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A square matrix $A = [a_{ij}]_1^n$ has dominant diagonal if $\forall_i\{|a_{ii}| > R_i = \sum_{j\neq i} |a_{ij}|\}$. A more complicated type of dominance is the following. Suppose for each i, there is assigned a set $I(i)$ (subset of $\{1, \cdots, n\}$), $i \in I(i)$: Define B_{ij} as the $I(i) \times I(i)$ submatrix of A that uses columns $I(i)$, and rows $\{I(i)\backslash i, j\}$, i.e., the set obtained from $I(i)$ by replacing the ith row by the jth row. Set $b_{ij} = \det B_{ij}$. Then $[b_{ij}]_1^n$ is a matrix, the elements of which are determinants of minor matrices of A. In an earlier paper, bounds for det A were derived in case $[b_{ij}]$ has dominant diagonal in the special case that $\{I(i)\}_i$ represents a partitioning of the indices into disjoint subsets.

In this article the general case is treated; $I(i)$ can be any subset of $\{1, \cdots, n\}$ that contains i. An identity is derived connecting $\det [b_{ij}]_1^n$ with $\det A$.

To establish the identity, a general multinomial identity is first derived, connecting determinants of certain submatrices of an $r \times 2r$ matrix of indeterminates. This result, reminiscent of Sylvester's determinantal identity, is used to bound $\det A$.

1. Application of a characterization of the determinant function.

LEMMA 1.01. *Let $A = [a_{ij}]_1^n$ be a matrix of complex numbers* [or indeterminates]; *let a function $\phi: A \to C$*[or $\phi: A \to C[a_{11}, \cdots, a_{nn}]$] *have the following properties for all $n \times n$ matrices A.*

(1.02) [1.03] *If any row* [*column*] *of A is replaced by the sum of that row* [*column*] *and a multiple of another row* [*column*], *$\phi(A)$ is unaltered.*

(1.04) *If any row of A is multiplied* (*throughout*) *by a constant α, $\phi(A)$ is multiplied α^r.*

Then $\phi(A)$ is a constant c_0 (independent of a_{ij}) multiplied by the rth power of det A.

Proof. The hypotheses (1.02, 1.03) guarantee that $\phi(A)$ is the same as $\phi(B)$, where B is any matrix obtainable from A by means of elementary transformations. It is known that $B = \text{diag}\,[\det A, 1, \cdots, 1]$ can be so obtained; see for example [1]. Thus $\phi(A)$ is some function of $\det A$; the conclusion of lemma 1.01 follows on applying hypothesis 1.04 to the matrix B: If $\phi(\alpha x) = \alpha^r \phi(x)$, then $\phi(x) \equiv c_0 x^r$, since $\phi(x)/x^r$ is constant.

An application of this result was made in [2], to which the reader should refer. In slightly changed notation, this application is as follows.

LEMMA 1.05. *Let* $A = [a_{ij}]_{i=1,j=1}^{r,\ 2r}$ *be an* $r \times 2r$ *matrix of indeterminates, let* $b_{ij} = \det A\begin{pmatrix} 1 & \cdots & r \\ 1 & \cdots & i-1, i+1, \cdots, r, j \end{pmatrix}$ *be the determinant of the* $r \times r$ *submatrix of* A *that uses columns* $\{1, \cdots, r\}\backslash i, j$. *This is the almost-principal submatrix of* A *in which the ith column is replaced by the jth column.* (For $j = i$, this is $A\begin{pmatrix} 1 & \cdots & r \\ 1 & \cdots & r \end{pmatrix}$. For $1 \leqq j \neq i \leqq r$, this submatrix has determinant 0.)

Then

$$X = \det\,[b_{ij}]_{i=1,j=r+1}^{r,\ 2r} = G_1^{r-1} \det\,[a_{ij}]_{i=1,j=r+1}^{r,\ 2r}\,, \tag{1.06}$$

where

$$G_1 = \det\,[a_{ij}]_{11}^{rr}\,.$$

Note that in 1.06, the column indices are $r+1, \cdots, 2r$.

To prove this Lemma, it is only necessary to observe that it is a multinomial identity, and that the hypotheses of Lemma 1.01 concerning the function X are satisfied.

1° if X is regarded as a function of $\{a_{ij}, 1 \leqq i, j \leqq r\}$;

2° if X is regarded as a function of $\{a_{ij}, 1 \leqq i \leqq r, r < j \leqq 2r\}$.

COROLLARY 1.07. *With the same hypothesis, the conclusion*

$$Y = \det\,[b_{ij}]_{i=1,j\in S}^{r} = G_1^{r-1} \det\,[a_{ij}]_{i=1,j\in S}^{r} \tag{1.08}$$

is valid, where S *is any set of* r *distinct positive integers not exceeding* $2r$.

Proof. Since 1.06 is a multinomial identity, the r^2 indeterminates $a_{ij}\,(j > r)$ on the right can simply be replaced by the r^2 indetеminates $a_{ij}\,(j \in S)$. But this replacement changes not only the range of j in the set variables $\{a_{ij}\}$, but also the range of j in the set $\{b_{ij}\}$, as the definition of b_{ij} shows.

LEMMA 1.09. *Suppose*

$$I(1) = \{1\}, I(2) = \{1, 2\}, \cdots, I(r) = \{1, 2, \cdots, r\} .$$

Let $B = \{b_{ij}\}_{i=1,j=1}^{r,\ 2r}$ *be defined as in* 1.05. *Then*

$$\begin{aligned} &\det B\binom{1, \cdots, r}{r+1, \cdots, 2r} \\ (1.10) \qquad &= a_{11} \det A\binom{12}{12} \det A\binom{123}{123} \cdots \det A\binom{12 \cdots r-1}{12 \cdots r-1} \\ &\times \det A\binom{1, 2, \cdots, r}{r+1, \cdots, 2r} . \end{aligned}$$

REMARK. This is again a multinomial identity in the $2r^2$ indeterminates a_{ij}. Therefore 1.09 has the Corollary

$$(1.11) \quad \det B\binom{1 \cdots r}{j_1 \cdots j_r} = a_{11} \det A\binom{12}{12} \det A\binom{123}{123} \cdots \det A\binom{1\,2 \cdots r}{j_1 \cdots j_r}$$

in view of the definition of b_{ij}.

Proof of Lemma 1.09. To show that a_{11} is a factor in (1.10), as shown, a_{21} times the first row is added to the second row. The second row becomes

$$(1.13) \qquad a_{11}a_{2,r+1},\ a_{11}a_{2,r+2},\ \cdots,\ a_{11}a_{2,r+j},\ \cdots$$

which obviously has a_{11} as a factor.

It is a little more complicated to show $\det \begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix}$ is also a factor, as is asserted in relation (1.10). The trick is to add to the third row $-\det \begin{pmatrix} a_{21}a_{22} \\ a_{31}a_{32} \end{pmatrix}$ times the first row as well as $a_{11}^{-1} \det \begin{pmatrix} a_{11}a_{12} \\ a_{31}a_{32} \end{pmatrix}$ times the second row (1.13). The new third row is

$$(1.14) \qquad \det \begin{pmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{pmatrix} [a_{3,r+1},\ a_{3,r+2},\ \cdots,\ a_{3,r+j},\ \cdots] ,$$

i.e., every element of that row has the common prefactor indicated.

The formal proof of (1.10) is inductive, as follows. As an induction hypothesis, assume that the left member of (1.10) can be written in the form

$$(1.15) \qquad a_{11} \det A\binom{12}{12} \cdots \det A\binom{12 \cdots k-1}{12 \cdots k-1} \det C_k ,$$

where C_k is the $r \times r$ matrix, the jth column of which is

$$\begin{array}{c} a_{1,r+j} \\ a_{2,r+j} \\ \vdots \\ a_{k,r+j} \end{array}$$

$$\det \begin{bmatrix} a_{11} \cdots a_{1k} a_{1,r+j} \\ a_{k1} \cdots a_{kk} a_{k,r+j} \\ a_{k+1,1} \cdots a_{k+1,r+j} \end{bmatrix} \\ \vdots \quad .$$

This has already been established for $k = 1, 2$. The inductive assertion is: the factor $\det A\binom{12 \cdots k}{12 \cdots k}$ splits off from $\det C_k$. To prove this, subtract from the $k+1$st row of the matrix C_k appropriate multiples of the preceding rows. The multiple of $a_{i,r+j}$ needed is precisely the cofactor of $a_{i,r+j}$ in C_k itself.

This completes inductive proof. To establish (1.10) in its entirety, a final visual check is needed of the circumstance that for $k = r$, the matrix C_r is indeded the matrix $A\binom{1 \cdots r}{r+1 \cdots 2r}$. See (1.05).

2. Some special factorizations.

THEOREM 2.01. *Let $A = [a_{ij}]$ be a matrix with r rows: $i = 1(1)r$, and $2r$ columns: $j = 1(1)r < j_1 < \cdots < j_r$. Suppose, for $i = 1, 2, \cdots, r-1$, $I(i) = \{1, 2, \cdots, r-1\}$; $I(r) = \{1, 2, \cdots, r\}$. For $j = j_1, j_2, \cdots, j_r$ set $B_{ij} = A\binom{I(i)}{I(i)\backslash i, j}$, $i = 1(1)r$.*

Denote $\det B_{ij}$ *by* b_{ij}; $B = [b_{ij}]$. *Then*

$$\det B = \pm\, C^{r-1} \det A\binom{1, 2, \cdots, r}{j_1 j_2 \cdots j_r};\ C = \det A\binom{I(1)}{I(1)}\ . \tag{2.02}$$

Proof. Consider the last row of B. The element b_{rj} in column "j" of this row is the determinant of the $r \times r$ matrix B_{rj}. If this determinant is expanded by minors of the elements $a_{rj}, a_{r1}, a_{r2}, \cdots a_{r,r-1}$ of the last row of B_{rj}, the result is

$$b_{rj} = \pm a_{rj} C \pm a_{r1} b_{1j} \pm a_{r2} b_{2j} \pm \cdots \pm a_{r,r-1} b_{r-1,j}\ . \tag{2.03}$$

Relation (2.03) shows that $\det B$ is not altered if every element b_{rj} of the last row of B is replaced by $\pm a_{rj} C$. (This replacement would merely omit from the last row of B a linear combination of the preceding rows.)

At this point it is clear that C is a factor of $\det B$, and that the other factor has the same first $r-1$ rows does B, and has last row a_{rj}. The conclusion of the theorem now follows by expanding $\det B$ by its last row and applying Corollary 1.07. See Lemmas 4.3, 4.4 of [2].

COROLLARY 2.04. *Suppose*

$$I(i) = \{1, 2, \cdots, r-k\} \text{ for } i = 1, 2, \cdots, r-k \, ;$$

and $I(i) = \{1, 2, \cdots, r-k, i\}$ *for* $i = r-k+1, \cdots, r$. *Then* (2.02) *holds; where C now means* $\det A\binom{1, 2, \cdots, r-k}{1, 2, \cdots, r-k}$.

2.05. Another special case is the case $I(1) = \{1, 2\}, I(2) = \{2, 3\}, I(3) = \{3, 1\}$. The formula

$$\text{(2.06)} \qquad \det B = G \det A, \qquad G = \det \begin{bmatrix} a_{11} & -a_{12} & 0 \\ 0 & a_{22} & -a_{23} \\ -a_{31} & 0 & a_{23} \end{bmatrix}$$

can be verified by appropriate devices. A generalization of (2.06) is the formula

$$\text{(2.07)} \qquad \det B\binom{1\ 2\ 3}{j_1 j_2 j_2} = G \cdot \det A\binom{1\ 2\ 3}{j_1 j_2 j_3},$$

valid for any 3×6 matrix A, with $I(i)$ defined as above. Among several valid proofs of this formula, the following is presented. It proves (2.07) as a special case of a still more general result.

THEOREM 2.08. *Let* $A = [a_{ij}]$ *be an* $r \times 2r$ *matrix,* $i = 1(1)r$, $j = 1(1)2r$. *Let B be the* $r \times r$ *matrix with* (i, j) *element* $b_{ij} = \det B_{ij}$, *where* $B_{ij} = A\binom{i \ \ i+1}{j \ \ i+1}$, $i = 1(1)r-1$, $B_{rj} = A\binom{r1}{j1}$; $j = r+1(1)2r$. *Then the relation*

$$\text{(2.09)} \quad \det B = G \det A\binom{1 \cdots r}{r+1 \cdots 2r}, \quad G = \det \begin{bmatrix} a_{11} & -a_{12} & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & & -a_{r-1,r} \\ -a_{r,1} & & & & a_{rr} \end{bmatrix}$$

holds; G is a bidiagonal matrix with $2r$ *nonzero elements.*

REMARK. This is the case $I(1) = \{1, 2\}, I(2) = \{2, 3\}, \cdots, I(r) = \{r, 1\}$.

Proof. Subtract a multiple of the first row of B from the second, then a multiple of the second from the third, $\cdots$, a multiple of the

$r - 1st$ from the last. The resulting matrix has the same determinant as B, and the multiples mentioned can be chosen so that this resulting matrix is, row by row,

$$\begin{array}{rr} a_{22}[a_{1j}] - a_{12}[a_{2j}] & \cdots 1 \\ (a_{22}a_{33}/a_{12})[a_{1j}] - a_{23}[a_{3j}] & \cdots 2 \\ (a_{22}a_{33}a_{44}/a_{12}a_{23})[a_{1j}] - a_{34}[a_{4j}] & \cdots 3 \\ \vdots & \vdots \\ (a_{11}a_{22} \cdots a_{rr}/a_{12} \cdots a_{r-1r} - a_{r1})[a_{1j}] & \cdots r\,. \end{array}$$

Now subtract a multiple of the new last row from each of the preceding rows; the first $r - 1$ rows of the new matrix are $-a_{12}[a_{2j}]$, $-a_{23}[a_{3j}]$, $\cdots$. This matrix obviously has determinant (2.09). ||

3. General factorization of $\det B$. The function $i \mapsto I(i)$ induces a (weak) separation of the indices $\{1, \cdots, n\}$ into agglomerated mutually exclusive sets $S(k)$, as follows.

DEFINITION 3.01. Let $i \mapsto I(i)$ be a function from the integers $\{1, \cdots, n\}$ to sets of these same integers, with the further property $i \in I(i)$ for all i. In the usual way, the sets $I(i)$ are now agglomerated into the smallest possible (minimal) *mutually exclusive* sets $S(k)$ so that: *Every $I(i)$ is in one or another of the sets $S(k)$.* Then $S(k)$ are the *mutually separated sets defined by the function I.* For example, the function

$$1 \longmapsto \{1\},\ 2 \longmapsto \{1, 2\},\ 3 \longmapsto \{1, 2, 3\},\ 4 \longmapsto \{4, 5\},\ 5 \longmapsto \{5, 6\},$$
$$6 \longmapsto \{6, 7\},\ 7 \longmapsto \{7\}$$

defines a separation of the indices $\{1, 2, 3, 4, 5, 6, 7\}$ into the mutually exclusive sets $S(1) = \{1, 2, 3\}$, $S(2) = \{4, 5, 6, 7\}$.

Parallel to the separation of Definition 3.01, there is a factorization of $\det B$ into a product of factors, one for each set $S(k)$. The kth factor is the determinant of a matrix; in general the elements of this matrix are again determinants of matrices: the elements of these matrices are elements a_{ij} of the matrix A, where $i, j \in S(k)$. The point is that the polynomial function $\det B$ of the elements of A factors into the product of multinomial factors; the kth factor is a polynomial in the indeterminates a_{ij}, where i, j belong only to the kth set $S(k)$ of indices. Besides these factors, $\det A$ also appears as a factor.

It there are two or more sets $S(k)$ in the separation, then $\det A$, but not $(\det A)^2$, is thus a factor of $\det B$. Even when the entire set $\{1, 2, \cdots, n\}$ of indices are connected through the sets I (there is but a single set S), the factor $\det A$ appears only to first power "in general." The exact meaning of "in general" is explained below.

The above remarks are summarized in the following theorem. Its proof, together with a more detailed atatement, unfold in § 4.

THEOREM 3.02. *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix of indeterminates; for* $i = 1(1)n$ *let* $I(i)$ *be a subset of the first* n *integers with* $i \in I(i)$. *Denote by* B_{ij} *the minor* $A\binom{I(i)}{I(i)\backslash i,\ j}$ *on rows* $I(i)$; *and on columns* $I(i)$, *but with index* i *replaced by* j. *Set* $b_{ij} = \det B_{ij}$; $B = [b_{ij}]$. *Thus* B *is an* $n \times n$ *matrix. Let the function* $I(i)$ *induce a separation of the indices* $\{1, \cdots, n\}$ *into* $s \geqq 1$ *mutually exclusive sets* $S_1, S_2, \cdots, S_s$. *Then* $\det B$, *which is obviously a polynomial function of the* n^2 *indeterminates* a_{ij} *with integer coefficients, can be factored in the form*

$$\det B = G \det A\ ,$$

where $G = M_1 M_2 \cdots M_s$, *and where each* M_k *is a multinomial in those indeterminates* a_{ij} *for which both indices* i, j *belong to the set* S_k. *In particular, det* A *is always a factor of det* B.

The details of the proof depend on the following lemma.

LEMMA 3.03. *Let* $A = [a_{ij}]$ *be an* $r \times 2r$ *matrix of indeterminates,* $i = 1(1)r$, $j = 1(1)2r$. *For each* i, *let* $I(i)$ *be a subset of the first* r *integers. Let* B_{ij}, b_{ij} *be defined formally as in Theorem* 3.02. B_1 *is the* $r \times r$ *matrix* $[b_{ij}]$, $1 \leqq i \leqq r < j \leqq 2r$. A_1 *is the* $r \times r$ *matrix* $[a_{ij}]_{1 \leqq i \leqq r < j \leqq 2r}$. (Note the range for j.)

Then the polynomial identity

$$\det B_1 = F \cdot \det A_1 \tag{3.04}$$

holds, where F is a multinomial with integer coefficients in the indeterminates $\{a_{ij}, 1 \leqq i, j \leqq r\}$.

REMARK 3.05. This lemma is more general than any of previous ones, since the sets $I(i)$ are more general.

COROLLARY 3.06. *Det* A_1 *is, but* $(\det A_1)^2$ *is not a factor of det* B_1.

Proof. The variables that figure in F are disjoint from those in B_1.

REMARK 3.07. This is the meaning of the phrase "in general" above.

COROLLARY 3.08. *Let* A_1, B_1 *redefined conformally. That is,* *without changing the sets $I(i)$, let the range for j in the definitions of A_1, B_1 be replaced by any range of r distinct integers, including some or all of the first r integers. Then* (3.04) *still holds.*

Proof. If some of the indices j in the polynomial $\det A_1$ are changed, the definition of b_{ij} shows that a conformal change is concurrently made in the polynomial $\det B_1$. In other words, the change amounts solely to a change of the names of the variables in (3.04). But (3.04) is a polynomial identity.

Under the change $a_{i,j} \to a_{i,j-r}$, $b_{i,j} \to b_{i,j-r}$ in (3.04), the factor $\det A_1$ could appear as a factor in F for suitable choice of $I(i)$. For example, if $I(i) \equiv \{1, 2, \cdots, r\}$, and if j runs through the range $1 \leqq j \leqq r$, then (3.04) becomes $\det B_1 = (\det A_1)^r$.

Proof of Lemma 3.03. To avoid difficulties with an algebraic sign, the columns of $B_{ij} \equiv A\binom{I(i)}{(I(i)\backslash i, j)}$ are to be thought of as written in a definite order: the jth column a_{ij} first, followed by the other columns in natural order. For example, if $I(1) = \{1, 2, 3\}$ then B_{1j} is the matrix

$$\begin{bmatrix} a_{1j} & a_{12} & a_{13} \\ a_{2j} & a_{22} & a_{23} \\ a_{3j} & a_{22} & a_{33} \end{bmatrix}.$$

Without this convention, the formula to be obtained for F would be determined only up to sign.

It will be instructive to carry through the proof in a special case, since a rather simple special case already embodies all the points of difficulty and interest. The case $I(1) = \{1, 2\}$, $I(2) = \{1, 2, 3\}$, $I(3) = \{1, 2, 3\}$ will serve as an illustration. The matrix B_1 has as jth column B_{1j}, where

$$B_{1j} = \begin{bmatrix} \det \begin{bmatrix} a_{1j} & a_{12} \\ a_{2j} & a_{22} \end{bmatrix} \\ \det \begin{bmatrix} a_{1j} & a_{11} & a_{13} \\ a_{2j} & a_{21} & a_{23} \\ a_{3j} & a_{31} & a_{33} \end{bmatrix} \\ \det \begin{bmatrix} a_{1j} & a_{11} & a_{12} \\ a_{2j} & a_{21} & a_{22} \\ a_{3j} & a_{31} & a_{32} \end{bmatrix} \end{bmatrix} \quad j = 4, 5, 6\,. \tag{3.09}$$

The first step in the proof is to border the 3×3 matrix B_1 with 3 rows and columns as shown below. The enlarged matrix B_2 clearly has the same determinant as B_1, except for the factor $(-1)^r$. Only the subscripts are printed; thus $1j$ is an abbreviation for a_{1j}. The reader must also supply the symbol det throughout: [ ] is an abbreviation for det [ ].

$$\begin{pmatrix} 14, & 15, & 16, & 1, & 0, & 0 \\ 24, & 25, & 26, & 0, & 1, & 0 \\ 34, & 35, & 36, & 0, & 0, & 1 \\ \begin{bmatrix} 14 & 12 \\ 24 & 22 \end{bmatrix}, & \begin{bmatrix} 15 & 12 \\ 25 & 22 \end{bmatrix}, & \begin{bmatrix} 16 & 12 \\ 26 & 22 \end{bmatrix}, & 0, & 0, & 0 \\ \begin{bmatrix} 14 & 11 & 12 \\ 24 & 21 & 23 \\ 34 & 31 & 33 \end{bmatrix}, & \begin{bmatrix} 15 & 11 & 13 \\ 25 & 21 & 23 \\ 35 & 31 & 33 \end{bmatrix}, & \begin{bmatrix} 16 & 11 & 13 \\ 26 & 21 & 23 \\ 36 & 31 & 33 \end{bmatrix}, & 0, & 0, & 0 \\ \begin{bmatrix} 14 & 11 & 12 \\ 24 & 21 & 22 \\ 34 & 31 & 32 \end{bmatrix}, & \begin{bmatrix} 15 & 11 & 12 \\ 25 & 21 & 22 \\ 35 & 31 & 32 \end{bmatrix}, & \begin{bmatrix} 16 & 11 & 12 \\ 26 & 21 & 22 \\ 36 & 31 & 32 \end{bmatrix}, & 0, & 0, & 0 \end{pmatrix}.$$

To show that the factor $\det A_1$ splits off from the determinant of this 6×6 matrix, it need only be noted that the matrix can be reduced to the form $\begin{bmatrix} A_1 & I \\ 0 & F_1 \end{bmatrix}$ by adding appropriate linear combinations of the first three rows to each of the last three. This argument is an alternative to a general argument of Loewy [3], who proved by another method that if $\det A_1 = 0$, then necessarily $\det B_1 = 0$. In the special case being expounded, $\det B_2 = -(\det F_1)(\det A_1)$, where F_1 is the 3×3 matrix

$$\begin{pmatrix} a_{22}, & -a_{12}, & 0 \\ \begin{bmatrix} 21 & 23 \\ 31 & 33 \end{bmatrix}, & -\begin{bmatrix} 11 & 13 \\ 31 & 33 \end{bmatrix}, & \begin{bmatrix} 11 & 13 \\ 21 & 23 \end{bmatrix} \\ \begin{bmatrix} 21 & 22 \\ 31 & 32 \end{bmatrix}, & -\begin{bmatrix} 11 & 12 \\ 31 & 32 \end{bmatrix}, & \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix} \end{pmatrix}.$$

The argument given above has general applicability. Formula (3.04) is established. The multinomial F is in fact the determinant of an $r \times r$ matrix. The (k, l) element of this matrix is the negative of the cofactor of $a_{l\,r+l}$ in $b_{k,r+l} = \det A\begin{pmatrix} I(k) \\ I(k)\backslash k,\ r+l \end{pmatrix}$, and is thus

$$f_{kl} = -(-1)^{1+\text{pos}\,l} \det A\begin{pmatrix} I(k)\backslash l \\ I(k)\backslash k \end{pmatrix},$$

where $\text{pos}\, l$ is the position of l in the set $I(k)$. If $l \notin I(k)$, then $f_{kl} = 0$, and conversely. For consistency, f_{kk} must be defined as 1 when $I(k) = \{k\}$.

COROLLARIES.

(3.09) $$\det B_1 = (-1)^r(\det F_1)(\det A_1)$$

3.10 [3] If $\det A_1 = 0$, then $\det B_1 = 0$.

3.11. If F_1 is a triangular matrix, then

(3.12) $$\det B_1 = -(-1)^r \Pi(\det G_1^{(i)}) \cdot (\det A_1), \text{ where}$$

(3.13) $$G_1^{(i)} = A\binom{I(i)\backslash i}{I(i)\backslash i}.$$

In particular, relation (1.10) follows; this proof differs from the first proof.

(3.14) In case $I(1)=\{1,2\}$, $I(2)=\{2,3\}$, $\cdots$, $I(i)=\{i, i+1\}$, $\cdots$, $I(n)=\{n, 1\}$, then formula

(3.15) $\det B_1 = G \cdot \det A_1$ holds, where $G = \det\begin{bmatrix} a_{11} & -a_{12} & \\ & a_{22} & -a_{23} \\ & \cdots & \end{bmatrix}$ is the determinant of the bidiagonal matrix shown. This proof is again different from the earlier proof of (2.09).

3.16. Note that the case $I(1) = \{1, 2, 3\}$, $I(2) = \{2, 3, 4\}$, $\cdots$ is considerably more complicated than the case (3.14); indeed while the first type of proof is more direct for the hypothesis (3.14), an attempt to generalize this proof to the case (3.16) is unrewarding.

3.17. *Relation* (1.06) *holds*.

The following proof of 1.06 is somewhat less direct than the original proof. The matrix F_1 is not triangular, so that the determinant $\det F_1$ does not factor for this simple reason. However F_1 is seen on inspection to be the $r-1st$ compound of the matrix $A\binom{I(1)}{I(1)}$; thus $\det F_1 = \det A\binom{I(1)}{I(1)}^{r-1}$. This proof requires a knowledge of the formula

(3.18) $\det C^{(t)} = (\det C)^e$, $e = \binom{r-1}{t-1}$, where $C^{(t)}$ is the tth compound of the $r \times r$ matrix C.

4. General factorization of det B (continued). In this section, Corollary 3.08 is applied to obtain a general formula for the determinant of the $n \times n$ matrix $B = [b_{ij}]$ defined in Theorem 3.02.

Since Theorem 3.02 holds for a matrix A of indeterminates, it holds in particular for a matrix A of complex numbers.

Proof of Theorem 3.02. The function $i \mapsto I(i)$ induces a separation of the indices $\{1, 2, \cdots, n\}$ into $s \geqq 1$ *mutually exclusive sets* $S(k)$ such that *every set* $I(i)$ *is in exactly one of the sets* $S(k)$, and the sets $S(k)$ cannot be further decomposed without destroying these properties.

In following the details of the proof, the reader may prefer to think of the indices of the sets $S(1), S(2), \cdots$ as occuring in natural order.

To continue the proof, the rows of B are partitioned into (mutually exclusive) sets $S(1), S(2), \cdots$ and $\det B$ is expanded according to the generalized Laplace expansion on these rows. Corollary 3.08 asserts that the determinants of all the $S(1) \times S(1)$ minor matrices on the set of rows with indices in $S(1)$ have a common factor M_1. The corollary asserts further that this common factor is a multinomial in the particular variables a_{ij} $(i, j \in S(1))$. Similarly for $S(2), \cdots$. Thus $M_1 M_2 \cdots M_s$ is a factor of $\det B$.

Besides the factor common to the determinants of all the $S(1) \times S(1)$ matrices, there is a factor, see (3.04), peculiar to the particular minor matrix. This peculiar factor is just what is needed, in the Laplace expansion of $\det B$, to produce $\det A$. The proof of Theorem 3.02 is complete.

Let A be a matrix of indeterminates. If there is more than one set $S(k)$, then $\det A$ is, but $(\det A)^2$ is not, a factor of $\det B$.

5. Applications. Theorem 3.02 can be used to obtain bounds for $\det A$ in case the matrix B has dominant diagonal. The details and results are similar to those of [2]. These results have one remarkable feature: This is the first occasion on which such bounds have been obtained for a "partitioning" of a matrix, in which the sets of rows in the "partitioning" overlap one another.

The results of this paper will be needed in any attempt to obtain minimal Geršgorin sets related to the Hoffman-Brenner theorem. If it can be accomplished, this will be an interesting generalization of the results of [5].

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