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Let L be a compact, distributive topological lattice of finite breadth and let A be a closed sublattice of L. It is shown that every closed congruence on A can be extended to a closed congruence on L. An example is provided to show that the requirement of finite breadth cannot be deleted.

The congruence extension property serves to characterize distributive lattices (cf. [4]). The definition of this property may be reformulated for topological lattices as follows: A topological lattice L has the congruence extension property if given any closed sublattice A of L and any closed congruence $[\varphi]$ on A there is a closed congrucence $[\Phi]$ on L such that $[\Phi] \cap (A \times A) = [\varphi]$. When this situation prevails we say that $[\varphi]$ has been extended to a closed congruence $[\Phi]$ on L. In this paper we prove that compact, distributive lattices of finite breadth have the congruence extension property. This fact is established by first showing that the lattice of closed congruences for such lattices is distributive. We also prove that the compact topological lattice $X = \times_{i=1}^{\infty}\{0, 1\}$ with coordinatewise operations does not have the congruence extension property.

1. Preliminaries. A finite subset B of a lattice is meet-irredundant if no proper subset of B has the same meet as B. The breadth of a lattice L, $Br(L)$, is the supremum of the cardinalities of its meet-irredundant sets. A chain is a lattice whose breadth is one. An element p of a lattice is prime if $x \wedge y \leqq p$ implies that $x \leqq p$ or $y \leqq p$. We shall use the notation that if $[\varphi]$ is a congruence then φ is the canonical map associated with $[\varphi]$.

A topological lattice is a Hausdorff topological space with a pair of continuous maps $\wedge, \vee : L \times L \rightarrow L$ such that $(L, \wedge, \vee)$ is a lattice. A point p of a subset A of a topological lattice L is a local maximum of A if there is an open subset U of L such that $(U \cap A) \cap (p \vee L) = \{p\}$. By A^* we shall mean the topological closure of A.

Suppose that L is a compact topological lattice. $\mathscr{C}(L)$ is the lattice of closed congruences on L (considered as subsets of $L \times L$) with operations $\wedge$ and $\vee$ defined by $[\varphi] \wedge [\theta] = [\varphi] \cap [\theta]$ and $[\varphi] \vee [\theta]$ is the smallest closed congruence on L which contains both $[\varphi]$ and $[\theta]$. $\mathscr{L}(L) = \{[\varphi] \in \mathscr{C}(L); Br(\varphi(L)) = 1\}$. For $[\varphi] \in \mathscr{L}(L)$ we define $M(\varphi) = \{x \in L; x = \vee \varphi^{-1}(\varphi(x))\}$ and $m(\varphi) = \{x \in L; x = \wedge \varphi^{-1}(\varphi(x))\}$.

Since $\varphi(L)$ is a chain both $M(\varphi)$ and $m(\varphi)$ are chains. Also since $\varphi(L)$ has the order topology when $M(\varphi)$ and $m(\varphi)$ are endowed with the order topology they are homeomorphic and isomorphic with $\varphi(L)$. Associated with φ there are two natural (algebraic) homomorphisms φ^1 and φ^0 where $\varphi^1(x) = \varphi^{-1}(\varphi(x)) \cap M(\varphi)$ and $\varphi^0(x) = \varphi^{-1}(\varphi(x)) \cap m(\varphi)$.

We say that a collection $\mathscr{P}$ of disjoint closed intervals of a compact topological lattice L is a partition of L if (a) $\cup \mathscr{P} = L$ and (b) $P_1, P_2 \in \mathscr{P}$ implies that $P_1 \wedge P_2 \subseteqq P_1$ or $P_1 \wedge P_2 \subseteqq P_2$. To every partition of L there corresponds a member of $\mathscr{L}(L)$. For $[\varphi] \in \mathscr{L}$(L) we shall let $\mathscr{P}_{[\varphi]}$ denote the partition of L corresponding to $[\varphi]$. It is easily proved that if $[\varphi]$ and $[\theta]$ are in $\mathscr{L}(L)$ then $\mathscr{P}_{[\varphi]\vee[\theta]} = \varphi^{-1}(\mathscr{Q})$ where $\mathscr{Q}$ is the smallest partition on $\varphi(L)$ which contains $\varphi(\mathscr{P}_{[\theta]})$.

Recall that a coordinate chain in a lattice L is a chain C such that (1) C consists only of prime elements of L and (2) C is closed with respect to arbitrary meets [2]. Note that if L is a compact topological lattice (2) is equivalent to (2′) C is closed with respect to decreasing nets.

LEMMA 1.1. *Let L be a compact topological lattice and let $[\varphi] \in \mathscr{L}(L)$. Then $M(\varphi)$ is a coordinate chain.*

Proof. Let $a \in M(\varphi)$ and suppose that for some pair $x, y \in L$, $x \wedge y \leqq a$. Since $\varphi(L)$ is a chain and φ is a homomorphism

$$\varphi(a) \geqq \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) = \min\{\varphi(x), \varphi(y)\}.$$

Let $\varphi(x) = \min\{\varphi(x), \varphi(y)\}$. Because $a \in M(\varphi)$ we have $a \geqq x$. Thus a is a prime element of L.

Let A be a decreasing net of elements of $M(\varphi)$. Since L is compact A converges to some element $a_0 \in L$. Suppose that for some pair $x, y \in L$, $a_0 \geqq x \wedge y$. Then $a \geqq x \wedge y$ for each $a \in A$. Since a is prime either $a \geqq x$ or $a \geqq y$. Thus we may assume that there is a cofinal subnet A' of A such that for each $a \in A'$, $a \geqq x$. Hence $a \wedge x = x$ for all $a \in A'$ and by the continuity of $\wedge$, $a_0 \wedge x = x$. Therefore a_0 is a prime element of L. Since A' is a decreasing net and it converges to a_0 it follows that $a_0 \in M(\varphi)$.

2. The congruence extension property on compact lattices.

LEMMA 2.1. *Let L be a compact topological lattice and let $[\alpha]$, $[\beta_1], [\beta_2], \cdots, [\beta_n] \in \mathscr{L}(L)$. Then $[\alpha] \vee (\bigwedge_{i=1}^n[\beta_i]) = \bigwedge_{i=1}^n([\alpha] \vee [\beta_i])$.*

Proof. To simplify notation we shall let $[\varphi] = [\alpha] \vee (\bigwedge_{i=1}^n[\beta_i])$. Note that since $[\varphi] \geqq [\alpha]$ we must have $[\varphi] \in \mathscr{L}(L)$. It is always the case that $[\varphi] \leqq \bigwedge_{i=1}^{n}([\alpha] \vee [\beta_i])$ so we need only prove that $[\varphi] \geqq \bigwedge_{i=1}^{n}([\alpha] \vee [\beta_i])$. This will be done by showing that if $(x, y) \notin [\varphi]$ then there is $i \in \{1, 2, \cdots, n\}$ such that $(x, y) \notin [\alpha] \vee [\beta_i]$ and consequently $(x, y) \notin \bigwedge_{i=1}^{n}([\alpha] \vee [\beta^i])$. $(x, y) \notin [\varphi]$ implies that $\varphi^1(x) \neq \varphi^1(y)$. We may assume that $\varphi^1(x) < \varphi^1(y)$.

(1) *If $a, b \in M(\varphi)$ with $a < b$ then there is $i \in \{1, 2, \cdots, n\}$ such that $\beta_i(a) < \beta_i(\varphi^0(b))$.*

If this were not the case then for each $i \in \{1, 2, \cdots, n\}$ we would have $\beta_i^1(\varphi^0(b)) \leqq \beta_i^1(a) \leqq \beta_i^1(b)$. This implies that

$$\varphi^0(b) \leqq \bigwedge_{i=1}^{n}\beta_i^1(\varphi^0(b)) \leqq \bigwedge_{i=1}^{n}\beta_i^1(a) \leqq \bigwedge_{i=1}^{n}\beta_i^1(b).$$

Then because $a \leqq \bigwedge_{i=1}^{n}\beta_i^1(a) \leqq \beta_i^1(a)$ for $i = 1, 2, \cdots, n$ we have $(a, \bigwedge_{i=1}^{n}\beta_i^1(a)) \in \bigwedge_{i=1}^{n}[\beta_i]$. For the same reason $(\varphi^0(b), \bigwedge_{i=1}^{n}\beta_i^1(\varphi^0(b)))$, $(b, \bigwedge_{i=1}^{n}\beta_i^1(b)) \in \bigwedge_{i=1}^{n}[\beta_i^1]$. Then since $(\varphi^0(b), b) \in [\varphi]$ and $\varphi^0(b) \leqq \bigwedge_{i=1}^{n}\beta_i^1(a) \leqq \bigwedge_{i=1}^{n}\beta \bigwedge_i^1(b)$ we have $(b, \bigwedge_{i=1}^{n}\beta_i^1(a)) \in [\varphi]$. Hence $(a, b) \in [\varphi]$. This is a contradiction so there must be an $i \in \{1, 2, \cdots, n\}$ with the property that $\beta_i(a) < \beta_i(\varphi^0(b))$.

Suppose that $\varphi([\varphi^1(x), \varphi^1(y)])$ is not connected then because $\varphi(L)$ is a chain there are $a, b \in M(\varphi)$ such that $\varphi^1(x) \leqq a < b \leqq \varphi^1(y)$, $L = (a \wedge L) \cup (\varphi^0(b) \vee L)$ and $\varphi(L) = \varphi(a \wedge L) \cup \varphi(\varphi^0(b) \vee L)$. From (1) above there is $i \in \{1, 2, \cdots, n\}$ such that $\beta_i(a) < \beta_i(\varphi^0(b))$. This implies that $\beta_i(L) = \beta_i(a \wedge L) \cup \beta_i(\varphi^0(v) \vee L)$ and $\beta_i(a \wedge L) \cap \beta_i(\varphi^0(b) \vee L) = \varnothing$. It then follows that $(a, b) \notin [\varphi] \vee [\beta_i]$. Then because $[\alpha] \leqq [\varphi]$, $x \leqq \varphi^1(x) \leqq$ a and $\varphi^0(b) \leqq \varphi^0(y) \leqq y$, we must have $(x, y) \notin [\alpha] \vee [\beta_i]$.

Thus we have disposed of the case in which $\varphi([\varphi^1(x), \varphi^1(y)])$ is not connected so we may assume throughout the rest of the proof that $\varphi([\varphi^1(x), \varphi^1(y)])$ is connected.

(2) *If $a \in M(\varphi) \cap [\varphi^1(x), \varphi^1(y)] \backslash \{\varphi^1(y)\}$ then there is $i \in \{1, 2, \cdots, n\}$ such that if $c \in M(\varphi)$ and $a < c$ then $\beta_i(a) < \beta_i(\varphi^0(c))$.*

Since $\varphi([\varphi^1(x), \varphi^1(y)])$ is connected a cannot be a local maximum of $M(\varphi)$. Hence there is a net B of elements of $M(\varphi)$ which converges to a such that if $b \in B$ then $a < b \leqq \varphi^1(y)$. Using (1) above and the fact that n is finite we may find an element $i \in \{1, 2, \cdots, n\}$ and a cofinal subnet B' of B such that if $b \in B'$ then $\beta_i(a) < \beta_i(\varphi^0(b))$. Then if $c \in M(\varphi)$ and $c > a$ there must be $b \in B'$ with the property that $a < b \leqq c$. Thus $\beta_i(a) < \beta_i(\varphi^0(b)) \leqq \beta_i(\varphi^0(c))$.

(3) *There is $i \in \{1, 2, \cdots, n\}$ and $a, b \in M(\varphi) \cap [\varphi^1(x), \varphi^1(y)]$ with $a < b$ such that if $c, d \in M(\varphi)$ and $a \leqq c < d \leqq b$ then $\beta_i(c) < \beta_i(\varphi^0(d))$.*

For $s \in M(\varphi) \cap [\varphi^1(x), \varphi^1(y)] \backslash \{\varphi^1(y)\}$ we let $\xi(s)$ be any element of $\{1, 2, \cdots, n\}$ with the the property that if $t \in M(\varphi)$ and $s < t$ then $\beta_{\xi(s)}(s) < \beta_{\xi(s)}(\varphi^0(t))$. From (2) above ξ is a function defined for each element of $M(\varphi) \cap [\varphi^1(x), \varphi^1(y)] \backslash \{\varphi^1(y)\}$. Let A_i be the closure of ξ^{-1} (i) in $M(\varphi)$ when $M(\varphi)$ is endowed with the order topology. With this topology $M(\varphi) \cap [\varphi^1(x), \varphi^1(y)]$ becomes a compact connected chain. Hence some A_i must contain a nondegenerate interval of $M(\varphi) \cap [\varphi^1(x), \varphi^1(y)]$. Let that interval be $[e, f] \cap M(\varphi)$. Then, still using the order topology on $M(\varphi)$, $\xi^{-1}(i)$ is dense in $[e, f]$. Hence there are $a, b \in \xi^{-1}(i)$ with $e \leqq a < b \leqq f$. $\xi^{-1}(i)$ is dense in $[a, b] \cap M(\varphi)$ so by an argument similar to that used in (2) $[a, b] \subset M(\varphi)$ has the desired properties. Thus (3) is proved.

Now let i be that member of $\{1, 2, \cdots, n\}$ secured in (3). Let $\mathscr{D}$ be the smallest partition of $\beta_i(L)$ which contains $\beta_i(\varphi_{[\varphi]})$. From (3) $\mathscr{D}$ and $\beta_i(\mathscr{P}_{[\varphi]})$ must coincide on $\beta_i(a, b])$. Thus $\beta_i(a) < \beta_i(b)$. Then

$$\beta_i(x) \leqq \beta_i(\varphi^1(x)) \leqq \beta_i(a) < \beta_i(\varphi^0(b)) \leqq \beta_i(\varphi^0(y)) \leqq \beta_i(y).$$

Hence $\beta_i(x)$ and $\beta_i(y)$ must be in different elements of the partition $\mathscr{Q}$. Thus we have $(x, y) \notin [\varphi] \vee [\beta_i]$. A fortiori $(x, y) \notin [\alpha] \vee [\beta_i]$ and our lemma is proved.

It is known (cf. [3]) the lattice of congruences for any lattice is distributive. For compact topological lattices we have

THEOREM 2.1. *Suppose that L is a compact distributive topological lattice of finite breadth. Then $\mathscr{C}(L)$ is a distributive lattice.*

Proof. It is an immediate consequence of Lemma 2.1 that for each $[\alpha] \in \mathscr{L}(L)$, $[\alpha] \vee \mathscr{C}(L)$ is a distributive lattice. We now claim that if $[\alpha] \in \mathscr{L}(L)$ then the map $[\varphi] \to [\alpha] \vee [\varphi]$ is a homomorphism of $\mathscr{C}(L)$ onto $[\alpha] \vee L$. Since this map obviously preserves joins we need only show that it also preserves meets. Suppose that $[\varphi]$ and $[\theta]$ are members of $\mathscr{C}(L)$. Homomorphisms cannot raise breadth so $Br(\varphi(L)) = m \leqq Br(L) \geqq n = Br(\theta(L))$. From Theorem 3.1 of [2] we know that there is a set of m elements of $\mathscr{L}(\varphi(L))$ which separates points in $\varphi(L)$ and a set of n elements of $\mathscr{L}(\theta(L))$ which separates points in $\theta(L)$. This implies that there are $[\varphi_1]$, $[\varphi_2]$, $\cdots$, $[\varphi_m]$, $[\theta_1]$, $[\theta_2]$, $\cdots$, $[\theta_n] \in \mathscr{L}(L)$ such that $\bigwedge_{i=1}^m [\varphi_i] = [\varphi]$ and $\bigwedge_{i=1}^n [\theta_i] = [\theta]$. These facts and Lemma 2.1 enable us to obtain

$$\begin{aligned}
[\alpha] \vee ([\varphi] \wedge [\theta]) &= [\alpha] \vee ([\varphi_1] \wedge [\varphi_2] \wedge \cdots \wedge [\varphi_m] \wedge [\theta_1] \wedge [\theta_2] \wedge \cdots [\theta_n]) \\
&= ([\alpha] \vee ([\varphi_1]) \wedge ([\alpha] \wedge [\varphi_2]) \wedge \cdots \wedge ([\alpha] \vee [\theta_n]) \\
&= \left(\bigwedge_{i=1}^m ([\alpha] \vee [\varphi_i])\right) \wedge \left(\bigwedge_{i=1}^n ([\alpha] \vee [\theta_i])\right) \\
&= \left([\alpha] \vee \left(\bigwedge_{i=1}^m [\varphi_i]\right)\right) \wedge \left([\alpha] \vee \left(\bigwedge_{i=1}^n [\theta_i]\right)\right) \\
&= ([\alpha] \vee [\varphi]) \wedge ([\alpha] \vee [\theta]).
\end{aligned}$$

Hence the map $[\varphi] \to [\alpha] \vee [\varphi]$ is a homomorphism.

Next we shall show that the collection of homomorphisms $[\varphi] \to [\alpha] \vee [\varphi]$ where $[\alpha] \in \mathscr{L}(L)$ separates points in $\mathscr{C}(L)$. Let $[\varphi], [\theta] \in \mathscr{C}(L)$. Suppose that $[\varphi] = \bigwedge_{i=1}^m [\varphi_i]$ and $[\theta] = \bigwedge_{i=1}^n [\theta_i]$ are representations of $[\varphi]$ and $[\theta]$ by elements of $\mathscr{L}(L)$ obtained as in the previous paragraph and suppose that no element in either representation is redundant. If $[\varphi] \neq [\theta]$ then we may assume that $[\varphi] \wedge [\theta] \neq [\theta]$. We claim that for some $i \in \{1, 2, \cdots, m\}$, $[\varphi_i] \vee [\varphi] \neq [\varphi_i] \vee [\theta]$. Suppose that this is not the case. Then for every $i \in \{1, 2, \cdots, m\}$, $[\varphi_i] = [\varphi_i] = [\varphi_i] \vee [\varphi] = [\varphi_i] \vee [\theta]$. Hence $[\varphi_i] \wedge [\theta] = [\theta]$ for every $i \in \{1, 2, \cdots, m\}$. This allows us to conclude that $[\varphi] \wedge [\theta] = (\bigwedge_{i=1}^{n}[\varphi_i]) \wedge [\theta] = [\theta]$ contrary to our assumption. Thus there are enough maps of the form $[\varphi] \to [\alpha] \vee [\varphi]$ where $[\alpha] \in \mathscr{L}(L)$ to separate points in $\mathscr{C}(L)$.

$\mathscr{C}(L)$ has enough homomorphisms onto distributive lattices to separate points. Thus $\mathscr{C}(L)$ can be embedded in a distributive lattice. Hence $\mathscr{C}(L)$ is distributive.

Note that our purpose in proving Theorem 2.1 is to aid in the proof of Theorem 2.2. No claims are made about the generality of Theorem 2.1.

THEOREM 2.2. *Let L be a compact, distributive topological lattice of finite breadth. Then L has the congruence extension property.*

Proof. Let A be a closed sublattice of L and let $[\varphi] \in \mathscr{C}(A)$. If $Br(L) = 1$ then $[\Phi] = \{(x, y)$: there exists $a, b \in A$ with $\varphi(a) = \varphi(b)$ and $x, y \in [a, b]\} \cup \Delta$ is a congruence which is an extension of $[\varphi]$. Suppose that $Br(L) = n \gg 1$. As a result of Theorem 3.1 of [2] L can be embedded in a lattice $L^\wedge = C_1 \times C_2 \times \cdots \times C_n$ where each C_i is a compact chain. Since this is the case we shall consider both A and L to be closed sublattices of $L^\wedge$ and proceed to show that $[\varphi]$ can be extended to a closed congruence on $L^\wedge$. Then a fortiori we will have proved that $[\varphi]$ can be extended to a closed congruence on L.

For each $i \in \{1, 2, \cdots, n\}$ define π_i to be the natural projection of $L^\wedge$ onto C_i and P_i to be the restriction of π_i to A. Because $\mathscr{C}(A)$ is a distributive lattice and

$$\bigwedge_{i=1}^{n}[P_i] = \Delta_A,\ [\varphi] = [\varphi] \vee \left(\bigwedge_{i=1}^{n}[P_i]\right) = \bigwedge_{i=1}^{n}([\varphi] \vee [P_i]).$$

Each C_i is a chain and $P_i(A)$ is a closed sublattice of C_i so the congruence on $P_i(A)$ associated with the natural map of $P_i(A)$ onto $A/[P_i] \vee [\varphi]$ can be extended to a closed congruence $[\rho_i]$ on C_i. Consequently $[\rho_i \circ \pi_i]$ is a member of $\mathscr{L}(L^\wedge)$ and it is also an extension of $[\varphi] \vee [P_i]$. We define $[\Phi] = \bigwedge_{i=1}^{n}[\rho_i \circ \pi_i]$. Then

$$[\Phi] \cap (A \times A) = \left(\bigwedge_{i=1}^{n}[\rho_i \circ \pi_i]\right) \cap (A \times A)$$

$$= \bigwedge_{i=1}^{n}([\rho_i^\circ\pi_i] \cap (A \times A))$$
$$= \bigwedge_{i=1}^{n}([p_i] \vee [\varphi])$$
$$= [\varphi].$$

Thus $[\Phi]$ is an extension of $[\varphi]$.

3. Properties of X. Let I be the usual topological lattice on the closed real interval $[0, 1]$. Let K be the standard representation of the Cantor set in I and let $N = \{1\text{-}1/n; n \in Z^+\} \cup \{1\}$. Then both K and N are closed sublattices of I. Let X be the usual topological lattice on $\times_{i=1}^{\infty}\{0, 1\}$. Define $W(X)$ to be the set of points of X having only finitely many nonzero coordinates.

Recall that the set of local minima of X is dense in X [6]. A sequence $\{x_n; n \in \omega\}$ is nondecreasing if $x_n \wedge x_{n+1} = x_n$ for all $n \in \omega$.

LEMMA 3.1. *Let $x \in X$ and let $w \in W(X)$. Then the following statements hold:*

(1) *$w \wedge X$ is finite.*

(2) *There is a nondecreasing sequence in $(x \wedge X) \cap W(X)$ which converges to x.*

(3) *$W(X)$ is the set of local minima of X. Hence $W(X)$ is dense in X.*

Proof. (1) is obvious and (3) is a direct consequence of (1) and (2). Thus it only remains to prove (2). For $x \in X$ we define a sequence $\{w_n; n \in \omega\}$ by having w_n agree with x for the first n coordinates and thereafter having all coordinates zero. Then each $w_n \in W(X)$ and $\{w_n; n \in \omega\}$ obviously converges to x.

Following [5] we say that a continuous homomorphism φ from a compact semilattice S onto a semilattice T has full cross-section if there is a closed sub-semilattice A of S such that φ restricted to A is an isomorphism of A onto T.

LEMMA 3.2. *Let φ be a continuous meet-homomorphism of X onto the compact zero-dimensional chain C. Then C is the continuous meet-homomorphic image of N. Moreover, φ has full cross-section.*

Proof. Let B be the set of local minima of C and let $A = \{x \in X; x$ is the least element of $\varphi^{-1}(b)$ for some $b \in B\}$. A is a chain and from Lemma 3.1, $A \subseteq W(X)$. Then since $B^* = C$ we have $\varphi(A^*) = C$. In view of Lemma 3.1, A^* must be the continuous meet-homomorphic image of N and A^* must define a full cross-section for φ.

THEOREM 3.1. *If there is a continuous meet-homomorphism of X onto Y, then Y is zero-dimensional.*

Proof. Suppose that φ is a continuous meet-homomorphism from X onto a semilattice S where dim $S \geqq 1$. Since dim $S \geqq 1, S$ must possess a nondegenerate component S_0. S_0 is a compact semilattice so it has a least element s_0. Choose $s \in S \backslash \{s_0\}$. The Rees-quotient sS/s_0S is a nondegenerate connected semilattice and it is the continuous meet-homomorphic image of X. As a consequence of Theorems 2.1, 3.1 and 4.3 of]7], sS/s_0S, and hence also X, must have a continuous meet-homomorphism onto I. Let η be one such homomorphism. $w \wedge X$ is finite for all $w \in W(X)$. Hence $\eta(W(X)) = 0$. This implies that $\eta(W(X)^*) = 0$. However, $X = W(X)^*$. Thus we have arrived at a contradiction. Therefore our theorem is proved.

We now show that X does not have the congruence extension property. In fact, we can say more.

COROLLARY 3.1. *There is a linearly ordered sublattice K of X and a closed congruence ζ on K such thxt ζ cannot be extended to a closed meet-congruence on X.*

Proof. From Theorem 2 of [8] K can be embedded in X. Thus we may consider K to be a sublattice of X. On K the relation defined by identifying the end points of complementary intervals is a member of $\mathscr{C}(K)$. Moreover $\zeta(K)$ is a compact, connected lattice. From Theorem 3.1 it follows that $[\zeta]$ cannot be extended to a closed meet-congruence on X.

We now provide several examples which are variants of the example used in Corollary 3.1.

Example 1. Let C denote the image of K in X under an embedding ρ where $\rho(0) = 0$ and $\rho(1) = 1$. X can be considered to be a sublattice of I^ω. Extend C to a maximal chain J in I^ω. The closed congruence $[\zeta]$ used in Corollary 3.1 can be extended to a closed congruence $[\zeta']$ on J. Then since $[\zeta]$ cannot be extended to a closed meet-congruence on X it follows that $[\zeta']$ cannot be extended to a closed meet-congruence on I^ω.

Example 2. In the previous example let $S = \cup \{x \wedge J; x \in X\}$. Then S is a compact, one-dimensional semilattice and J is the unique thread from the zero of S to the identity of S. Applying the reasoning used in Example 1 the congruence $[\zeta']$ on J cannot be extended to S. S has the additional property that J cannot be the continuous (semi-lattice) homomorphic retract of S. Note that this will remain true if I is replaced by any standard thread.

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