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OSCILLATORY PROPERTIES OF SOLUTIONS OF EVEN ORDER DIFFERENTIAL EQUATIONS

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Consider the following n th order nonlinear differential equation

$$(1) \quad x^{(n)} + f(t, x, x', \dots, x^{(n-1)}) = 0.$$

All functions considered will be assumed continuous and all the solutions of (1), continuously extendable through the entire nonnegative real axis. A nontrivial solution of (1) is called oscillatory if it has zeros for arbitrarily large t and equation (1) is called oscillatory if all of its solutions are oscillatory. A nontrivial solution of (1) is called nonoscillatory if it has only a finite number of zeros on $[t_0, \infty)$ and equation (1) is called nonoscillatory if all of its solutions are nonoscillatory. In this paper, theorems on oscillation and nonoscillation are presented.

Recently, J. S. W. Wong [8] posed a definition called strongly continuous with which he proved some theorems to (1) for $n = 2$. The proof is based on that of his earlier results [7]. We introduce more general definition. A function $f(t, x_1, \dots, x_n)$ is called generalized strongly continuous from the left at x_{1c} if $f(t, x_1, x_2, \dots, x_n)$ is jointly continuous in t and $x_i (i = 1, 2, \dots, n)$ and for $\varepsilon > 0$ there exist $\delta > 0$, $T \geq 0$, and $x_i \in [x_{1c} - \delta, x_{1c}]$ such that for all $x_i \in [x_{1c} - \delta, x_{1c}]$, and for all x_i satisfying $|x_i - k_i| \leq \delta$ (k_i is any constant) for $i = 2, \dots, n$,

$$(1 - \varepsilon)f(t, x_i, k_2, \dots, k_n) \leq f(t, x_1, \dots, x_n) \leq (1 + \varepsilon)f(t, x_{1c}, k_2, \dots, k_n),$$

for all $t \geq T$.

Generalized strong continuity from the right is defined analogously. A function $f(t, x_1, \dots, x_n)$ is said to be generalized strongly continuous if it is generalized strongly continuous both from the left and from the right. If $f = f(t, x_1)$, then our definition is the same as Wong's one. For example, $f(t, x_1, \dots, x_n) = a(t)f(x_1, \dots, x_n)$ is generalized strongly continuous.

2. Oscillation and nonoscillation theorems.

THEOREM 1. *Assume that n is even and that*

- (α) $f(t, c, k_2, \dots, k_n)$ is bounded for any constant c and $k_i (i = 2, \dots, n)$ and $x_1 f(t, x_1, \dots, x_n) > 0$ ($x_1 \neq 0$).

Let $f(t, x_1, \dots, x_n)$ be generalized strongly continuous from the left

for $x_1 > 0$ and generalized strongly continuous from the right for $x_1 < 0$. Then, a necessary and sufficient condition for equation (1) to have a bounded nonoscillatory solution is

$$(2) \quad \left| \int_0^\infty t^{n-1} f(t, c, k_2, \dots, k_n) dt \right| < \infty$$

($c \neq 0$) and $k_i (i = 2, \dots, n)$ are some constants).

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1), which must eventually be of one sign. Then we may assume that $x(t) > 0$ for $t \geq T > 0$. Since $x(t) > 0$, then $f(t, x, x', \dots, x^{(n-1)}) > 0$ for $t \geq T$, we see from (1) and the assumption that $x(t)$ is bounded:

$$\begin{aligned} x^{(n)} &\leq 0, \quad x^{(n-1)} \geq 0, \quad x^{(n-2)} \leq 0, \quad \dots, \quad x' \geq 0, \\ \lim_{t \rightarrow \infty} x^{(i)}(t) &= 0, \quad i = 1, 2, \dots, n-1. \end{aligned}$$

From this and the fact $x(t)$ is positive and bounded implies that $x(t)$ tends to a finite limit $L > 0$. Integrating (1), we obtain for sufficiently large t

$$\begin{aligned} x^{(n-i)}(t) &= (-1)^{n-i-1} \int_t^\infty \frac{(s-t)^{i-1}}{(i-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds \\ &\quad (\text{for } i = 1, 2, \dots, n-1). \end{aligned}$$

In particular,

$$x(t) = L - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds.$$

By the generalized strong continuity of $f(t, x_1, x_2, \dots, x_n)$ implies that there exist $0 < \delta < L$, and $L_\delta \in [L - \delta, L]$ such that for all x_i satisfying $|x_i - k_i| \leq \delta (i = 2, \dots, n)$ and for all $x_1 \in [L - \delta, L]$,

$$f(t, x_1, x_2, \dots, x_n) \geq \frac{1}{2} f(t, L_\delta, k_2, \dots, k_n).$$

Choose T sufficiently large, we can restrict the solution $x(t)$ to satisfy $L - \delta \leq x(t) \leq L$ for all $t \geq T$, and that

$$|x^{(i)}(t) - 0| \leq \delta (i = 1, 2, \dots, n-1),$$

for all $t \geq T$.

Thus, we obtain for $t \geq T$,

$$0 < \frac{1}{2} f(t, L_\delta, 0, \dots, 0) < f(t, x(t), x'(t), \dots, x^{(n-1)}(t)).$$

Accordingly, we obtain

$$\begin{aligned}
 (3) \quad x(t) &= L - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds \\
 &\leq L - \frac{1}{2(n-1)!} \int_t^\infty (s-t)^{n-1} f(s, L_\delta, 0, \dots, 0) ds.
 \end{aligned}$$

Since $x(t)$ is bounded, we obtain

$$(4) \quad \int_t^\infty (s-t)^{n-1} f(s, L_\delta, 0, \dots, 0) ds < \infty,$$

which implies

$$\int_t^\infty s^{n-1} f(s, L_\delta, 0, \dots, 0) ds < \infty.$$

Conversely, we show that if (2) holds for some constant $c > 0$ and $k_i (i = 2, \dots, n)$, then there exists a nonnegative continuous bounded solution to the following integral equation:

$$\begin{aligned}
 (5) \quad x_{n-1}(t) &= k_n + \int_t^\infty f(s, x_0(s), x_1(s), \dots, x_{n-1}(s)) ds \\
 x_{n-2}(t) &= k_{n-1} - \int_t^\infty (s-t) f(s, x_0(s), x_1(s), \dots, x_{n-1}(s)) ds \\
 x_{n-3}(t) &= k_{n-2} + \int_t^\infty \frac{(s-t)^2}{2!} f(s, x_0(s), x_1(s), \dots, x_{n-1}(s)) ds \\
 &\quad \vdots \\
 x_0(t) &= c - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_0(s), x_1(s), \dots, x_{n-1}(s)) ds.
 \end{aligned}$$

We define $E = \{0, 1, 2, \dots, n-1\}$. Let a positive number T be chosen such that

$$(6) \quad \max_{i \in E} \frac{1}{(n-1-i)!} \int_t^\infty (s-t)^{n-1-i} f(s, c, k_2, \dots, k_n) ds \leq \frac{c}{M}$$

where $M (> 2)$ is some constant.

We define, for N a positive interger $N \geq T$: for $t \geq N$,

$$\begin{aligned}
 (7) \quad x_{n-1,N}(t) &= k_n \\
 x_{n-2,N}(t) &= k_{n-1} \\
 &\quad \vdots \\
 x_{1,N}(t) &= k_2 \\
 x_{0,N}(t) &= c
 \end{aligned}$$

and for $T \leq t \leq N$,

$$\begin{aligned}
 x_{n-1,N}(t) &= k_n + \int_{t+(1/N)}^{\infty} f(s, x_{0,N}(s), x_{1,N}(s), \dots, x_{n-1,N}(s)) ds \\
 x_{n-2,N}(t) &= k_{n-1} - \int_{t+(1/N)}^{\infty} \left(s - t - \frac{1}{N} \right) f(s, x_{0,N}(s), \dots, x_{n-1,N}(s)) ds \\
 &\quad \vdots \\
 x_{1,N}(t) &= k_2 + \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N)^{n-2}}{(n-2)!} f(s, x_{0,N}(s), \dots, x_{n-1,N}(s)) ds \\
 x_{0,N}(t) &= c - \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N)^{n-1}}{(n-1)!} f(s, x_{0,N}(s), \dots, x_{n-1,N}(s)) ds .
 \end{aligned}
 \tag{8}$$

This formula defines $x_{i,N}(t)$ for $i = 0, 1, \dots, n-1$, successively on the intervals $[N - (K/N), N - (K-1)/N]$ for $k = 1, 2, \dots, N(N-T)$: hence $x_{i,N}(t)$, $i = 0, 1, \dots, n-1$, are defined on $[T, \infty)$.

For $N - (1/N) \leq t < \infty$, we have by (6)

$$\begin{aligned}
 |x_{i,N}(t) - k_{i+1}| &\leq \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N))^{n-1-i}}{(n-1-i)!} f(s, x_{0,N}(s), \dots, x_{n-1,N}(s)) ds \\
 &\leq \int_{t+(1/N)}^{\infty} \frac{(s - t - (1/N))^{n-1-i}}{(n-1-i)!} f(s, c, k_2, \dots, k_n) ds \\
 &\leq \frac{c}{M} (i = 1, 2, \dots, n-1)
 \end{aligned}$$

and also

$$0 < c - \frac{c}{M} \leq x_{0,N}(t) \leq c .$$

By easy induction, we have

$$\begin{aligned}
 0 &\leq |x_{i,N}(t) - k_{i+1}| \leq \frac{c}{M} i = 1, 2, \dots, n-1 , \\
 0 &< c - \frac{c}{M} \leq x_{0,N}(t) \leq c
 \end{aligned}
 \tag{9}$$

on the entire interval $[T, \infty)$. Consequently, for $t \geq T$, since f is generalized strongly continuous and $f(t, c, k_2, \dots, k_n)$ is bounded, we have

$$\begin{aligned}
 |x_{n-1,N}(t)| &= f\left(t + \frac{1}{N}, x_{0,N}\left(t + \frac{1}{N}\right), \dots, x_{n-1,N}\left(t + \frac{1}{N}\right)\right) \\
 &\leq \frac{3}{2} f\left(t + \frac{1}{N}, c, k_2, \dots, k_n\right) \\
 &\leq \frac{3}{2} K \quad (K \text{ is constant}) ,
 \end{aligned}
 \tag{10}$$

$$|x'_{i,N}(t)| = |x_{i+1,N}(t) - k_{i+2}| \leq \frac{c}{M}, \quad (i = 0, 1, \dots, n - 2).$$

Since the family $\{x_{i,N}(t)\}(i = 0, 1, 2, \dots, n - 1)$, is uniformly bounded and equicontinuous on $[T, B]$ (B is arbitrary), we extract from $\{x_{i,N}(t)\}$ ($i = 0, 1, \dots, n - 1$), a uniformly convergent subsequence $\{x_{i,k}(t)\}$,

$$\lim_{k \rightarrow \infty} x_{i,k}(t) = \bar{x}_i, \quad (\text{on } [T, B], \text{ for } i = 0, 1 \dots, n - 1).$$

For any large number $B > T$, we may write (8) in the form

$$(11) \quad x_{i,k}(t) = d + (-1)^{i+1} \left\{ \int_{t+(1/k)}^B \frac{(s-t-(1/k))^{n-1-i}}{(n-1-i)!} f(s, x_{0,k}(s), \dots, x_{n-1,k}(s)) ds + \phi_k(B) \right\},$$

where,

$$\phi_k(B) = \int_B^\infty \frac{(s-t-(1/k))^{n-1-i}}{(n-1-i)!} f(s, x_{0,k}(s), \dots, x_{n-1,k}(s)) ds, \\ (d = c \text{ for } i = 0, d = k_{i+1} \text{ for } i = 1, 2 \dots, n - 1).$$

For fixed B we let k tend to infinity in (11) and obtain

$$\liminf_{k \rightarrow \infty} \phi_k(B) \leq (-1)^i (-\bar{x}_i + d + (-1)^{i+1} \int_t^B \frac{(s-t)^{n-1-i}}{(n-1-i)!} f(s, \bar{x}_0, \dots, \bar{x}_{n-1}) ds) \leq \limsup_{k \rightarrow \infty} \phi_k(B).$$

From (6) and (9) and f is generalized strongly continuous, we obtain

$$(12) \quad 0 \leq \phi_k(B) = \int_B^\infty \frac{(s-t-(1/k))^{n-1-i}}{(n-1-i)!} f(s, x_{0,k}(s), \dots, x_{n-1,k}(s)) ds \\ \leq \frac{3}{2} \int_B^\infty \frac{(s-t)^{n-1-i}}{(n-1-i)!} f(s, c, k_2, \dots, k_n) ds < \infty.$$

By (2), the integral in (12) tend to zero as $B \rightarrow \infty$. Thus, we conclude that $\bar{x}_i(t)$ ($i = 0, 1, \dots, n - 1$), is a solution of (5) and also $\bar{x}_0(t)$ is a bounded nonoscillatory solution of (1).

THEOREM 2. *Assume that n is even and that*

$$(\beta) \quad x_1 f(t, x_1, x_2, \dots, x_{n-1}, \gamma) > 0 \quad (x_1 \neq 0), \text{ where } \gamma \text{ is constant.}$$

Let $f(t, x_1, x_2, \dots, x_{n-1}, \gamma)$ be generalized strongly continuous from the left for $x_1 > 0$, and generalized strongly continuous from the right for $x_1 < 0$. Then, a necessary and sufficient condition for equation

$$(1') \quad x^{(n)} + f(t, x, x', \dots, x^{(n-2)}, \gamma) = 0 \quad (\gamma \text{ is constant}),$$

to have a bounded nonoscillatory solution is

$$(2') \quad \left| \int_t^\infty t^{n-1} f(t, c, k_2, \dots, k_{n-1}, \gamma) dt \right| < \infty$$

($c \neq 0$) and k_i ($i = 2, \dots, n-1$) are some constant).

Proof. The necessity follows from the necessary part of the proof of Theorem 1.

Conversely, we show that if (2') holds for some constant $c > 0$ and k_i ($i = 2, \dots, n-1$), then there exists a nonnegative continuous bounded solution to the following integral equation:

$$(13) \quad \begin{aligned} x_{n-2}(t) &= k_{n-1} - \int_t^\infty (s-t) f(s, x_0(s), \dots, x_{n-2}(s), \gamma) ds \\ x_{n-3}(t) &= k_{n-2} + \int_t^\infty \frac{(s-t)^2}{2!} f(s, x_0(s), \dots, x_{n-2}(s), \gamma) ds \\ &\vdots \\ x_0(t) &= c - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, x_0(s), \dots, x_{n-1}(s), \gamma) ds. \end{aligned}$$

Let a positive number T be chosen such that

$$(14) \quad \max_{i \in E} \int_T^\infty \frac{(s-t)^{n-1-i}}{(n-1-i)!} f(s, c, k_2, \dots, k_{n-1}, \gamma) ds \leq \frac{c}{M}$$

($M > 2$) is some constant).

We define, for N a positive integer $N \geq T$: for $t \geq N$,

$$\begin{aligned} x_{n-2,N}(t) &= k_{n-1} \\ &\vdots \\ x_{1,N}(t) &= k_2 \\ x_{0,N}(t) &= c \end{aligned}$$

and for $T \leq t \leq N$

$$\begin{aligned} x_{n-2,N}(t) &= k_{n-1} - \int_{t+(1/N)}^\infty \left(s-t - \frac{1}{N} \right) f(s, x_{0,N}(s), \dots, x_{n-2,N}(s), \gamma) ds \\ &\vdots \\ x_{1,N}(t) &= k_2 + \int_{t+(1/N)}^\infty \frac{(s-t - (1/N))^{n-2}}{(n-2)!} f(s, x_{0,N}(s), \dots, x_{n-2,N}(s), \gamma) ds \\ x_{0,N}(t) &= c - \int_{t+(1/N)}^\infty \frac{(s-t - (1/N))^{n-1}}{(n-1)!} f(s, x_{0,N}(s), \dots, x_{n-2,N}(s), \gamma) ds. \end{aligned}$$

As same as the proof of Theorem 1, $x_{i,N}(t)$ ($i = 0, 1, \dots, n-2$), are defined on $[T, \infty)$ and that for $i = 0, 1, \dots, n-3$,

$$|x'_{i,N}(t)| = |x_{i+1,N}(t) - k_{i+2}| \leq \frac{c}{M}$$

and for $i = n - 2$,

$$|x'_{i,N}(t)| = \int_t^\infty f(s, x_{0,N}(s), \dots, x_{n-2,N}(s), \gamma) ds \leq \frac{c}{M}.$$

Hence the family $\{x_{i,N}(t)\}$ ($i = 0, 1, \dots, n - 2$) is uniformly bounded and equicontinuous on $[T, B]$ (B is arbitrary). Using an argument similar to that given in the proof of Theorem 1, we have a bounded nonoscillatory solution of (1').

REMARK. For $n = 2$, Theorem 2 coincides with Theorem 3 in [8].

COROLLARY 1. Suppose that n is even and that

$$x_1 f(t, x_1, \dots, x_n) > 0 \quad (x_1 \neq 0).$$

Let $f(t, x_1, \dots, x_n)$ be generalized strongly continuous from the left for $x_1 > 0$ and generalized strongly continuous from the right for $x_1 < 0$ and that

$$\left| \int_{t_0}^\infty t^{n-1} f(t, c, k_2, \dots, k_n) dt \right| = +\infty \quad (c \neq 0 \text{ and } k_i (i = 2, \dots, n)$$

are any constants). Then, every bounded solution of (1) is oscillatory.

Proof. The proof of Corollary 1 follows immediately from the necessary part of Theorem 1.

COROLLARY 2 [1, Theorem 1]. Consider

$$(15) \quad x^{(2n)} + p(t)g(x, x', x^{(2)}, \dots, x^{(2n-1)}) = 0$$

under the following assumption:

$$(i) \quad p: I \rightarrow R_+ = (0, +\infty), \quad I = [t_0, +\infty), \quad t_0 \geq 0, \quad p \in C[t_0, +\infty),$$

and

$$(A) \quad \int_{t_0}^{+\infty} t^{2n-1} p(t) dt = +\infty$$

is satisfied;

$$(ii) \quad g: R^{2n} \rightarrow R = (-\infty, \infty), \quad x_1 g(x_1, x_2, \dots, x_{2n}) > 0 \quad \text{for } x_1 \neq 0,$$

and continuous on R^{2n} ;

then, under the above conditions, every bounded solution of (15) is oscillatory.

Proof. The function $p(t)g(x_1, x_2, \dots, x_{2n})$ is generalized strongly continuous. Hence, Corollary 2 is included in Corollary 1.

COROLLARY 3 [4, Theorem 1]. *Under the assumption (α') $p(t)$ is bounded and eventually nonnegative:*

$$x_1 g(x_1, x_2, \dots, x_{2n}) > 0 \quad (x_1 \neq 0), \quad \text{for } (x_1, x_2, \dots, x_{2n}) \in R^{2n},$$

a necessary and sufficient condition that (15) have a bounded nonoscillatory solution is

$$(16) \quad \int^{\infty} t^{2n-1} p(t) dt < \infty.$$

Proof. The function $p(t)g(x_1, x_2, \dots, x_n)$ is generalized strongly continuous, hence Corollary 3 is included in Theorem 1.

COROLLARY 4 [4, Theorem 2]. *Under the assumption (β') $p(t)$ is eventually nonegative and $x_1 g(x_1, x_2, \dots, x_{2n-1}, c) > 0$ $x_1 \neq 0$, for $(x_1, x_2, \dots, x_{2n-1}, c) \in R^{2n}$ where c is constant, a necessary and sufficient condition that the differential equation*

$$(17) \quad x^{(2n)} + p(t)g(x, x', \dots, x^{(2n-2)}, c) = 0, \quad n \geq 1,$$

c is constant, have a bounded nonoscillatory solution is (16).

Proof. The proof follows immediately from Theorem 2.

THEOREM 3. *Consider*

$$(18) \quad x^{(2n)} + p(t)g(x) = 0,$$

under the following assumptions:

(i) $p: I \rightarrow R_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geq 0$, $p \in C[t_0, +\infty)$,

(ii) $g: R \rightarrow R = (-\infty, +\infty)$, $g \in C'(-\infty, +\infty)$, $xg(x) > 0$ for $x \neq 0$, and continuous on R ;

$g'(x) \geq 0$ for $|x| \in [K, +\infty)$ (K is some positive constant), and

$$\int_{\varepsilon}^{+\infty} \frac{1}{g(u)} du < +\infty, \quad \int_{-\varepsilon}^{-\infty} \frac{1}{g(u)} du < +\infty$$

for every $\varepsilon > 0$;

then a necessary and sufficient condition that every solution of (18) is oscillatory is

$$(19) \quad \int_{t_0}^{+\infty} t^{2n-1} p(t) dt = +\infty.$$

Proof. From Theorem 2, if $\int_{t_0}^{+\infty} t^{2n-1}p(t)dt < +\infty$, then (18) is nonoscillatory. Hence we conclude that if every solution of (18) is oscillatory, then $\int_{t_0}^{+\infty} t^{2n-1}p(t)dt = +\infty$.

Conversely $\int_{t_0}^{+\infty} t^{2n-1}p(t)dt = +\infty$, then every solution of (18) is oscillatory in the case $n > 1$ [1] and $n = 1$ [6; 7].

THEOREM 4. *Assume that n is even and that (β) . Let $f(t, x_1, \dots, x_{n-1}, \gamma)$ be generalized strongly continuous from the left for $x_1 > 0$, and generalized strongly continuous from the right for $x_1 < 0$. Then, a necessary and sufficient condition for every bounded solution of (1') to be oscillatory is*

$$(20) \quad \left| \int_{t_0}^{\infty} t^{n-1}f(t, c, k_2, \dots, k_{n-1}, \gamma)dt \right| = +\infty$$

($c(\neq 0)$ and $k_i(i = 2, \dots, n - 1)$ are any constant).

Proof. Assume that (20) does not hold, then (2') holds for some $c \neq 0$ and $k_i(i = 2, \dots, n - 1)$. Hence by Theorem 2, equation (1') has a bounded nonoscillatory solution, so clearly condition (20) is necessary. Conversely, let $x(t) > 0$ be a nonoscillatory solution of (1'). In view of the arguments of Theorem 1, $x(t)$ must be nondecreasing and the limit is finite. Hence the argument in the proof of Theorem 1 is applicable, which shows that leads a contradiction.

THEOREM A [5]. *If in addition to the hypothesis of Corollary 3 (or Corollary 4), for some $r > 1$ and n is even,*

$$(21) \quad \liminf_{|x_1| \rightarrow +\infty} \frac{|g(x_1, x_2, \dots, x_n)|}{|x_1|^r} > 0$$

then a necessary and sufficient condition that all solution of (15) (or (17)) be oscillatory is

$$(22) \quad \int_{t_0}^{\infty} t^{n-1}p(t)dt = +\infty .$$

THEOREM B [2]. *Consider*

$$(23) \quad x^{(n)} + p(t)g(x, x', \dots, x^{(n-1)}) = 0$$

with n even, and moreover,

- (i) $p: I \rightarrow R_+ = (0, +\infty)$, $I = [t_0, +\infty)$, $t_0 \geq 0$.
- (ii) $g: R^n \rightarrow R = (-\infty, +\infty)$, and such that Condition (G):
 $x_1g(x_1, x_2, \dots, x_n) > 0$ for every $(x_1, x_2, \dots, x_n) \in R^n$ with $x_1 \neq 0$,

and for every $(x_1, x_2, \dots, x_n) \in R^n$, and every $\lambda \geq K$ (=fixed positive constant), $g(-x_1, -x_2, \dots, -x_n) = -g(x_1, x_2, \dots, x_n)$, and $g(\lambda, \lambda x_2, \dots, \lambda x_n) = \lambda^r g(1, x_2, \dots, x_n)$, where $\gamma = q/r$, q r odd positive integers relatively prime;

then under any one of the following conditions, all solutions of (23) are oscillatory:

- (a) $0 < \gamma < 1$, $\int_{t_0}^{\infty} t^{\gamma(n-1)} p(t) dt = +\infty$;
- (b) $\gamma = 1$, $\int_{t_0}^{\infty} t^{n-1-\varepsilon} p(t) dt = +\infty$,
for some ε with $0 < \varepsilon < 1$;
- (c) $\gamma > 1$, $\int_{t_0}^{\infty} t^{n-1} p(t) dt = +\infty$.

Kartsatos [2, Remark 3] posed a problem that under what additional assumptions on the function g , the conditions of Theorem B are also necessary for the theorem to hold. In case $0 < \gamma < 1$, the condition (a) is also necessary for theorem B to hold. When $\gamma = 1$ (this is the linear case), it is well known that condition (b) can not be necessary. Consider the Euler equation. Thus, we answer the problem for $\gamma > 1$.

THEOREM 5. *In addition to the assumptions of Theorem A, assume $p(t)$ being bounded. Then (c) is necessary for all solutions of (23) to be oscillatory.*

Proof. As $p(t)$ being bounded, the proof follows immediately from Theorem 1.

THEOREM 6. *Consider the equation*

$$(24) \quad x^{(n)} + p(t)g(x, x', \dots, x^{(n-2)}, \delta) = 0 \text{ (for } n \text{ even),}$$

where δ is constant. Then (c) is necessary for all solutions of (24) to be oscillatory.

Proof. The proof follows immediately from Theorem 2.

REMARKS. Theorem 5 and Theorem 6 are proved also from Theorem A, since from Condition (G), we see

$$\liminf_{|x_1| \rightarrow +\infty} \frac{|g(x_1, x_2, \dots, x_{n-1})|}{|x_1|^\gamma} = \liminf_{|x_1| \rightarrow +\infty} \frac{|x_1|^r g(1, (x_2/x_1), \dots, (x_{n-1}/x_1))}{|x_1|^\gamma} > 0 .$$

In case $0 < \gamma < 1$, Ličko and Švec [3] proved Theorem 6 with

$g = x^\gamma$ ($0 < \gamma < 1$).

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