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**A GELFAND REPRESENTATION THEORY FOR  $C^*$ -ALGEBRAS**

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## A GELFAND REPRESENTATION THEORY FOR $C^*$ -ALGEBRAS

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Recent work by the author which was independently duplicated in part by Giles and Kummer has made it possible to generalize the Gelfand representation theorem for abelian  $C^*$ -algebras to the non-abelian case. Let  $A$  be a  $C$ -algebra with unit. If  $A$  is abelian, it can be identified with the algebra of all continuous complex-valued functions on its maximal ideal space (with the hull-kernel topology). A less precise way of looking at this result would be to say that an abelian  $A$  is completely recoverable from the set of maximal ideals and a certain structure thereon (in this case, a topology). If we use the latter description as the basis for a theory applicable to non-abelian  $A$ , we find immediately that two changes are necessary. The set of maximal ideals is replaced by the set of maximal left ideals, and secondly, the structure defined thereon will not be a topology, though it will have many similar properties when viewed correctly. This paper shows how the  $C^*$ -algebra is recovered from the maximal left ideals (with structure).

I. Preliminaries. Consider the  $W^*$ -algebra  $A^{**}$ , the second Banach space dual of  $A$  [9, p. 236]. There exists a central projection  $z \in A^{**}$  which is the supremum of all the minimal projections in  $A^{**}$  [3, p. 278]. Set  $M = zA^{**}$ . The minimal projections of  $M$  are in one to one correspondence with the maximal left ideals of  $A$  [3, p. 280 and 9, p. 48], so that we can define a structure on this set of minimal projections instead of directly on the maximal left ideals. Naturally the first thing we "build" is the algebra  $M$ . We then single out a class  $L$  of projections in  $M$  as the  $q$ -open projections as follows. First note that we can consider  $A \subset M$  since  $A \subset A^{**}$  and  $A \rightarrow zA$  is a  $*$ -isomorphism [9, p. 39]. (Also we can view  $M$  as the direct sum of irreducible representations of  $A$ , one from each equivalence class.) A projection  $p$  in  $M$  is  $q$ -open if there exists a closed left ideal  $I$  of  $A$  such that the weak\* closure  $\bar{I}$  of  $I$  in  $M$  is of the form  $Mp$ . The  $q$ -open projections are analogous to the open sets of a topology.

If  $A$  were abelian,  $M$  would be the algebra of all bounded complex function on its maximal ideal space  $K$ . The  $q$ -open projections would be characteristic functions of open sets of  $K$  for the hull-kernel topology. A self-adjoint operator  $b$  in  $M$  actually lies in  $A$  ( $\subset M$ ) if and only if the spectral projections of  $b$  corresponding to open sets of real numbers are  $q$ -open projections in the above sense.

This is a restatement of Gelfand's theorem since a function is continuous if and only if its inverse images of open sets are open.

We may now state an identical theorem for the non-abelian case. The proof follows immediately from the addendum to [4] and Theorem II.17 of [3].

**THEOREM I.1.** *A self-adjoint operator  $b \in M$  lies in  $A(\subset M)$  if and only if each spectral projection of  $b$  which corresponds to an open subset of the real numbers is also a  $q$ -open projection.*

This theorem says that we may reconstruct  $A$  from its set of maximal left ideals together with the above defined structure. As a corollary we note that if two algebras  $A_1$  and  $A_2$  have "isomorphic structures" then they are isomorphic.

**COROLLARY I.2.** *Let  $A_1$  and  $A_2$  be  $C^*$ -algebras with  $M_i = z_i A_i^{**}$  and  $L_i$  the  $q$ -open projections in  $M_i$  ( $i = 1, 2$ ). If there exists a  $*$ -isomorphism  $\varphi: M_1 \rightarrow M_2$  which maps  $L_1$  onto  $L_2$ , then  $\varphi|A_1$  is an isomorphism of  $A_1$  onto  $A_2$ .*

This paper extends these results to  $C^*$ -algebras without unit with appropriate modifications suggested by the abelian case. A number of other "topological" results are proved, and counter-examples are given to close off several tempting avenues of approach.

To complete our terminology, we shall assume from now on that  $A$  is a  $C^*$ -algebra which may not have a unit. The above discussion still applies to get  $z \in A^{**}$  and we set  $M = zA^{**}$ . Identify  $A$  and  $zA \subset M$  and call  $M$  the pure state  $q$ -space of  $A$ . (The terminology is lifted from [11].) We have already defined  $q$ -open projections in  $M$ , and their complements (in  $M$ ) are called  $q$ -closed.  $\tilde{A}$  will denote the algebra  $A$  with unit adjoined as in [9, p. 7]. Note that  $A$  is a closed two-sided ideal in  $\tilde{A}$  of co-dimension one. Thus  $\tilde{A}^* \cong A^* \oplus \{\lambda f_\infty\}$ , where  $f_\infty$  is the unique pure state of  $\tilde{A}$  which vanishes on  $A$ . Also the pure state  $q$ -space  $\tilde{M}$  of  $\tilde{A}$  is:  $\tilde{M} \cong M \oplus \{\lambda 1_\infty\}$ , with  $f_\infty(1_\infty) = 1$ . In view of Theorem II.17 of [3] all the properties of open or closed projections in  $A^{**}$  (as considered in [3 and 4]) carry over immediately to corresponding properties of  $q$ -open or  $q$ -closed projections in  $M$ .

**II. The problem of compactness.** Although the notion of compactness is vaguely introduced in [3], it is clear that a theory which claims to generalize locally compact Hausdorff spaces should generalize the notion of a compact set.

**DEFINITION II.1.** A projection  $p \in M$  is  $q$ -compact if  $p$  is  $q$ -closed

and there exists  $b \in A^+ (= \{a \in A: a \geq 0\})$  with  $bp = p$ .

There are a number of conditions equivalent to compactness for a set in a locally compact Hausdorff space. It would be desirable to show that many of them can be extended to equivalent conditions for  $q$ -compactness. The most desirable such condition would be:

*Conjecture II.2.* A regular [10, p.408] projection  $p \in M$  is compact if for every family  $\{p_\alpha\}$  of  $q$ -closed projections such that the family  $\{p_\alpha \wedge p\}$  has the finite intersection property, then  $p \wedge \bigwedge_\alpha p_\alpha \neq 0$ .

We shall prove this for certain  $p$  in Theorem II.6. The conjecture is false without the assumption of regularity (see Example IV.5).

**LEMMA II.3.** Suppose  $B$  is a  $C^*$ -algebra,  $b \in B^+$ ,  $p \in B^{**}$  a projection and  $\{a_\alpha\} \subset B$  an increasing net of positive elements with  $\|b^{1/2} - b^{1/2}a_\alpha\| \xrightarrow{\alpha} 0$ . If  $b \geq p$  (considering  $B \subset B^{**}$ ), then  $\|p - a_\alpha p\| \xrightarrow{\alpha} 0$ .

*Proof.* Since  $\|b^{1/2} - b^{1/2}a_\alpha\| \xrightarrow{\alpha} 0$ , clearly  $\|(1 - a_\alpha)b(1 - a_\alpha)\| \xrightarrow{\alpha} 0$ . Since  $(1 - a_\alpha)b(1 - a_\alpha) \geq (1 - a_\alpha)p(1 - a_\alpha)$ , we get

$$\|(1 - a_\alpha)p(1 - a_\alpha)\| = \|(1 - a_\alpha)p\|^2 = \|p - a_\alpha p\|^2 \longrightarrow 0.$$

**LEMMA II.4.** If  $p$  is  $q$ -closed for  $A$  and we consider  $\tilde{A}$  and  $\tilde{M}$  as above with  $M \subset \tilde{M}$  (hence  $p \in \tilde{M}$ ) and there exists  $b \in A^+$  with  $b \geq p$ , then  $p$  is  $q$ -closed in  $\tilde{M}$ .

*Proof.* Let  $K = (pA^*p)^+$ . Then  $K$  is  $\sigma(A^*, A)$  closed by [3, II.2]. If  $K$  is not  $\sigma(\tilde{A}^*, \tilde{A})$  closed, then there is a net  $\{f_\alpha\} \subset K$  with  $\|f_\alpha\| \subset K$  with  $\|f_\alpha\| = 1$  and  $f_\alpha \xrightarrow{\alpha} f$ ,  $\sigma(\tilde{A}^*, \tilde{A})$ , for some  $f \in \tilde{A}^*$  with  $\|f\| = f(1) = 1$ . Since  $\tilde{A}^* = A^* \oplus \{\lambda f_\infty\}$ , we get  $f = f_0 + \lambda f_\infty$  where  $f_0 \in A^{**}$  and  $\lambda \geq 0$ . For any  $c \in A$  with  $c \geq p$ ,

$$f_0(c) = f(c) = \lim_{\alpha} f_\alpha(c) \geq \overline{\lim}_{\alpha} f_\alpha(p) = 1$$

since each  $f_\alpha \in K$ .

Now if  $1 \in A$ , then  $A^*$  is  $\sigma(\tilde{A}^*, \tilde{A})$  closed in  $\tilde{A}^*$ , so the conclusion of this lemma is immediate. If  $1 \notin A$ , let  $\{a_\gamma\} \subset A^+$  be an increasing approximate unit. Then, by Lemma II.3,  $\{a_\gamma\}$  is an approximate unit for  $p$  also. Thus given  $\varepsilon > 0$  there exists  $c \in A$  with  $c \geq p$  and  $\|c\| \leq 1 + \varepsilon$  by Theorem 1.2 of [2]. Hence  $f_0(c) \geq 1$  by the above. Since  $\varepsilon > 0$  was arbitrary,  $\|f_0\| = 1$ , so  $\lambda = 0$ , since

$$\|f\| = \|f_0\| + |\lambda| = 1.$$

Thus  $f \in A^*$ . Since  $\{f_\alpha\} \subset K$ ,  $K$  is  $\sigma(A^*, A)$  closed, and  $f_\alpha \xrightarrow{\alpha} f$  in the  $\sigma(\tilde{A}^*, \tilde{A})$  topology, we see that  $f \in K$ , so  $K$  is  $\sigma(\tilde{A}^*, \tilde{A})$  closed.

**THEOREM II.5.** *If  $p$  is  $q$ -closed and there exists  $b \in A$  with  $b \geq p$ , then  $p$  is  $q$ -compact.*

*Proof.* Since  $p$  is  $q$ -closed for  $\tilde{A}$  by Lemma II.4, there exist  $\{b_\alpha\} \subset \tilde{A}$ ,  $b_\alpha = a_\alpha + \lambda_\alpha 1$  with  $a_\alpha \in A$ ,  $1 \geq b_\alpha \geq p$  and  $b_\alpha \downarrow p$  in  $M$  [5, proof of Prop. 1]. Thus each  $b_\alpha$  (and hence  $a_\alpha$ ) commutes with  $p$ . Since  $f_\infty(b_\alpha) \xrightarrow{\alpha} 0$ , there exists  $\alpha_0$  with  $f_\infty(b_{\alpha_0}) < 1/2$ . Thus  $\lambda_{\alpha_0} < 1/2$  since  $f_\infty(a_{\alpha_0}) = 0$ . Let  $g(t)$  be a continuous function which has  $g(t) = 1$  for  $t \geq 1/2$ ,  $g(0) = 0$ ,  $0 \leq g(t) \leq 1$  for all  $t$ . Then  $g(a_{\alpha_0}) \geq p$ . (Since  $a_{\alpha_0}, b_{\alpha_0}$  and  $p$  all commute, we may view them as functions on a common locally compact space; this makes the assertion clear.) Since  $g(a_{\alpha_0}) \in A$ , the theorem follows.

The construction in the proof of last theorem will not work for all projections  $p$  in  $M$  having only the property that  $p \leq b \in A$ , even though it easily works whenever  $p$  is central.

**THEOREM II.6.** *Suppose  $1 \in A$  and  $A$  is separable. Then Conjecture II. 2 holds for central projections  $p \in M$ .*

*Proof.* Suppose  $p$  satisfies the intersection condition of Conjecture II.2. We need only show  $p$  is  $q$ -closed since  $1 \in A$ . If it is not  $q$ -closed, let  $\bar{p}$  be its closure [3, II. 11] and let  $q \leq \bar{p} - p$  be a minimal projection. As in [1] there exists a strictly positive element  $a_0$  in  $\{a \in A : aq = qa = 0\} = I$ , so we let  $p_n$  be the spectral projection of  $a_0$  corresponding to the interval  $[0, 1/n]$ . Since  $\bigwedge_n p_n \wedge p = 0$ , there is some  $n_0$  with  $p_{n_0} p = 0$  by hypothesis. Since  $p$  is central, the spectral projection  $x$  of  $a_0$  corresponding to  $[1/n_0, \infty)$  is  $q$ -closed and  $x \geq p$ . This contradicts  $xq = 0$  and  $q \leq \bar{p}$ .

**THEOREM II.7.** *If  $p$  is  $q$ -compact, then  $p$  satisfies the intersection condition of Conjecture II.2.*

*Proof.* Since  $p$  is also  $q$ -closed in  $\tilde{M}$  by Lemma II.4, the theorem follows from [3, II.10] for if  $\{p_\alpha\}$  are  $q$ -closed in  $M$ , then their  $q$ -closures  $\{\bar{p}_\alpha\}$  in  $\tilde{M}$  have no larger  $M$  component. (Recall that  $\tilde{M} = M \oplus \{\lambda 1_\infty\}$  with  $1_\infty M = \{0\}$ .) Thus if  $p \wedge \bigwedge_{\alpha \in J} \bar{p}_\alpha \neq 0$  for all finite sets  $J$ ,  $p \wedge \bigwedge_\alpha \bar{p}_\alpha \neq 0$ , so  $p \wedge \bigwedge_\alpha p_\alpha \neq 0$ , since  $p \wedge \bar{p}_\alpha = p \wedge p_\alpha$ .

Next we move in a different direction for a characterization of

$M$ . If  $A$  were an abelian  $C^*$ -algebra of functions containing the constants and separating the points of the topological space  $\Omega$ , then  $A$  consists of all continuous functions on  $\Omega$  if and only if  $\Omega$  is compact. Following [11] we define a  $q$ -space to be an atomic  $W^*$ -algebra. If  $M_1$  is a  $q$ -space and  $A \subset M_1$ , is a weak\* dense  $C^*$ -subalgebra with  $1 \in A$ , we can define a  $q$ -open projection in  $M_1$  as a sup of range projections of elements of  $A$ . Naturally  $q$ -closed projections are complements of  $q$ -open projections. If  $M_1 = M$ , the two definitions coincide.

**THEOREM II.8.** *If  $A$  is separable and  $A \subset M_1$  as above, then there is an  $A$ -preserving  $*$ -isomorphism between  $M_1$  and  $M$  if and only if the  $q$ -closed projections of  $M_1$  satisfy the intersection condition of Conjecture II.2.*

*Proof.* If  $M_1$  is  $*$ -isomorphic to  $M$  under an  $A$ -preserving map the verification is routine. Now suppose the  $q$ -closed projections of  $M_1$  satisfy the intersection condition. If every pure state of  $A$  extends to a normal state of  $M_1$ , there is a natural isomorphism between  $M_1$  and  $M$  which preserves  $A$  because of the definition of  $M$  as a subset of  $A^{**}$ . Thus let  $f$  be a pure state of  $A$  with no normal extension to  $M_1$ . Let  $\{a_j\} \subset A$  be an increasing positive abelian [1] approximate unit for  $\{a \in A: f(a^*a + aa^*) = 0\}$ . Then let  $p_{j_n}$  be the spectral projection of  $a_{j_n}$  corresponding to the interval  $(1/n, \infty)$ . Clearly  $\bigvee_{j,n} p_{j_n} = 1$  in  $M_1$ , for if not, then  $(1 - \bigvee_{j,n} p_{j_n})$  would be one-dimensional, hence  $f$  could be extended to a normal functional on  $M_1$  with support  $(1 - \bigvee_{j,n} p_{j_n})$ . But  $\{(1 - p_{j_n})\}$  is a decreasing net of closed projections in  $M_1$  with  $\bigwedge_{j,n} (1 - \bigvee p_{j_n}) = 0$ . Thus  $(1 - p_{j_n}) = 0$  for some  $j$  and  $n$ . Hence  $a_j$  is invertible, so  $f = 0$ , a contradiction.

### III. The Gelfand representation.

**LEMMA III.1.** *If  $p$  is  $q$ -closed,  $p_1$  is  $q$ -compact, and  $p_1 p = 0$ , then there exists  $a \in A^+$  with  $\|a\| = 1$ ,  $ap = 0$ , and  $ap_1 = p_1$ .*

*Proof.* Set  $A_1 = \{a \in A: ap = pa = 0\}$ . Consider  $\tilde{A}_1 \subset \tilde{A}$ . By Lemma II.4,  $p_1$  is  $q$ -closed for  $\tilde{A}$ . Thus the unit ball of  $p_1 A^* p_1 = p_1 \tilde{A}_1^* p_1 = p_1 \tilde{A}_1^* p_1$  is compact for the  $\sigma(\tilde{A}^*, \tilde{A})$  topology, hence also for the weaker  $\sigma(\tilde{A}_1^*, \tilde{A}_1)$ . Thus  $p_1 \tilde{A}_1^* p_1$  is  $\sigma(\tilde{A}_1^*, \tilde{A})$  closed, so  $p_1$  is  $q$ -closed for  $\tilde{A}_1$  [3, II.2]. Now by [4, I.1] there exists  $a \in \tilde{A}_1^+$  with  $\|a\| = 1$ ,  $ap_1 = p_1$  and  $ap_2 = 0$ , where  $p_2$  is the one dimensional projection in  $\tilde{M}_1$  which supports the pure state  $f_\infty$  which vanishes on  $A_1$ . Since  $ap_2 = 0$ ,  $a \in A_1$ , so  $ap = 0$ .

This last Lemma generalizes Urysohn's Lemma. We now define an analog for a continuous function.

**DEFINITION III.2.** A self-adjoint operator  $b \in M$  is *q-continuous* if each spectral projection of  $b$  corresponding to an open subset of the spectrum of  $b$  is also *q-open*.

Now we can state our best Gelfand representation theorem.

**THEOREM III.3.** *The self-adjoint elements of  $A$  are exactly those q-continuous elements  $b$  of  $M$  such that the spectral projections of  $b$  corresponding to closed subsets of the spectrum of  $b$  which don't contain 0 are q-compact (i.e.,  $b$  "vanishes at  $\infty$ ").*

*Proof.* Consider  $A \subset \tilde{A}$ ,  $M \subset \tilde{M}$ . If  $b \in \tilde{A}$ , then  $b \in A$ , since  $b \in M$ . But if  $p$  is the spectral projection of  $b$  corresponding to an open subset  $U$  of the spectrum of  $b$ , we consider two cases. First if  $0 \notin U$ , then  $p \in M$ , hence  $p$  is *q-open* since it is *q-open* for  $A$  by hypothesis. Secondly if  $0 \in U$ , then the complement of  $U$  is closed and doesn't contain 0, thus the spectral projection corresponding to it is *q-compact* for  $A$ , hence *q-closed* for  $\tilde{A}$  by Lemma II.4. Thus  $b$  is *q-continuous* for  $\tilde{A}$  and Theorem I.1 applies.

For the abelian case it is well-known that if  $B$  is a  $C^*$ -algebra of continuous bounded functions on a locally compact Hausdorff space  $\Omega$  such that the smallest topology on  $\Omega$  making all  $b \in B$  continuous agrees with the given topology, then  $B$  contains all continuous functions vanishing at  $\infty$  on  $\Omega$ . A similar result is true in general.

**THEOREM III.4.** *Let  $A_1$  be a  $C^*$ -subalgebra of  $M$  such that the q-open projections for  $A_1$  in  $M$  are the same as the q-open projections for  $A$ . Then  $A_1 \supset A$  and  $A_1 = A$  if  $1 \in A$ .*

*Proof.* Let  $A_2 = A \cap A_1$ . If  $p$  is *q-open* for  $A$ , then  $p = \bigvee_{\alpha} p_{\alpha}$  where  $p_{\alpha}$  is *q-open* with *q-compact* closure. For each  $\alpha$ ,  $p_{\alpha}$  is also  $A_1$  open, so there exists a net  $\{a_{\alpha}^{\gamma}\} \subset A_1$  with  $0 \leq a_{\alpha}^{\gamma} \uparrow p_{\alpha}$ . By hypothesis each  $a \in A_1$  is *q-continuous*, and since  $p_{\alpha}$  has compact closure, Theorem III.3 applies to give  $\{a_{\alpha}^{\gamma}\} \subset A$ , hence in  $A_2$ . Thus  $p$  is  $A_2$  open. We now apply Theorem III.3 of [3] and get  $A_2 = A$ . (Theorem III.3 of [3] is stated for algebras with unit, but considering  $\tilde{A}_2$  and  $\tilde{A}$  we get the result.)

Now if  $1 \in A$ , Theorem I.1 gives that  $A_1 \subset A$ , so  $A_1 = A$ .

Recall that one way of constructing the double centralizer  $M(A)$  of  $A$  is to let  $M(A)$  be the idealizer of  $A$  in  $A^{**}$ , i.e.,

$$M(A) = \{b \in A^{**}: bA + Ab \subset A\}.$$

We first prove a lemma bringing  $M(A)$  into  $M$ .

LEMMA III.5. *The mapping  $b \rightarrow bz$  is a  $*$ -isomorphism of  $M(A)$  into  $M$ .*

*Proof.* Suppose  $b \geq 0$  in  $M(A)$  and  $zb = 0$ . Then let  $a \in A$  with  $0 < a \leq b$ . Then  $za = 0$  since  $za \leq zb = 0$ . This means  $a = 0$ , a contradiction.

From now on consider  $M(A)$  as a subalgebra of  $M$ . A tempting conjecture would be;

Conjecture III.6. The self-adjoint elements of  $M(A)$  are exactly the  $q$ -continuous elements of  $M$ .

Our next result is one half of the conjecture.

THEOREM III.7. *Every self-adjoint element of  $M(A)$  is  $q$ -continuous.*

*Proof.* Let  $\{a_n\} \subset A$  be a positive increasing approximate unit for  $A$ . Let  $b \in M(A)$  be self-adjoint and let  $U$  be an open subset of the spectrum of  $b$  with  $p$  the spectral projection of  $b$  corresponding to  $U$ . Let  $\{b_n\}$  be a sequence of continuous functions of  $b$  with  $0 \leq b_n \uparrow p$ . Then  $\{b_n^{1/2} a_n b_n^{1/2}\}$  is a net in  $A$  which is  $\leq p$  and converges to  $p$ . Thus  $p$  is  $q$ -open for  $A$ .

In [7] Dixmier introduces the ideal center of a  $C^*$ -algebra which is a  $C^*$ -subalgebra of  $M(A)$  containing  $A$ . Dixmier constructs it in  $A^{**}$  but Lemma III.5 assures us the idea carries over to  $M$  as well. We can characterize it in the obvious way.

COROLLARY III.8. *The ideal center of  $A$  consists of exactly those central elements of  $M$  which are  $q$ -continuous.*

*Proof.* We need to show that if  $d$  is central in  $M$  and  $p$ -continuous and  $a \in A$ , then  $da \in A$ . Clearly we need only consider  $d, a \geq 0$  and  $\|d\| = \|a\| = 1$ . For  $\lambda > 0$ , the spectral projection  $p$  of  $(da)$  corresponding to the interval  $[\lambda, \infty)$  is less than or equal to the



spectral projection of  $a$  corresponding to  $[\lambda, \infty)$  which is  $q$ -compact since  $a \in A$ . By III.3 we need only show  $ad$  is  $q$ -continuous.

To show that  $(ad)$  is  $q$ -continuous, let  $(\alpha, \beta)$  be an open interval and consider  $a$  and  $d$  as real functions on  $\sigma(ad)$  (the spectrum of  $ad$ ). Then let  $t_0 \in K = \{t: a(t)d(t) \in (\alpha, \beta)\}$ . For sufficiently small  $\varepsilon$  and  $\delta$  we have  $U \cap V \subset K$ , where  $U = \{t: a(t_0) - \varepsilon < t < a(t_0) + \varepsilon\}$  and  $V = \{t: d(t_0) - \delta < t < d(t_0) + \delta\}$ . Since  $K$  is a union of open sets of the form  $U \cap V$ , the spectral projection  $p$  of  $ad$  in  $M$  corresponding to  $K$  is a union of projections corresponding to sets of the form  $U \cap V$ . But for any  $U$  and  $V$  as above, the spectral projections of  $(ad)$  corresponding to  $U$  and  $V$  are both  $q$ -open and they commute. Hence their intersection corresponds to  $U \cap V$  and it is  $q$ -open [3, II.7]. Thus  $p$  is a union of  $q$ -open projections, hence it is  $q$ -open [3, II.5].

**IV. Assorted results and examples.** One interesting question is: What are all the different  $C^*$ -algebras which have a factor for their pure state  $p$ -space? If  $M$  is countably decomposable, then the question was answered in [13] where it was shown that the  $C^*$ -algebra must consist of exactly the compact operators in  $M$  (i.e., the  $C^*$ -algebra generated by the minimal projections). We can slightly extend this result.

**THEOREM IV.1.** *Suppose  $M$  is a factor. Then  $A$  consists of exactly the compact operators in  $M$  if any  $q$ -open projection  $p$  is countably decomposable.*

*Proof.* Let  $A_0 = \{a \in A: ap = pa = a\}$ . Then the pure state  $q$ -space  $M_0$  of  $A_0$  is  $pMp$ . By [13]  $A_0$  consists of the compact operators in  $pMp$ . Thus  $A$  contains all the compact operators in  $M$  by [9, p.85]. But if  $A$  is strictly larger than the compact operators, then they form an ideal in  $A$ , so  $A$  has at least two inequivalent irreducible representations. This contradicts the assumption that  $M$  is a factor.

Next is a theorem of the Stone-Weierstrass type.

**THEOREM IV.2.** *Let  $B \subset A$  be a  $C^*$ -subalgebra which separates the pure states of  $A$  and 0. If  $pBp$  is norm closed in  $M$  for each  $q$ -closed projection  $p$  for  $A$ , then  $B = A$ .*

*Proof.* By [3, III.2]  $M$  is also the pure state  $q$ -space for  $B$ . Let  $p_1$  be the  $B$ -closure of  $p$  in  $M$  (i.e., the smallest projection  $\geq p$  which is  $q$ -closed for  $B$ ). If  $p_1 > p$ , then there is a minimal projection  $p_2$  in  $M$  with  $p_2 \leq p_1 - p$ . Let  $\{b_\alpha\} \subset B$  with  $1 \geq b_\alpha \downarrow p_2$  in  $M$ . Then

$\|p_1 b_\alpha p_1\| = 1$  for all  $\alpha$ , but  $\|p b_\alpha p\| \xrightarrow{\alpha} 0$  since  $p$  is  $q$ -closed. By [3, II.12] the map  $B \rightarrow p_1 B p_1$  has closed range, and by hypothesis the map  $p_1 B p_1 \xrightarrow{\varphi} p B p$  has closed range also. But since  $p_1$  is the  $q$ -closure of  $p$  for  $B$ , the map  $\varphi$  is 1-1. Thus  $\varphi^{-1}$  is continuous by the closed graph theorem, and this contradicts  $\|p_1 b_\alpha p_1\| = 1, \|p b_\alpha p\| \xrightarrow{\alpha} 0$ .

The most difficult aspect of the  $q$ -theory is the existence of non-regular projections, even in the best of circumstances [4, I.2]. The next result shows that some interesting projections are regular.

**PROPOSITION IV.3.** *If  $p'$  is finite-dimensional, then  $p$  is regular.*

*Proof.* Let  $p_1$  be the  $q$ -closure of  $p$ . Then  $p'_1$  is finite dimensional, so  $p'_1$  is  $q$ -closed [3, II.8]. Hence  $p_1$  is  $q$ -open and  $q$ -closed, so  $p'_1 \in A$  by [3, II.18]. By considering  $p_1 A p_1$ , we can assume  $p_1 = 1$ . Let  $b \in A$  with  $\|b\| = 1$  and suppose  $\|bp\| < 1$ . This would be the case if  $p$  were not regular. Since  $\|b^*b\| = 1$  and  $\|b^*bp\| < 1$ , we can assume  $b > 0$ . Let  $p_2$  be the spectral projection of  $b$  corresponding to the open interval  $(\delta, \infty)$ , where  $\|bp\| < \delta < 1$ . Then  $p_2$  is  $q$ -open and  $p_2 \neq 0$ , so  $p_2 \wedge p \neq 0$  as follows. If  $p_2 \wedge p = 0$ , then  $p'_2 \vee p' = 1$ . Since  $p'$  is finite dimensional, this implies that  $p_2$  is finite dimensional. But then  $p_2 \in A$ , so we can get a minimal projection  $p_3 \in A$  with  $p_3 \leq p'$ . This contradicts  $\bar{p} = 1$ . Now if  $g$  is a pure state of  $A$  with  $g(p_2 \wedge p) = 1$ , then

$$g(bp) = g(b) = g(p_2 b p_2) \geq g(\delta p_2) = \delta.$$

This contradicts the definition of  $\delta$ .

The next proposition and example show how badly behaved non-regular projections can be and how reasonable regular projections are.

**PROPOSITION IV.4.** *If  $p \in M$  is regular,  $f$  a pure state of  $A$ ,  $b \in A$  with  $b \geq p$  and  $f(b) = 0$ , then  $f(\bar{p}) = 0$  ( $\bar{p}$  = closure of  $p$ ).*

*Proof.* Let  $\{a_\alpha\}$  be an increasing positive approximate unit for  $\{a \in A: f(a^*a + aa^*) = 0\}$ . By Lemma II.3 and by [2, I.2] we can get  $\{b_n\} \subset A$  with  $b_n \geq p$ ,  $\|b_n\| \leq 1 + 1/n$ ,  $f(b_n) = 0$ . Let  $p_1$  be the support projection of  $f$ . If  $f(\bar{p}) \neq 0$ , then there exists a pure state  $g$  of  $A$  with  $g(\bar{p}) = 1$  and  $g(p_1) \neq 0$ . By regularity and [10, 6.1] there exists a net  $\{g_\gamma\}$  of states of  $A$  with  $g_\gamma \xrightarrow{\gamma} g$ ,  $\sigma(A^*, A)$ , and  $g_\gamma(p) = 1$  for all  $\gamma$ . Let  $b_0$  be a limit point of  $\{b_n\}$  for the weak\* topology of  $M$ , clearly  $\|b_0\| \leq 1$ . Since  $g_\gamma(b_n) \geq g_\gamma(p) = 1$  for all  $\gamma$

and all  $n$ , then  $g(b_n) \geq 1$  for all  $n$ . Hence  $g(b_0) \geq 1$ . But  $\|b_0 + p_1\| = 1$  since  $b_n p_1 = 0$  for all  $n$  implies  $b_0 p_1 = 0$  (and similarly  $p_1 b_0 = 0$ ). Hence  $g(b_0 + p_1) \geq 1 + g(p_1) > 1$ , contradicting the assumption that  $\|g\| = 1$ .

EXAMPLE IV.5. Let us work in the direct sum  $\sum_{n=1}^{\infty} \oplus B(H_n)$  of matrix algebras where dimension  $H_n = 2$  for all  $n$ . Set

$$a = \sum_{n=1}^{\infty} \begin{pmatrix} 1/n & 0 \\ 0 & 0 \end{pmatrix}, \quad p = \sum_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \sum_{n=1}^{\infty} \begin{pmatrix} 1 - \gamma_n & (\gamma_n - \gamma_n^2)^{1/2} \\ (\gamma_n - \gamma_n^2)^{1/2} & \gamma_n \end{pmatrix}$$

where  $\{\gamma_n\}_{n=1}^{\infty}$  is an enumeration of the rationals between 0 and 1 which contains each rational an infinite number of times. Set  $b = p + q$  and let  $A$  be the  $C^*$ -algebra generated by  $a$  and  $b$ . Let  $p_1$  be the range projection of  $a$  in  $M$ .

*Conclusions from the example.* (1)  $b \geq p_1$  but there is no  $d \in A^+$  with  $dp_1 = p_1$  (c.f., [12] page 11, line 11). (2) If  $f$  is the pure state at  $\infty$  for  $A$ , then  $f(b) = 0$  but  $f(\bar{p}_1) \neq 0$ , so  $p_1$  is nonregular by Proposition IV.4. (3) Let  $p_2$  be the support projection of  $f$ . Then  $p_1 + p_2$  satisfies the intersection condition of Conjecture II.2, but  $p_1 + p_2$  is not  $q$ -closed.

If  $\varphi: A_1 \rightarrow A_2$  is a  $*$ -homomorphism of  $A_1$  onto  $A_2$ , we may easily extend it to a normal  $*$ -homomorphism of  $M_1$  onto  $M_2$ . However if  $\varphi$  is not onto, this extension may not be possible. The natural representation of the continuous function on the interval  $[0, 1]$  into the algebra of all bounded operators on  $L^2[0, 1]$  by  $\varphi(f)h = fh$  has no such extension (the proof was communicated to me by R. Giles). In order to place  $q$ -theory into a category theory setting, one must restrict the class of allowable “*morphisms*” between two  $C^*$ -algebras. The following restriction is empty in the abelian case.

PROPOSITION IV.6. A  $*$ -homomorphism  $\varphi$  taking the  $C^*$ -algebra  $A_1$  into the  $C^*$ -algebra  $A_2$  has a normal extension  $\tilde{\varphi}: M_1 \rightarrow M_2$  (necessarily unique) if and only if  $\varphi$  is continuous for the topologies generated by the seminorms  $\|a\|_f = f(a^*a)$  for all pure states  $f$  of  $A_1$  (or  $A_2$  for the topology on  $A_2$ ).

*Proof.* It  $\tilde{\varphi}$  exists, the continuity is automatic for  $\tilde{\varphi}$ , hence for  $\varphi$ . The converse follows immediately from [14, p. 3 of appendix].

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# Pacific Journal of Mathematics

Vol. 39, No. 1

May, 1971

Charles A. Akemann, <i>A Gelfand representation theory for <math>C^*</math>-algebras</i> . . . .	1
Sorrell Berman, <i>Spectral theory for a first-order symmetric system of ordinary differential operators</i> . . . . .	13
Robert L. Bernhardt, III, <i>On splitting in hereditary torsion theories</i> . . . . .	31
J. L. Brenner, <i>Geršgorin theorems, regularity theorems, and bounds for determinants of partitioned matrices. II. Some determinantal identities</i> . . . . .	39
Robert Morgan Brooks, <i>On representing <math>F^*</math>-algebras</i> . . . . .	51
Lawrence Gerald Brown, <i>Extensions of topological groups</i> . . . . .	71
Arnold Barry Calica, <i>Reversible homeomorphisms of the real line</i> . . . . .	79
J. T. Chambers and Shinnosuke Oharu, <i>Semi-groups of local Lipschitzians in a Banach space</i> . . . . .	89
Thomas J. Cheatham, <i>Finite dimensional torsion free rings</i> . . . . .	113
Byron C. Drachman and David Paul Kraines, <i>A duality between transpotence elements and Massey products</i> . . . . .	119
Richard D. Duncan, <i>Integral representation of excessive functions of a Markov process</i> . . . . .	125
George A. Elliott, <i>An extension of some results of Takesaki in the reduction theory of von Neumann algebras</i> . . . . .	145
Peter C. Fishburn and Joel Spencer, <i>Directed graphs as unions of partial orders</i> . . . . .	149
Howard Edwin Gorman, <i>Zero divisors in differential rings</i> . . . . .	163
Maurice Heins, <i>A note on the Löwner differential equations</i> . . . . .	173
Louis Melvin Herman, <i>Semi-orthogonality in Rickart rings</i> . . . . .	179
David Jacobson and Kenneth S. Williams, <i>On the solution of linear G.C.D. equations</i> . . . . .	187
Michael Joseph Kallaher, <i>On rank 3 projective planes</i> . . . . .	207
Donald Paul Minassian, <i>On solvable <math>O^*</math>-groups</i> . . . . .	215
Nils Øvrelid, <i>Generators of the maximal ideals of <math>A(\bar{D})</math></i> . . . . .	219
Mohan S. Putcha and Julian Weissglass, <i>A semilattice decomposition into semigroups having at most one idempotent</i> . . . . .	225
Robert Raphael, <i>Rings of quotients and <math>\pi</math>-regularity</i> . . . . .	229
J. A. Siddiqi, <i>Infinite matrices summing every almost periodic sequence</i> . . . .	235
Raymond Earl Smithson, <i>Uniform convergence for multifunctions</i> . . . . .	253
Thomas Paul Whaley, <i>Multiplicity type and congruence relations in universal algebras</i> . . . . .	261
Roger Allen Wiegand, <i>Globalization theorems for locally finitely generated modules</i> . . . . .	269