$ if $p \geq u$, and $x_{pq}(s_{ui}) = 0$ if only if $p \geq u$. Further, $x_{pq}(t_{vj}) = c$ if $v \leq p \leq j < p + q$ and 0 otherwise. If $\chi(S)$ denotes the characteristic function of the subset $S$ of $[0; 3]$, it turns out that

$$x^p = \chi(\{s_{pi}: i < 2^p\} \cup \{2 + s_{pi}: i < 2^p\}) + c\chi(\{t_{vj}: v \leq p \leq j\})$$

and that

$$x^0 = \chi(\{s_{pi}: p \in \omega, i < 2^p\} \cup \{2 + s_{pi}: p \in \omega, i < 2^p\}).$$

**Lemma 3.1.** Let $Q$ be the norm-closed cone in $X$ generated by $\{x_{pq}: p, q \in \omega\}$. Then $Q$ coincides with

$$Q_0 = \{\Sigma_p \Sigma_q a_{pq} x_{pq}: a_{pq} \geq 0, \Sigma_p \Sigma_q a_{pq} < \infty\},$$

where the indicated summations are over the set $\omega$ of all positive integers.

**Proof.** It is clear that $Q_0$ is a cone containing $\{x_{pq}: p, q \in \omega\}$ and contained in $Q$. If $\{z_n\}$ is a sequence in $Q_0$ which converges in norm to some $z \in X$, then each $z_n$ has the form $z_n = \Sigma_p \Sigma_q a_{npq} x_{pq}$ with $a_{npq} \geq 0$ and $\Sigma_p \Sigma_q a_{npq} < \infty$. As noted in [7] the limit $\lim_n a_{npq} = a_{pq}$ exists for all $p, q$; indeed, in the notation of [7],
\[ a_{pq} = c^{-1}(x(t_{pp} - 2^{-2p-q-2}) - x(t_{pp} - 2^{-2p-q-1})). \]

Clearly each \( a_{pq} \geq 0 \), and if \( r, s \in \omega \) then
\[ \Sigma_{p \leq r} \Sigma_{q \leq s} a_{pq} = \lim_n \Sigma_{p \leq r} \Sigma_{q \leq s} a_{npq} \leq \lim_n z_n(s_{ii}) = x(s_{ii}); \]

hence \( \Sigma_p \Sigma_q a_{pq} \leq x(s_{ii}) \) and \( z = \Sigma_p \Sigma_q a_{pq} x_{pq} \in Q_0 \).

Let \( \varepsilon > 0 \) be given. It follows from [7, p. 1196] that each \( x_{pq} \) is
continuous and vanishes at 0 and at \( 2 - 2^{-1} \) and hence that each element
of \( X \) shares these properties. Since \( s_{pi} \to 0 \), there exists \( p_i \in \omega \)
such that \( z(s') < \varepsilon \) and \( x(s') < \varepsilon \) for \( s' = s_{p_{i+1,1}} \). Since \( \|z_n - x\| \to 0 \), there
exists \( n' \) such that \( z_n(s') < \varepsilon \) for all \( n > n' \). Thus, by [7],
\[ \Sigma_{p > p_i} \Sigma_q a_{pq} = z(s') < \varepsilon \] and \( \Sigma_{p > p_i} \Sigma_q a_{npq} = z_n(s') < \varepsilon \) for \( n > n' \). Further,
since \( t_{ij} \to 2 - 2^{-1} \), there exists by continuity \( q_i \geq p_i \) such that \( z(t_{i,q_i}) < \varepsilon \) and \( x(t_{i,q_i}) < \varepsilon \); hence there exists \( n'' \geq n' \) such that \( z_n(t_{i,q_i}) < \varepsilon \) for all \( n > n'' \). It follows from [7] that
\[ \Sigma_{p \leq p_i} \Sigma_{q > q_i} a_{pq} \leq \Sigma_{p \leq q_{i-1}} \Sigma_{q > q_i - p} a_{pq} = c^{-1}z(t_{i,q_i}) < \varepsilon \]

and similarly \( \Sigma_{p \leq p_i} \Sigma_{q > q_i} a_{npq} \leq c^{-1}z_n(t_{i,q_i}) < \varepsilon \) for all \( n > n'' \). Moreover,
since \( a_{npq} \to a_{pq} \), there exists \( n_i \geq n'' \) such that \( \Sigma_{p \leq p_i} \Sigma_{q \leq q_i} |a_{pq} - a_{npq}| < \varepsilon \) for all \( n > n_i \). Hence for \( n > n_i \) the triangle inequality implies that
\[ \|z - z_n\| \leq \|\Sigma_{p > p_i} \Sigma_q a_{pq} x_{pq}\| + \|\Sigma_{p \leq p_i} \Sigma_{q > q_i} a_{npq} x_{pq}\| \]
\[ + \|\Sigma_{p \leq p_i} \Sigma_{q > q_i} a_{pq} x_{pq}\| + \|\Sigma_{p \leq p_i} \Sigma_{q > q_i} a_{npq} x_{pq}\| \]
\[ + \|\Sigma_{p \leq p_i} \Sigma_{q > q_i} (a_{pq} - a_{npq}) x_{pq}\| \]
\[ < 5c\varepsilon, \]

since \( \|x_{pq}\| = c \) for all \( p, q \). Thus \( \|z - z_n\| \to 0 \) and therefore \( x = z \in Q_0 \), proving that \( Q_0 \) is norm-closed.

**Lemma 3.2.** Let \( Q_1 = \{\Sigma_p b_p x^p : b_p \geq 0, \Sigma_p b_p < \infty\} \). Then \( L_1(Q) = Q + Q_1 \).

**Proof.** Since \( L_1(Q) \) is a norm-closed cone in \( B([0; 3]) \) by [6, Theorem 1, p. 192] and Theorem 2.1, and since \( \{x^p\}_p \subset L_1(Q) \), it is clear that \( Q + Q_1 \subset L_1(Q) \). If \( \{z_n\} \) is a bounded sequence in \( Q \) which
is pointwise convergent to some \( z \in L_1(Q) \), each \( z_n \) has the form \( z_n = \Sigma_p \Sigma_q a_{npq} x_{pq} \) with \( a_{npq} \geq 0 \) and \( \Sigma_p \Sigma_q a_{npq} < \infty \). As in the proof of Lemma 3.1, for all \( p, q \in \omega \) the limit \( a_{pq} = \lim_n a_{npq} \) exists. For all \( p, q_i \in \omega \),
\[ \Sigma_{q \leq q_i} a_{pq} = \lim_n \Sigma_{q \leq q_i} a_{npq} \leq \lim_n c^{-1}z_n(t_{pp}) = c^{-1}z(t_{pp}); \]

hence \( \Sigma_q a_{pq} \leq c^{-1}z(t_{pp}) \) for each \( p \in \omega \). Let \( b_p = c^{-1}z(t_{pp}) - \Sigma_q a_{pq} \) for each \( p \), and note that all the numbers \( a_{pq} \) and \( b_p \) are nonnegative.

For \( n, p \in \omega \) let \( u_{np} = \Sigma_q a_{npq} x_{pq} \) and \( u_p = \Sigma_q a_{pq} x_{pq} + b_p x^p \). For each \( p \), if \( t \in [0; 3] \) and \( t \) is not of the form \( s_{p_i}, 2 + s_{p_i}, \) or \( t_{vj} \) with \( v \leq p \)
\[ \leq j, \text{ in the notation of } [7, \text{ p. } 1196], \; x_{pq}(t) = 0 \text{ for all sufficiently large } q \] and hence \( x_{p}(t) = 0 \), so that \( u_{np}(t) \xrightarrow{n} u_{p}(t) \). If \( t = s_{pi} \) or \( t = 2 + s_{pi} \), then

\[ u_{np}(t) = \Sigma q a_{npq} = c^{-1}z_{a}(t_{pp}) \longrightarrow c^{-1}z(t_{pp}) = u_{p}(t). \]

Finally, if \( v \leq p \leq j \), then

\[ u_{np}(t_{si}) = c\Sigma q_{j-p} a_{npq} \longrightarrow z(t_{pp}) - c\Sigma q_{j-p} a_{pq} = c[b_{p} + \Sigma q_{j-p} a_{pq}] = u_{p}(t_{si}), \]

proving that \( \{u_{np}\} \) converges pointwise to \( u_{p} \) on \([0; 3]\).

For each \( r \in \omega \),

\[ \Sigma p \leq r(\Sigma q a_{pq} + b_{p}) = c^{-1}\Sigma p \leq r z(t_{pp}) \]

\[ = c^{-1}\lim_{n}\Sigma p \leq r z_{n}(t_{pp}) = \lim_{n}\Sigma p \leq r \Sigma q a_{npq} \]

\[ \leq \lim_{n} z_{n}(s_{i1}) = z(s_{i1}), \]

Hence \( \Sigma_{p} u_{p} \in Q + Q_{i} \). Let \( w = z - \Sigma_{p} u_{p} \); then \( w \) is easily seen to be a Baire function of the first class on \([0; 3]\) and hence by \([8, \text{ p. } 143]\) \( w \) must have a point \( t \) of continuity in \([2; 3]\).

At each point of the form \( t = 2 + s_{ri} \) with \( i \) odd, \( u_{p}(t) = u_{p}(s_{i1}) \) for each \( p \geq r \) and hence

\[ w(t) = \lim_{n}(\Sigma p < r u_{np}(t) + \Sigma p \geq r \Sigma q a_{npq}) - \Sigma p u_{p}(t) \]

\[ = \lim_{n}(z_{n}(s_{i1}) - \Sigma p < r u_{np}(s_{i1}) - \Sigma p \geq r u_{p}(t)) \]

\[ = z(s_{i1}) - \Sigma p u_{p}(s_{i1}) = w(s_{i1}). \]

Since the set of such points \( t \) is dense in \([2; 3]\), \( w(t_{i}) = w(s_{i1}) \). On the other hand, it follows from \([7]\) that for each point of the form \( s = 2 + s_{ri} \pm 2c_{ri1} \) with \( i \) odd, \( x_{pq}(s) = 0 \) whenever \( p \geq r \), and hence

\[ w(s) = \lim_{n}(\Sigma p < r u_{np}(s) - \Sigma p < r u_{p}(s)) = 0. \]

Since the set of such points \( s \) is also dense in \([2; 3]\), it follows that \( w(t_{i}) = 0 \) and hence that \( w(s_{i1}) = 0 \).

For each \( r \in \omega \) let \( w_{r} = z - \Sigma p < r u_{p} \). Then \( w_{r} \to w \) in the norm topology, and \( w_{r} \) is the pointwise limit of \( \{\Sigma p \geq r u_{np}\} \). Hence

\[ ||w_{r}|| \leq \lim \sup_{n}||\Sigma p \geq r u_{np}|| \leq c\lim_{n}\Sigma p \geq r u_{np}(s_{i1}) = cw_{r}(s_{i1}) \]

and consequently

\[ ||w|| = \lim_{r}||w_{r}|| \leq c\lim_{r}w_{r}(s_{i1}) = cw(s_{i1}) = 0. \]

Therefore \( w = 0 \) and \( z = \Sigma_{p} u_{p} \in Q + Q_{i} \), completing the proof of the lemma.

**Note.** The last paragraph of the previous proof shows that if
\{z_n\} is a bounded pointwise convergent sequence in \(Q\), then in the notation of that proof for each \(\varepsilon > 0\) there exist \(p_1, n_1 \in \omega\) such that \(\Sigma_{p \geq p_1} \Sigma_q a_{npq} < \varepsilon\) for all \(n \geq n_1\). Indeed, given \(\varepsilon > 0\) there exists \(p_1\) such that \(cw_{p_1}(s_{11}) < \varepsilon\). Since \(\limsup_n ||\Sigma_{p \geq p_1} u_{np}|| \leq cw_{p_1}(s_{11})\), there exists \(n_1\) such that for each \(n \geq n_1\)

\[
\Sigma_{p \geq p_1} \Sigma_q a_{npq} = (\Sigma_{p \geq p_1} u_{np})(s_{11}) \leq ||\Sigma_{p \geq p_1} u_{np}|| < \varepsilon.
\]

**LEMMA 3.3.** Let \(Q_2 = \{c_0 x^0 : c_0 \geq 0\}\). Then \(L_2(Q) = L_\Omega(Q) = Q + Q_1 + Q_2\).

**Proof.** Clearly \(Q + Q_1 + Q_2\) is a cone containing \(L_i(Q)\) and contained in \(L_2(Q)\). To prove the lemma it suffices to show that \(L(Q + Q_1 + Q_2) \subseteq Q + Q_1 + Q_2\). If \(\{z_n\}\) is a bounded sequence in \(Q + Q_1 + Q_2\) which is pointwise convergent to a function \(z\), then each \(z_n\) has the form

\[
z_n = y_n + \Sigma_{p} b_{np} x^p + c_n x^0
\]

where \(y_n \in Q, b_{np} \geq 0, c_n \geq 0,\) and \(\Sigma_{p} b_{np} < \infty\). Since \(\{z_n\}\) is bounded, the diagonal process yields a subsequence \(\{z_{n_j}\}\) of \(\{z_n\}\) such that \(c_0 = \lim c_{n_j}\) and \(b = \lim b_{n_jp}\) exist and \(b_p = \lim b_{n_jp}\) exists for each \(p \in \omega\). It is easily seen from [7, p. 1196] that these limits are finite and nonnegative, that \(\Sigma_{p} b_{p} \leq b\), and that the sequence \(\{\Sigma_{p} b_{np} x^p + c_n x^0\}\) is pointwise convergent to \(\Sigma_{p} b_{p} x^p + (c_0 + b - \Sigma_{p} b_{p}) x^0\). Hence also \(\{y_{n_j}\}\) is pointwise convergent, and by Lemma 3.2 its pointwise limit is in \(Q + Q_1\). Since \(z\) is the pointwise limit of \(\{z_{n_j}\}\), it follows that \(z \in Q + Q_1 + Q_2\).

**REMARK.** It is clear from [7] that the representation of each \(z \in L_\Omega(Q)\) in the form \(\Sigma_{p} \Sigma_q a_{pq} x_{pq} + \Sigma_{p} b_{p} x^p + c_0 x^0\) is unique.

4. Given an arbitrary countable ordinal \(\alpha \geq 2\) and a number \(c \geq 1\), we now construct a separable Banach space \(X_\alpha\) containing a cone \(P_\alpha\) for which there exists \(z_\alpha \in L_\alpha(P_\alpha)\) such that \(||z_\alpha|| = 1\) but such that if \(\{w_n\}\) is a bounded sequence in \(\bigcup_{\beta < \alpha} L_\beta(P_\alpha)\) converging pointwise to \(z_\alpha\), then \(\lim_n ||w_n|| \geq c\).

Let \(B_\alpha\) be the countable set \(\{(2, 1)\} \cup \{(\beta, \gamma) : \alpha \geq \beta > \gamma \geq 2\}\). Then there exists a one-to-one mapping \(\nu_\alpha\) from \(D_\alpha\) onto \(B_\alpha\), where \(D_\alpha = \{1, \cdots, 2^{-1}(\alpha^2 - 3\alpha + 4)\}\) if \(\alpha < \omega\) and \(D_\alpha = \omega\) if \(\alpha \geq \omega\), such that \(\nu_\alpha(1) = (2, 1)\). Let \(U = \{0\} \cup \{n^{-1} : n \in D_\alpha\}\) and let \(S_\alpha\) be the compact subset \([0; 6] \times U\) of \(E^2\). For each real function \(z\) defined on \(S_\alpha\) and each \(u \in U\), let

\[
z^1.u(t) = z(t, u), \quad z^2.u(t) = z(t + 3, u)
\]
for \( t \in [0; 3] \). Further, let \( \mathcal{S}_\alpha \) be the set of all type \( - \alpha \) generalized sequences \( s = (s_\beta: 1 \leq \beta \leq \alpha) \) of positive integers.

Letting \( x_{pq} \) be as in § 3 and noting by [7] that \( x_{pq}(0) = x_{pq}(3) = 0 \) for \( p, q \in \omega \), we easily verify that for each \( s \in \mathcal{S}_\alpha \) the function \( x_s \) defined by

\[
x^{1,u}_s = \begin{cases} x_{s_\beta} & \text{if } u > 0, u^{-1} \leq s, \nu_s(u^{-1}) = (\beta, \gamma) \\ 0 & \text{if } u > 0, u^{-1} > s \\ 0 & \text{if } u = 0 \\ \end{cases}
\]

is an element of \( C(S_\alpha) \). Let \( X_\alpha \) be the norm-closed subspace and \( P_\alpha \) the norm-closed cone in \( C(S_\alpha) \) generated by \( \{ x_s: s \in \mathcal{S}_\alpha \} \). Since \( S_\alpha \) is compact metric, \( C(S_\alpha) \) is separable [3, p. 340] and hence also \( X_\alpha \) is separable. Note that \( ||x_s|| = c \) for each \( s \in \mathcal{S}_\alpha \).

For \( 1 \leq \delta \leq \alpha \) and \( s \in \mathcal{S}_\alpha \) let \( z_{s, \delta} \) be defined on \( S_\alpha \) by

\[
z^{1,\delta}_{s, \delta} = u^{-1} z^{2,\delta}_{s, \delta} = \begin{cases} x_{s_{\beta}} & \text{if } u > 0, \nu_s(u^{-1}) = (\beta, \gamma), \beta > \gamma > \delta \\ x^{\delta} & \text{if } u > 0, \nu_s(u^{-1}) = (\beta, \gamma), \beta > \delta \geq \gamma \\ x^{0} & \text{if } u > 0, \nu_s(u^{-1}) = (\beta, \gamma), \delta \geq \beta > \gamma \\ 0 & \text{if } u = 0 \\ \end{cases}
\]

Thus \( ||z_{s, \delta}|| = 1 \) for each \( s \in \mathcal{S}_\alpha \). In fact, \( z_{s, \delta} \) is independent of \( s \in \mathcal{S}_\alpha \) and we simply write \( z_\delta \) instead of \( z_{s, \delta} \).

**Lemma 4.1.** For each \( s \in \mathcal{S}_\alpha \) and \( 1 \leq \delta \leq \alpha \), \( z_{s, \delta} \in L_\delta(P_\alpha) \).

**Proof.** If \( \delta = 1 \) and \( s \in \mathcal{S}_\alpha \), then for each \( q \in \omega \) let \( s^q \in \mathcal{S}_\alpha \) be defined by

\[
s^q_\beta = \begin{cases} q & \text{if } \beta = 1 \\ s^q_\beta & \text{if } 1 < \beta \leq \alpha \\ \end{cases}
\]

It is easy to verify that \( \{ x_{s^q_\beta} \}_{q=1}^\omega \) is a bounded sequence in \( P_\alpha \) converging pointwise to \( z_{s,1} \), so that \( z_{s,1} \in L_1(P_\alpha) \).

Proceeding by transfinite induction, assume that \( 1 < \delta \leq \alpha \) and that \( z_{s, \epsilon} \in L_\epsilon(P_\alpha) \) for each \( s \in \mathcal{S}_\alpha \) and \( 1 \leq \epsilon < \delta \). Let \( s \in \mathcal{S}_\alpha \) be given, and let \( t^q \in \mathcal{S}_\alpha \) be defined for each \( q \in \omega \) by

\[
t^q_\beta = \begin{cases} s^q_\beta & \text{if } \delta \neq \beta \leq \alpha \\ q & \text{if } \beta = \delta \end{cases}
\]

If \( \delta \) is not a limiting ordinal, then \( \delta \) has an immediate predecessor \( \delta - 1 \), and it is straightforward to show that the bounded sequence
\( \{z_{t,s-1}\}_{t=1}^{\infty} \) in \( L_{t-1}(P_a) \) converges pointwise to \( z_{s,a} \) on \( S_a \). On the other hand, if the countable ordinal \( \delta \) is limiting, there exists an increasing sequence \( \{r_\alpha\}_{\alpha=1}^{\infty} \) of ordinals whose limit is \( \delta \), and it can be verified that the bounded sequence \( \{z_{t,s-1}\}_{t=1}^{\infty} \) in \( \bigcup_{i<\delta} L_i(P_a) \) is pointwise convergent to \( z_{s,a} \). Thus the lemma is proved inductively. In particular, our proof has shown that \( z_{s,a} \), whose norm is 1, is the pointwise limit of a sequence of elements of norm \( c \) in \( \bigcup_{\beta<\delta} L_\beta(P_a) \).

Note that if \( 1 \leq \delta \leq \Omega \), \( z \in L_\delta(P_a), \) \( i \in \{1, 2\} \), and \( u \in U \), then \( z^i,u \in L_\delta(Q) \subseteq L_\alpha(Q) = Q + Q_1 + Q_2 \) by Lemma 3.3, and trivially \( z^i,0 = 0 \).

**Lemma 4.2.** Let \( 1 \leq \delta \leq \Omega \) and \( z \in L_\delta(P_a) \) with
\[
z^1,i = \Sigma_p \Sigma_q \alpha_{pq} x_{pq} + \Sigma_p b_p x^p + c_0 x^0.
\]
Then also \( y \in L_\delta(P_a) \), where
\[
y^1,i = y^2,i = \Sigma_p (b_p + \Sigma_q \alpha_{pq}) x^p + c_0 x^0,
\]
y^1,0 = y^2,0 = 0, and \( uy^1,u = y^2,u = z^2,u \) for each \( u \in U \setminus \{0, 1\} \).

**Proof.** The proof will be by induction on \( \delta \). If \( \delta = 1 \), then \( z^1,i \in L_1(Q) = Q + Q_1 \) and hence \( c_0 = 0 \). There exists a bounded sequence \( \{w_n\} \) in \( P_a \) which converges pointwise to \( z \) on \( S_a \). Since the finite linear combinations with nonnegative coefficients of elements in \( \{z_s: s \in \mathcal{S}_a\} \) are norm-dense in \( P_a \), each \( w_n \) can be assumed to have the form
\[
w_n = \Sigma_i \omega r_{ni} x_{(s^{ni})},\text{ where each } s^{ni} \in \mathcal{S}_a, \text{ each } r_{ni} \geq 0, \text{ and for each } n \text{ there exist only finitely many } i \text{ such that } r_{ni} > 0.\]
If \( t^{ni} \in \mathcal{S}_a \) is defined for all \( n, i \in \omega \) by \( (t^{ni})_\beta = (s^{ni})_\beta \) for \( 2 \leq \beta \leq \alpha \) and \( (t^{ni})_1 = n \), then the sequence \( \{w'_n\} \), where \( w'_n = \Sigma_i \omega r_{ni} x_{(s^{ni})} \), is clearly a bounded sequence in \( P_a \). It will now be shown that \( \{w'_n\} \) converges pointwise to \( y \).

For each \( u \in U \setminus \{0, 1\}, \nu(u(w^{-1})) = (\beta, \gamma) \) for some \( \beta, \gamma \) such that \( \beta > \gamma \geq 2 \), and hence for each \( n \geq w^{-1}, \)
\[
w'_n,1,u = w^{-1} w'_n,2,u = \Sigma_i \omega r_{ni} x_{(s^{ni})} \beta_1 \gamma = \Sigma_i \omega r_{ni} x_{(s^{ni})} \beta_1 \gamma = w^{-1} w'_n,1,u;
\]
therefore,
\[
w'_n,1,u(t) \longrightarrow n \longrightarrow w^{-1} x^{2,u}(t) = y^{1,u}(t) \text{ and } w'_n,2,u(t) \longrightarrow z^{2,u}(t) = y^{2,u}(t) \text{ for all } t \in [0, 3].
\]

Since the situation for \( u = 0 \) is trivial, it remains only to consider the case in which \( u = 1 \). Given \( n, p, q \in \omega \) let
\[
a_{npq} = \Sigma \{r_{ni}: (s^{ni})_2 = p, (s^{ni})_1 = q\}.
\]
Thus each \( a_{npq} \geq 0 \), and for each \( n \) there are only finitely many pairs \( (p, q) \) for which \( a_{npq} > 0 \). Since \( w'^1,n = \Sigma_p \Sigma_q a_{npq} x_{pq} \) for each \( n \), it follows from the proof of Lemma 3.2 and the note following that proof that
\[ \lim_n \alpha_n p q = \alpha p q \text{ for each } p, q; \]

\[ \lim_n \Sigma \alpha_n p q = e^{-1} \xi^{1,1}(t_{pp}) = \Sigma \alpha p q + b_p \]

for each \( p \); and that \( \lim \sup_n \Sigma p \geq r \Sigma \alpha_n p q \to 0 \) as \( r \to \infty \). Thus given \( \varepsilon > 0 \), there exist \( r \) and \( n_1 \) such that \( \Sigma p \geq r (\Sigma \alpha p q + b_p) < \varepsilon/3c \) and \( \Sigma p \geq r \Sigma \alpha_n p q < \varepsilon/3c \) for all \( n > n_1 \). Now \( w_n^{1,1} = \Sigma_p (\Sigma \alpha_n p q) x_{pn} \), and for each \( t \in [0; 3] \) there exists \( n_2(t) > n_1 \) such that

\[ |(\Sigma \alpha_n p q) x_{pn}(t) - (\Sigma \alpha p q + b_p) x^p(t)| < \frac{\varepsilon}{3r} \]

for each \( n > n_2(t) \) and \( p < r \). It follows easily by the triangle inequality that

\[ |w_n^{1,1}(t) - \Sigma_p (b_p + \Sigma \alpha p q) x^p(t)| < \varepsilon \]

for each \( n > n_2(t) \). Thus

\[ w_n^{1,1}(t) = w_n^{2,1}(t) \to y^{1,1}(t) = y^{2,1}(t) \]

for all \( t \), completing the proof for \( \delta = 1 \).

Now let \( \delta > 1 \) and assume that the statement of the lemma is true for each ordinal \( \varepsilon \) such that \( 1 \leq \varepsilon < \delta \). If \( z \in L_\varepsilon(P_\alpha) \), there exists a bounded sequence \( \{w_n\} \subset \bigcup_{\varepsilon < \delta} L_\varepsilon(P_\alpha) \) which converges pointwise to \( z \).

By the induction hypothesis the sequence \( \{y_n\} \) is contained in \( \bigcup_{\varepsilon < \delta} L_\varepsilon(P_\alpha) \), where, if

\[ w_n^{1,1} = \Sigma_p, q \alpha_n p q x_{pq} + \Sigma_p b_n p x^p + c_n x^0, \]

then

\[ y_n^{1,1} = y_n^{2,1} = \Sigma_p (b_n p + \Sigma \alpha_n p q) x^p + c_n x^0, \]

and \( y_n^{1,0} = y_n^{2,0} = 0 \) and \( uy_n^{1,0} = y_n^{2,0} = w_n^{1,0} \) for \( u \neq 0, 1 \). An easy induction argument shows that \( ||f^{2,0}\| \leq ucf^{1,1}(s_{t_1}) \) for each \( u \in U \) and \( f \in L_\varepsilon(P_\alpha) \), and from this result it follows that the sequence \( \{y_n\} \) is bounded. To see that \( \{y_n\} \) converges pointwise to \( y \), note first that \( y_n^{1,0} = y_n^{2,0} = 0 = y^{1,0} = y^{2,0} \) for each \( n \). Next, if \( u \neq 0, 1 \) and \( t \in [0; 3] \), then

\[ uy_n^{1,*}(t) = y_n^{2,*}(t) = w_n^{2,*}(t) \to z^{2,*}(t) = uy_n^{1,*}(t) = y^{2,*}(t). \]

For \( u = 1 \), since \( y_n^{1,1} = y_n^{2,1} \) and \( y^{1,1} = y^{2,1} \), it remains only to show that \( y_n^{1,1}(t) \to y^{1,1}(t) \) for each \( t \in [0; 3] \). If \( t \) is not of the form \( s_{pi} \), \( 2 + s_{pi} \), or \( t_{pj} \) with \( v \leq j \), then \( y_n^{1,1}(t) = 0 = y^{1,1}(t) \). If \( t = s_{pi} \) or \( 2 + s_{pi} \) with \( i \) odd, then

\[ y_n^{1,1}(t) = w_n^{1,1}(t) - \Sigma_{p < p_1} \Sigma q \alpha_n p q x_{pq}(t) \]

and
\[ y^{i,i}(t) = z^{i,i}(t) - \Sigma_{p < p_1} \Sigma_{q \in \sigma} a_{pq} x_{pq}(t); \]

since \( w_n^{i,i}(t) \to z^{i,i}(t) \) and \( a_{n pq} \to a_{pq} \) (as noted in the proof of Lemma 3.1), and since there exists \( q \) such that \( x_{pq}(t) = 0 \) whenever \( p < p_1, q > q_1 \), it follows that \( y_n^{i,i}(t) \to y^{i,i}(t) \). Finally, if \( t = t_{\nu j} \) with \( 1 \leq \nu \leq j \), then

\[
y_n^{i,i}(t) = w_n^{i,i}(t) + c \Sigma_{p = v}^{j} \Sigma_{q = 1}^{j} a_{npq}
\to z^{i,i}(t) + c \Sigma_{p = v}^{j} \Sigma_{q = 1}^{j} a_{npq} = y^{i,i}(t).
\]

This completes the induction step and hence the proof of the lemma.

**Lemma 4.3.** Let \( 0 \leq \delta \leq \Omega \) and \( z \in L_\delta(P_\alpha) \). Then \( z^{i,u} \leq u^{-1} z^{i,u} \) for each \( u \in U \setminus \{0\} \). If

\[
z^{i,i} = \Sigma_{p} \Sigma_{q} a_{pq} x_{pq} + \Sigma_{p} b_{p} x_{p} + c_{o} x_{0}
\]

and if \( q_1 \in \omega \), then

\[
z^{i,u} \leq u^{-1} z^{i,u} - c \Sigma_{p} \Sigma_{q < q_1} a_{pq}
\]

for each \( u \geq q_1^{-1} \).

**Proof.** The first assertion is immediate by induction on \( \delta \). For the second assertion suppose first that \( z \) has the form \( z = \Sigma_{s \in \sigma} d_{s} x_{s} \) where \( \sigma \) is a finite subset of \( \sigma_a \) and \( d_{s} \geq 0 \) for each \( s \). Then \( z^{i,i} = \Sigma_{p} \Sigma_{q < q_1} a_{pq} x_{pq} \), where

\[
a_{pq} = \Sigma\{d_{s}: s \in \sigma, s_{1} = p, s_{1} = q\}.
\]

Thus \( \Sigma_{p} \Sigma_{q < q_1} a_{pq} = \Sigma\{d_{s}: s \in \sigma, s_{1} < q_{1}\} \) and hence if \( u \geq q_1^{-1} \) and \( \nu_{s}(u^{-1}) = (\beta, \gamma) \), then

\[
z^{i,u} = u \Sigma_{s \in \sigma} d_{s} x_{s_{1},s_{1}} = u z^{i,u} + u \Sigma_{s_{1} < q_{1}} d_{s} x_{s_{1},s_{1}}
\leq u (z^{i,u} + \Sigma_{s_{1} < q_{1}} d_{s} x_{s_{1},s_{1}}) \leq u (z^{i,u} + c \Sigma_{p} \Sigma_{p < q_{1}} a_{pq})
\]

as desired.

Next, suppose \( z \) is the pointwise limit of a bounded sequence \( \{w_{n}\}_{n \in \omega} \) in \( L_\delta(P_\alpha) \) such that each \( w_{n} \) has the desired property; i.e., for each \( u \geq q_1^{-1} \),

\[
w^{i,i}_{n} \geq u^{-1} w^{i,u}_{n} - c \Sigma_{p} \Sigma_{q < q_1} a_{npq}
\]

where

\[
w^{i,i}_{n} = \Sigma_{p} \Sigma_{q} a_{npq} x_{pq} + \Sigma_{p} b_{p} x_{p} + c_{o} x_{0}.
\]

By the proof of Lemma 3.3 there is a subsequence \( \{w_{n_{s}}\} \) of \( \{w_{n}\} \) such that \( \{\Sigma_{p} \Sigma_{q} a_{npq} x_{pq}\} \) is pointwise convergent, and by the note following
Lemma 3.2 for each \( \zeta > 0 \) there exist \( p_i \) and \( i \) such that for each \( i > i_1 \),
\[
\Sigma_{p \geq p_1} \Sigma_q a_{n_i p q} < c \zeta.
\]
Since \( a_{n_i p q} \rightarrow a_{pq} \) for each \( p \) and \( q \), there exists \( i_2 > i_1 \) such that for each \( i > i_2 \),
\[
\Sigma_{p < p_1} \Sigma_{q < q_1} a_{n_i p q} < \Sigma_{p < p_1} \Sigma_{q < q_1} a_{pq} + \zeta.
\]
Hence, for each \( i > i_2 \),
\[
\Sigma_p \Sigma_{q < q_1} a_{n_i p q} < \Sigma_p \Sigma_{q < q_1} a_{pq} + (1 + c) \zeta.
\]
For each \( t \in [0; 3] \) and \( u \geq q_1^{-1} \),
\[
z^{',u}(t) = \lim_i w^{',u}_{n_i}(t) = \lim_{i} (w^{-1}w^{',u}_{n_i}(t) - c \Sigma_p \Sigma_{q < q_1} a_{n_i p q})
\geq u^{-1}z^{',u}(t) - c \Sigma_p \Sigma_{q < q_1} a_{pq} + (1 + c) \zeta.
\]
Since \( \zeta \) can be arbitrarily small,
\[
z^{',u} \geq u^{-1}z^{',u} - c \Sigma_p \Sigma_{q < q_1} a_{pq}
\]
for each \( u \geq q_1^{-1} \), as desired.

The preceding paragraphs provide both the base step and the inductive step for the proof of the second assertion of the lemma.

**Lemma 4.4.** Let \( G \) be the set of all \( z \in L_\delta(P_{a}) \) such that \( z^{1,1} \in Q_1 + Q_2 \). If \( z \in G \), then \( z^{1,u} = u^{-1}z^{2,u} \) for each \( u \in U \setminus \{0\} \).

**Proof.** In the notation of Lemma 4.3, \( a_{pq} = 0 \) for all \( p, q \) and hence \( \Sigma_p \Sigma_{q < u^{-1}} a_{pq} = 0 \). The present result now follows immediately from Lemma 4.3.

**Lemma 4.5.** \( L_\delta(P_{a}) \cap G = \begin{cases} L_{\delta^{-1}}(L_1(P_{a}) \cap G) \text{ if } 1 \leq \delta < \omega \\ L_\delta(L_1(P_{a}) \cap G) \text{ if } \omega \leq \delta \leq \Omega. \end{cases} \)

**Proof.** The result is trivial for \( \delta = 1 \). Let \( 1 < \delta < \omega \) and assume the result is true for all \( \epsilon < \delta \). Then for each \( z \in L_\epsilon(P_{a}) \cap G \) it follows from Lemma 4.4 that \( z^{1,u} = u^{-1}z^{2,u} \) for each \( u \neq 0 \). Since \( z \in G \), it follows that \( z \) is identical with the \( y \) occurring in the statement of Lemma 4.2 and hence is the pointwise limit of the bounded sequence \( \{y_n\} \subset G \cap \bigcup_{\epsilon < \delta} L_{\epsilon}(P_{a}) \) which appears in the inductive step of the proof of Lemma 4.2. By the inductive hypothesis
\[
\{y_n\} \subset \bigcup_{\epsilon < \delta} L_{\epsilon^{-1}}(L_1(P_{a}) \cap G) = L_{\delta^{-1}}(L_1(P_{a}) \cap G)
\]
and hence \( z \in L_{\delta}(L(P_a) \cap G) \). Conversely, if \( z \in L_{\delta}(L(P_a) \cap G) \), then \( z \) is the pointwise limit of a bounded sequence \( \{w_n\} \subset L_{\delta}(L(P_a) \cap G) \). By the inductive hypothesis \( L_{\delta}(L(P_a) \cap G) = L_{\delta}(P_a) \cap G \). Hence clearly \( z \in L_{\delta}(P_a) \), and also \( z \in G \) by the proof of Lemma 3.3. Thus the proof is complete for \( \delta < \omega \).

Now let \( \omega \leq \delta \leq \Omega \) and assume the result is true for all \( \varepsilon < \delta \). As in the previous case each \( z \in L_{\delta}(P_a) \cap G \) is the pointwise limit of a bounded sequence \( \{y_n\} \subset G \cap \bigcup_{\varepsilon<\delta} L_\varepsilon(P_a) \). By the inductive hypothesis \( \{y_n\} \subset \bigcup_{\varepsilon<\delta} L_\varepsilon(L(P_a) \cap G) \), and hence \( z \in L_{\delta}(L(P_a) \cap G) \). Conversely, if \( z \in L_{\delta}(L(P_a) \cap G) \), then \( z \) is the pointwise limit of a bounded sequence \( \{w_n\} \subset \bigcup_{\varepsilon<\delta} L_\varepsilon(P_a) \) and hence \( z \in G \cap L_{\delta}(P_a) \), completing the proof of the lemma.

**Lemma 4.6.** Let \( \{w_n\} \) be a bounded sequence in \( \bigcup_{\varepsilon<\omega} L_\varepsilon(P_a) \) which converges pointwise on \( S_a \) to the function \( z_a \) defined earlier in the present section. If

\[
 w_n^{t_1} = \Sigma_p \Sigma_q a_{n pq} x_{pq} + \Sigma_p b_{np} x^p + c_n x^0
\]

for each \( n \in \omega \), then \( \lim_n \Sigma_p \Sigma_q a_{n pq} = 0 \).

**Proof.** If the conclusion is not true, then as in the proof of Lemma 3.3 a subsequence \( \{w_n^t\} \) of \( \{w_n\} \) exists such that \( \inf_t \Sigma_p \Sigma_q a_{n pq} > 0 \) and such that the limits \( c_0 = \lim_n c_n \), \( b = \lim_n \Sigma_p b_{np} \), \( b_p = \lim_n b_{np} \), and \( a_p = \lim_n \Sigma_q a_{n pq} \) all exist (\( p \in \omega \)). Since \( z_a^{t_1} = x^0 \) by definition of \( z_a \), the coefficient of each \( x_{pq} \) in the unique expansion of \( z_a^{t_1} \) must vanish and it is easily verified that \( \{\Sigma_p b_{np} x^p + c_n x^0\} \) and \( \{\Sigma_p \Sigma_q a_{n pq} x_{pq}\} \) converge pointwise to \( \Sigma_p b_{np} x^p + (c_0 + b - \Sigma_p b_p) x^0 \) and \( \Sigma_p \Sigma_q a_{n pq} x_{pq} \) respectively, as in the proofs of Lemmas 3.3 and 3.2 (note that the symbol \( b_p \) is used differently in those two proofs). Hence

\[
 z_a^{t_1} = \Sigma_p (a_p + b_p) x^p + (c_0 + b - \Sigma_p b_p) x^0.
\]

Now the uniqueness of the expansion of \( z_a^{t_1} \) shows that \( a_p + b_p = 0 \) for each \( p \) and \( c_0 + b - \Sigma_p b_p = 1 \). Since \( a_p \) and \( b_p \) are nonnegative, they must both vanish for each \( p \) and hence \( c_0 + b = 1 \). Now

\[
 1 = z_a^{t_1}(s_{t_1}) = \lim_n (\Sigma_p \Sigma_q a_{n pq} + \Sigma_p b_{np} + c_n)
  = \lim_n \Sigma_p \Sigma_q a_{n pq} + b + c_0
\]

and hence \( \lim_n \Sigma_p \Sigma_q a_{n pq} = 0 \), contradicting our assumption and thus proving the lemma.

**Theorem 4.1.** If \( \{w_n\} \) is a bounded sequence in \( \bigcup_{\varepsilon<\omega} L_\varepsilon(P_a) \) which converges pointwise to \( z_a \), then there exists a sequence
\{y_n\} \subset G \cap \bigcup_{a<\alpha} L_a(P_a) \text{ such that } \|y_n - w_n\| \to 0.

**Proof.** Each $w_n^{1,1}$ has the form

$$w_n^{1,1} = \Sigma_p \Sigma_q a_{npq} x_{pq} + \Sigma_p b_{np} x_p + c_n x_0.$$ 

By Lemma 4.2 there exists a sequence $\{y_n\} \subset \bigcup_{a<\alpha} L_a(P_a)$ such that

$$y_n^{1,1} = y_n^{w_n^{1,1}} = \Sigma_p (b_{np} + \Sigma_q a_{npq}) x_p + c_n x_0,$$

and $y_n^{0,0} = y_n^{w_n^{0,0}} = 0$ and $uy_n^{1,u} = y_n^{w_n^{1,u}} = w_n^{1,u}$ for each $u \neq 0, 1$. Since obviously $\{y_n\} \subset G$, if remains only to show that $\lim_{n \to \infty} \|y_n - w_n\| = 0$.

First note that $(y_n - w_n)^{1,0} = 0$ and $(y_n - w_n)^{2,u} = 0$ for all $u \neq 1$.

For each real $r > 0$ there exists by Lemma 4.6 an $n_r \in \omega$ such that $\Sigma_p \Sigma_q a_{npq} < r$ for all $n > n_r$. For each $u \neq 0$ there exists $q_u \in \omega$ such that $u \geq q_u^{-1}$ and hence by Lemma 4.3,

$$u^{-1} w_n^{2,u} - cr < u^{-1} w_n^{1,u} - c\Sigma_p \Sigma_q a_{npq} x_p + c_n x_0 \leq w_n^{1,u} \leq u^{-1} w_n^{2,u}$$

for each $n > n_r$. Since $y_n^{1,u} = w_n^{2,u}$ for each $u \neq 1$,

$$\|y_n - w_n\|^2 = \|u^{-1} w_n^{2,u} - c\Sigma_p \Sigma_q a_{npq} x_p\| < 2cr$$

for each $n > n_r$ and $u \neq 0, 1$.

Finally, since $z^{1,1} = z^{2,1}$ for each $z \in L_a(P_a)$,

$$\|y_n - w_n\|^2 = \|y_n - w_n\|^2 = \||\Sigma_p (\Sigma_q a_{npq} x_p - \Sigma_q a_{npq} x_{pq})\| < 2cr$$

for each $n > n_r$.

We have now shown that $\|y_n - w_n\| < 2cr$ for each $n > n_r$, completing the proof of the theorem.

**Lemma 4.7.** Let $\zeta$ be a countable ordinal, and let $y \in L_\zeta(L_1(P_a) \cap G)$. Let $\zeta' = \zeta + 1$ if $\zeta < \omega$ and $\zeta' = \zeta$ if $\zeta \geq \omega$. If $u \in U\setminus\{0\}$ and $\nu_a(u^{-1}) = (\beta, \gamma)$ with $\beta > \gamma > \zeta'$, then $y^{1,u}$ is continuous and hence has the form $y^{1,u} = \Sigma p \Sigma q a_{prq} x_{pq}$. If also $v \in U\setminus\{0\}$ and $\nu_a(v^{-1}) = (\gamma, \delta)$ with $\beta > \gamma > \delta > \zeta'$, then for each $r \in \omega$, $\Sigma_p a_{pr} = \Sigma q a_{rq}$.

**Proof.** The proof will be by induction on $\zeta$. If $y \in L_0(L_1(P_a) \cap G) = L_1(P_a) \cap G$, there is a bounded sequence $\{w_n\} \subset P_a$ which converges pointwise to $y$. The sequence $\{w_n\}$ can be chosen so that each $w_n$ is a finite linear combination of elements of $\{x_s: s \in \mathcal{S}_a\}$, and hence there exists a countable subset $\sigma$ of $\mathcal{S}_a$ such that each $w_n$ has the form $w_n = \Sigma_{s \in \sigma} b_{ns} x_s$, where each $b_{ns}$ is nonnegative and for each $n$ only a finite number of the $b_{ns}$ are nonzero. If $u \neq 0$ and $\nu_a(u^{-1}) = (\beta, \gamma)$, then
\[ w_n^w = u \sum_{s \in \sigma} b_n^s x_{s^w}^r = u \sum_p \sum_q a_{npq}^w x_{pq}, \]

where
\[ a_{npq}^w = \sum \{ b_n^s : s^w = p, s^r = q \}. \]

Now \( y^w = u^{-1} y^w \) by Lemma 4.4 since \( y \in G \); hence \( y^w \) is the pointwise limit of the bounded sequence \( \{ \Sigma_p \Sigma_q a_{npq}^w x_{pq} \} \). The function \( y^w \) is in \( L_1(Q) \) and hence has the form
\[ y^w = \Sigma_p \Sigma_q a_{npq}^w x_{pq} + \Sigma_p b_p^w x_p; \]
by the proof of Lemma 3.2, \( a_{npq}^w = \lim_n a_{npq}^w \) for all \( p, q \) and
\[ b_p^w = c^{-1} y^w(t_p) - \Sigma_q a_{pq}^w = \lim_n \Sigma_q a_{npq}^w - \Sigma_q a_{pq}^w \]
for all \( p \).

Now assume further that \( \nu_a(u^{-1}) = (\beta, \gamma) \) with \( \gamma > 1 \), and let \( \lambda = 2 \) if \( \gamma > 2 \) and \( \lambda = 1 \) if \( \gamma = 2 \). Then \( (\gamma, \lambda) \in B_\alpha \) so there exists \( v_1 \in U \setminus \{ 0 \} \) such that \( \nu_a(v_1^{-1}) = (\gamma, \lambda) \). Since \( \{ \Sigma_p \Sigma_q a_{npq}^w x_{pq} \} \) and \( \{ \Sigma_p \Sigma_q a_{npq}^w x_{pq} \} \) are bounded pointwise convergent sequences in \( Q \), it follows from the note following Lemma 3.2 that for each real \( \epsilon > 0 \) there exist integers \( p_i \) and \( n_i \) such that \( \Sigma_{p > p_i} \Sigma_q a_{npq}^w < \epsilon \) and \( \Sigma_{p > p_i} \Sigma_q a_{npq}^w < \epsilon \) for all \( n \geq n_i \).

Since
\[ \Sigma_{p > p_i} \Sigma_q a_{npq}^w = \Sigma \{ b_n^s : s^w > p \} = \Sigma_{p > p_i} \Sigma_q a_{npq}^w < \epsilon \]
for each \( n \geq n_i \), it follows that if \( f_n = \Sigma_{p \leq p_i} \Sigma_q a_{npq}^w x_{pq} \),
\[ || u^{-1} w_n^w - f_n || \leq c \Sigma \{ a_{npq}^w : p > p_i \text{ or } q < p_i \} > 2 c \epsilon \]
for each \( n \geq n_i \). Since \( || f_n || \leq || u^{-1} w_n^w || \leq u^{-1} \sup_n || w_n || \) for each \( n \), it follows that for each \( n \geq n_i \), \( f_n \) belongs to the compact subset
\[ \mathcal{C}_{u, p_1} = \{ \Sigma_{p \leq p_1} \Sigma_{q \leq p_1} k_{pq} x_{pq} : k_{pq} \geq 0, \Sigma_{p \leq p_1} \Sigma_{q \leq p_1} k_{pq} \leq u^{-1} \sup_n || w_n || \} \]
of \( C[0; 3] \). By compactness some subsequence \( \{ f_{n_k} \} \) of \( \{ f_n \} \) must converge to an element \( f \) of \( \mathcal{C}_{u, p_1} \), and since \( u^{-1} w_n^w \) converges pointwise to \( y^w \), it follows that \( || y^w - f || \leq 2 c \epsilon \). Thus, for each \( \epsilon > 0 \) there exists an \( f \in C[0; 3] \), depending on \( \epsilon \), such that \( || y^w - f || \leq 2 c \epsilon \). Since \( C[0; 3] \) is complete in norm, \( y^w \in C[0; 3] \) and must therefore be equal to \( \Sigma_p \Sigma_q a_{pq}^w x_{pq} \).

Now if \( 0 \neq v \in U \) and \( \nu_a(v^{-1}) = (\gamma, \delta) \) with \( \gamma > \delta > 1 \), then for all \( n \) and \( r \),
\[ \Sigma_p a_{npq}^w = \Sigma \{ b_n^s : s^w = r \} = \Sigma_q a_{nrq}^w. \]
Since \( y^w = \Sigma_p \Sigma_q a_{pq}^w x_{pq} \), it follows that
\[
\sum_q a_{rq} = c^{-1} y_i^u(t_{rr}) = \lim_n c^{-1} p^{-1} w_n^u(t_{rr}) = \lim_n \sum_q a_{nqr}^u = \lim_n \sum_p a_{npq}^u.
\]

On the other hand the bounded sequence \(\{\sum_p \sum_q a_{npq}^w x_{pq}\}\) converges pointwise to \(y^{1,u} = \sum_p \sum_q a_{npq}^w x_{pq}\). By the note following Lemma 3.2, for each \(\epsilon > 0\) there exist \(p, n_i\) such that \(\sum_{p > p_i} \sum_q a_{npq}^w < \epsilon\) for all \(n \geq n_i\) and also \(\sum_{p > p_i} \sum_q a_{npq}^w < \epsilon\). Hence

\[
\left| \sum_p a_{pr}^w - \lim_n \sum_p a_{npq}^u \right| < 2\epsilon + \left| \sum_{p \leq p_i} a_{pr}^w - \lim_n \sum_{p \leq p_i} a_{npq}^w \right| = 2\epsilon.
\]

Since \(\epsilon\) is an arbitrary positive number,

\[
\sum_p a_{pr}^w = \lim_n \sum_p a_{npq}^u = \sum_q a_{nqr}^w.
\]

This completes the proof of the lemma for \(\zeta = 0\).

For the induction step let \(0 < \zeta < \Omega\), assume the desired result holds for each \(\eta < \zeta\), and let \(y, \zeta', u, \beta, \) and \(7\) be as in the statement of the lemma. Then there exists a bounded sequence \(\{y_n\}\) in \(U_{\eta < \zeta} L_0(L_1(P_\alpha) \cap G)\) which converges pointwise to \(y\). Since \(1 < \zeta' < \gamma \leq \alpha\), there exists \(n_1 \in U \setminus \{0\}\) such that \(\nu_a(v^{-1}) = (\gamma, \zeta')\). For each \(n\) there exists \(y_n \in \zeta_n \in L_{\eta_n}(L_1(P_\alpha) \cap G)\), and it follows that \(\beta > \gamma > \zeta' > \eta_n^\alpha\) for each \(n\), where \(\eta_n^\alpha\) is defined in terms of \(\eta_n\) as \(\zeta'\) was defined in terms of \(\zeta\). By the induction assumption \(y_n^{1,u}\) and \(y_n^{1,\zeta'}\) are continuous and have the form \(y_n^{1,u} = \sum_p \sum_q a_{npq}^w x_{pq}\) and \(y_n^{1,\zeta'} = \sum_p \sum_q a_{npq}^w x_{pq}\), and \(\sum_p a_{npq}^w = \sum_q a_{nqr}^w\) for all \(n\) and \(r\).

As in the proof for \(\zeta = 0\), for each \(\epsilon > 0\) there exist \(n_i\) and \(p_i\) such that \(\sum_{p > p_i} \sum_q a_{npq}^w < \epsilon\) and \(\sum_{p > p_i} \sum_q a_{npq}^w < \epsilon\) for all \(n \geq n_i\). Hence, since \(\sum_p a_{npq}^w = \sum_q a_{nqr}^w\) for all \(n\) and \(r\), it follows that for \(n \geq n_i\), the distance between \(y_n^{1,u}\) and the compact subset

\[
D_{p_i} = \{\sum_{p \leq p_i} \sum_{q \leq q_i} k_{pq} x_{pq}: k_{pq} \geq 0, \sum_{p \leq p_i} \sum_{q \leq q_i} k_{pq} \leq \sup_n \|y_n^{1,u}\|\}
\]

of \(C[0; 3]\) is less than \(2\epsilon\). Since \(\{y_n^{1,w}\}\) converges pointwise to \(y^{1,u}\), the compactness of \(\mathcal{D}_{p_i}\) implies that \(\|y^{1,u} - w\| \leq 2\epsilon\) for some continuous \(w\) depending on \(\epsilon\). Then the completeness of \(C[0; 3]\) implies that \(y^{1,u} \in C[0; 3]\) and therefore, since also \(y^{1,u} \in L_i(Q)\), that \(y^{1,u}\) has the form \(\sum_p \sum_q a_{npq}^w x_{pq}\).

If also \(0 = v \in U\) and \(\nu_a(v^{-1}) = (\gamma, \delta)\) with \(\beta > \gamma > \delta > \zeta'\), then \(y^{1,v}\) and each \(y_n^{1,w}\) are continuous and have form corresponding to \(y^{1,u}\) and \(y_n^{1,u}\) respectively. Further, by the induction assumption, \(\sum_p a_{npq}^w = \sum_q a_{nqr}^w\) for all \(n\) and \(r\). Hence

\[
\sum_q a_{rq}^w = c^{-1} y_i^w(t_{rr}) = \lim_n c^{-1} y_n^{1,u}(t_{rr}) = \lim_n \sum_q y_n^{1,u}_{nqr} = \lim_n \sum_q a_{npq}^w.
\]

Exactly as in the last part of the proof for \(\zeta = 0\) it is seen that
\[ \sum_p a_{pr}^n = \lim_n \sum_p a_{pr}^n. \] This completes the proof of the induction step and hence of the lemma.

**Lemma 4.8.** If \( y \in L_\zeta(L(\varepsilon_\eta) \cap G) \) for some countable \( \zeta \) and if \( u, v \in U \setminus \{0\} \) with \( \nu_\eta(u^{-1}) = (\beta, \gamma) \) and \( \nu_\eta(v^{-1}) = (\beta, \delta) \) for certain ordinals \( \beta, \gamma, \delta \) then in the expression

\[ y^{1,u} = \sum_p \sum_q a_{pq}^n x_{pq} + \sum_p b_p^u x_p + c^u x^0 \]

and the corresponding expression for \( y^{1,v} \) it must be true that \( y^{1,u}(2^{-1}) = y^{1,v}(2^{-1}), c^u = c^v, \) and \( b_p^u + \sum_q a_{pq}^u = b_p^v + \sum_q a_{pq}^v \) for each \( p. \)

**Proof.** By Lemma 4.5, \( y \in G. \) Hence, by Lemma 4.4, \( y^{1,u} = u^{-1} y^{2,u} \) and \( y^{1,v} = v^{-1} y^{2,v}. \)

If \( \zeta = 0, \) then \( y \) is the pointwise limit of a bounded sequence \( \{y_n\} \) of functions of the form \( y_n = \sum_{s \in \sigma_n} b_{ns} x_s, \) where \( \sigma_n \) is a finite subset of \( S^n \) and each \( b_{ns} \) is nonnegative. For each \( p \) and \( n, \)

\[ u^{-1} y_n^{2,u}(t_{pp}) = c \sum b_{ns} : s_{\beta} = p \]

Since \( \{y_n^{2,u}\} \) converges pointwise to \( y^{2,u}, \)

\[ y^{1,u}(t_{pp}) = u^{-1} y^{2,u}(t_{pp}) = v^{-1} y^{2,v}(t_{pp}) = y^{1,v}(t_{pp}) \]

for each \( p, \) and hence it follows immediately that

\[ b_p^u + \sum_q a_{pq}^u = c^{-1} y^{1,u}(t_{pp}) = c^{-1} y^{1,v}(t_{pp}) \]

\[ = b_p^v + \sum_q a_{pq}^v \]

for each \( p. \) Since \( y^{1,u} \) and \( y^{1,v} \) are Baire functions of the first class, \( c^u = 0 = c^v. \) Hence

\[ y^{1,u}(2^{-1}) = \sum_p (b_p^u + \sum_q a_{pq}^u) = y^{1,v}(2^{-1}). \]

For the induction step let \( \zeta > 0 \) and assume the statement of the lemma holds for each \( \gamma < \zeta. \) By hypothesis there exists a bounded sequence \( \{y_\eta\} \) in \( \bigcup_{\gamma < \zeta} L_\eta(L(\varepsilon_\eta) \cap G) \) which converges pointwise to \( y. \) Under the usual notation the relations

\[ b_{np}^u + \sum_q a_{npq}^u = b_{np}^v + \sum_q a_{npq}^v, \]

\( c_n^u = c_n^v, \) and \( y_n^{1,u}(2^{-1}) = y_n^{1,v}(2^{-1}) \) must hold for all \( n \) and \( p. \) It is seen immediately that \( y^{1,u}(2^{-1}) = y^{1,v}(2^{-1}) \) and \( y^{1,u}(t_{pp}) = y^{1,v}(t_{pp}) \) for all \( p, \) from which the remaining desired relations for \( y^{1,u} \) and \( y^{1,v} \) follow. The proof is thus complete.

**Theorem 4.2.** Let \( \zeta \) be a countable ordinal, and let \( \zeta' \) be defined as in Lemma 4.7. If \( y \in L_\zeta(L(\varepsilon_\eta) \cap G) \) and \( 0 \neq u \in U \) with \( \nu_\eta(u^{-1}) = (\beta, \gamma) \)
and $\beta > \zeta'$, then $y^{i,n} \in Q + Q_i$.

Proof. If $\zeta = 0$, then $y \in L_1(P_\alpha)$ and hence trivially $y^{i,n} \in L_1(Q)$, which is equal to $Q + Q_i$ by Lemma 3.2.

If $\zeta > 0$ and the desired result is true for each $\eta < \zeta$, then $2 \leq \zeta' < \beta \leq \alpha$ and hence there exists $v \in U\setminus\{0\}$ such that $\nu_\alpha(v^{-1}) = (\beta, \zeta')$. There exists a bounded sequence $\{y_n\}$ in $\bigcup_{\eta < \zeta} L_\beta(L_1(P_\alpha) \cap G)$ which converges pointwise to $y$. Since $\beta > \zeta' > \eta'$ for each $\eta < \zeta$ it follows from Lemma 4.7 that each $y^{i,n}_\alpha$ is continuous and hence belongs to $Q$. Hence $y^{i,n} \in L_1(Q) = Q + Q_i$. Thus in the usual notation for $y^{i,n}$ and $y^{i,n}$ it follows that $c^* = 0$, but then also $c^* = 0$ by Lemma 4.8, hence $y^{i,n} \in Q + Q_i$, and the proof is complete.

The following theorem justifies the claim made at the beginning of the present section.

**Theorem 4.3.** The element $z_\alpha \in L_\alpha(P_\alpha)$ has the property that $||z_\alpha|| = 1$ but that if $\{w_n\}$ is a bounded sequence in $\bigcup_{\beta < \alpha} L_\beta(P_\alpha)$ converging pointwise to $z_\alpha$, then $\lim_n ||w_n|| \geq c$.

Proof. By Lemma 4.1 and the remarks preceding it we know that $z_\alpha \in L_\alpha(P_\alpha)$ and $||z_\alpha|| = 1$. If $\{w_n\}$ is a bounded sequence in $\bigcup_{\beta < \alpha} L_\beta(P_\alpha)$ converging pointwise to $z_\alpha$, then by Theorem 4.1 there exists a sequence $\{y_n\}$ in $G \cap \bigcup_{\beta < \alpha} L_\beta(P_\alpha)$ such that $||y_n - w_n|| \to 0$. Clearly $\lim_n ||w_n|| = \lim_n ||y_n||$. Now by Lemma 4.5,

$$\{y_n\} \subset \begin{cases} \{L_{\alpha-n}(L_1(P_\alpha) \cap G) & \text{if } 2 \leq \alpha < \omega \\ \bigcup_{\beta < \alpha} L_\beta(L_1(P_\alpha) \cap G) & \text{if } \omega \leq \alpha < \Omega. \end{cases}$$

Defining $\zeta'$ as in Lemma 4.7, one sees easily that each $y_{\alpha,n} \in L_{\zeta_n}(L_1(P_\alpha) \cap G)$ for some $\zeta_n$ such that $\alpha > \zeta_n$. Now there exists $v_1 \in U\setminus\{0\}$ such that $\nu_\alpha(v_1^{-1}) = (\alpha, \gamma)$ for some $\gamma < \alpha$; for example, take $\gamma = 1$ if $\alpha = 2$ and $\gamma = 2$ if $\alpha > 2$. Then by Theorem 4.2, $y_{\alpha,n+1}^{i,n} \in Q + Q_i = L_1(Q)$ for each $n$. Now $z_{\alpha,n}^{i,n} = x^0$ by definition, and hence $\lim_n ||y_{\alpha,n}^{i,n}|| \geq c$ by Theorem 1 of [7]. It follows that

$$\lim_n ||w_n|| = \lim_n ||y_n|| \geq \lim_n ||y_{\alpha,n}^{i,n}|| \geq c.$$

**Corollary 4.1.** Let $T$ be the mapping of Theorem 2.1 for the space $X_\alpha$, and let $G_\alpha = Tz_\alpha$. Then $G_\alpha \in K_\alpha(J_{X_\alpha}P_\alpha)$ and $||G_\alpha|| = 1$, but if $\{F_n\}$ is a sequence in $\bigcup_{\beta < \alpha} K_\beta(J_{X_\alpha}P_\alpha)$ such that $F_n \rightharpoonup^* G_\alpha$, then $\lim_n ||F_n|| \geq c$.

Proof. It is immediate from Theorem 2.1 that $G_\alpha \in K_\alpha(J_{X_\alpha}P_\alpha)$ and $||G_\alpha|| = 1$. If $\{F_n\} \subset \bigcup_{\beta < \alpha} K_\beta(J_{X_\alpha}P_\alpha)$ and $F_n \rightharpoonup^* G_\alpha$, then by Theorem 2.1 the sequence $\{T^{-1}F_n\}$ is in $\bigcup_{\beta < \alpha} L_\beta(P_\alpha)$ and $||T^{-1}F_n|| = ||F_n||$ for each
n. Now \( \sup_n \| T^{-1} F_n \| = \sup_n \| F_n \| < \infty \) since \( \{F_n\} \) is \( w^* \)-convergent. For each \( t \in S_\alpha \) let \( f_t \in X^*_\alpha \) be defined as in the proof of Theorem 2.1. Then

\[
(T^{-1} F_n)(t) = F_n(f_t) \longrightarrow G_\alpha(f_t) = z_\alpha(t)
\]

for each \( t \), and hence

\[
\lim_n \| F_n \| = \lim_n \| T^{-1} F_n \| \geq c.
\]

5. Our main theorems will now be proved through consideration of product spaces, as defined in [2, p. 31], of spaces of the type \( X_\alpha \). Since \( X_\alpha, P_\alpha, \) and \( G_\alpha \) depend on the given number \( c \geq 1 \) as well as on \( \alpha \), the objects mentioned will henceforth be indicated with double subscripts as \( X_{c,\alpha}, P_{c,\alpha}, \) and \( G_{c,\alpha} \) respectively. Recall that if \( I \) is a set and \( X_s \) is a Banach space for each \( s \in I \), then the product spaces \( \Pi_{s \in I} X_s^* \) and \( \Pi_{s \in I} X_s^{**} \) are respectively the dual and bidual of the Banach space \( \Pi_{s \in I} X_s \) under the natural identifications.

**Theorem 5.1.** For each countable ordinal \( \alpha \geq 2 \) let \( Y_\alpha \) be the Banach space \( \Pi_{\omega \in \omega} X_{n_{\omega},\alpha} \) and let

\[
Q_\alpha = \bigcap_{n_{\omega} \in \omega} \{ y \in Y_\alpha : y(n) \in P_{n_{\omega},\alpha} \}.
\]

Then \( Y_\alpha \) is separable, and \( Q_\alpha \) is a norm-closed cone in \( Y_\alpha \) such that \( K_\alpha(J_{Y_\alpha} Q_\alpha) \) is not norm-closed in \( Y^{**}_\alpha \).

**Proof.** It is evident that \( Y_\alpha \) is separable and \( Q_\alpha \) is a closed cone in \( Y_\alpha \). An easy transfinite induction argument shows that for each \( n \) the functional \( F_n \) belongs to \( K_\alpha(J_{Y_\alpha} Q_\alpha) \), where \( F_n(n) = G_{n_{\omega},\alpha} \) and \( F_n(i) = 0 \) for all \( i \neq n \). Hence \( \sum_{n_{\omega} = 1}^{m,n_{\omega}} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha) \) for each positive integer \( m \), and therefore \( \sum_{n_{\omega} = 1}^{m,n_{\omega}} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha) \). If \( \{H_k\} \) were a sequence in \( \bigcup_{\beta < \alpha} K_\beta(J_{Y_\alpha} Q_\alpha) \) such that \( H_k \xrightarrow{w^*} \sum_n n^{-1} F_n \), then for each \( i \in \omega \) it would follow that

\[
[H_k(i)]_k \subset \bigcup_{\beta < \alpha} K_\beta(J_{X_{n_{\omega},\alpha}} P_{n_{\omega},\alpha})
\]

and

\[
H_k(i) \xrightarrow{w^*} \sum_n n^{-1} F_n(i) = i^{-1} G_{i^{\omega},\alpha}.
\]

It would then result by Corollary 4.1 that

\[
\lim_k \| H_k \| \geq \lim_k \| H_k(i) \| \geq i,
\]

but then since \( i \) is arbitrary the sequence \( \{H_k\} \) would be unbounded in norm, contradicting the fact that a \( w^* \)-convergent sequence in \( Y^{**}_\alpha \) must be bounded [3, p. 60]. Hence \( \sum_n n^{-1} F_n \in K_\alpha(J_{Y_\alpha} Q_\alpha) \), and the proof
is complete.

**THEOREM 5.2.** For each countable ordinal $\alpha \geq 2$ there exists a separable Banach space $W_\alpha$ containing a norm-closed cone $R_\alpha$ such that if $2 \leq \beta \leq \alpha$, then $K_\beta(J_{W_\alpha}R_\alpha)$ is not norm-closed in $W_\alpha^{**}$.

**Proof.** Let $A_\alpha = \{\beta: 2 \leq \beta \leq \alpha\}$ and for each $\beta \in A_\alpha$ let $Y_\beta$ and $Q_\beta$ be as defined in Theorem 5.1. Let $W_\alpha = \Pi_{\beta \in A_\alpha} Y_\beta$ and $R_\alpha = \bigcap_{\beta \in A_\alpha} \{w \in W_\alpha: w(\beta) \in Q_\beta\}$. Then the Banach space $W_\alpha$ is separable since $A_\alpha$ is countable, and $R_\alpha$ is clearly a norm-closed cone in $W_\alpha$. For each $\beta \in A_\alpha$ there exists by Theorem 5.1 a sequence $\{\psi_{\beta,n}\}$ in $K_\beta(J_{Y_\beta}Q_\beta)$ which converges in norm to an element $\psi_{\beta,0} \in Y_\beta^{**}$ not in $K_\beta(J_{Y_\beta}Q_\beta)$. If $\psi_{\beta,n}$ is defined for each integer $n \geq 0$ by $\psi_{\beta,n}(\gamma) = 0$ for $\gamma \neq \beta$ and $\psi_{\beta,n}(\beta) = \phi_{\beta,n}$, it is easily shown that $\{\psi_{\beta,n}\}_{n \geq 0} \subset K_\beta(J_{W_\alpha}R_\alpha)$ and $\{\psi_{\beta,n}\}$ converges in norm to $\psi_{\beta,0}$, but that $\psi_{\beta,0} \notin K_\beta(J_{W_\alpha}R_\alpha)$. Hence for each $\beta \in A_\alpha$, $K_\beta(J_{W_\alpha}R_\alpha)$ fails to be norm-closed in $W_\alpha^{**}$.

**THEOREM 5.3.** There exists a Banach space $Z$ containing a norm-closed cone $P$ such that if $\beta$ is a countable ordinal $\geq 2$, then $K_\beta(J_ZP)$ fails to be norm-closed in $Z^{**}$.

**Proof.** The proof is almost identical with that of Theorem 5.2. Let $A = \{\beta: 2 \leq \beta < \Omega\}$, $Z = \Pi_{\beta \in A} Y_\beta$, and $P = \bigcap_{\beta \in A} \{z \in Z: z(\beta) \in Q_\beta\}$. Since $A$ is uncountable, the Banach space $Z$ is nonseparable. It is clear that $P$ is a closed cone in $Z$. The proof that $K_\beta(J_ZP)$ fails to be norm-closed in $Z^{**}$ for each $\beta \in A$ is identical with the corresponding part of the proof of Theorem 5.2, in which it was shown that $K_\beta(J_{W_\alpha}R_\alpha)$ fails to be norm-closed in $W_\alpha^{**}$ for each $\beta \in A_\alpha$.

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