ON REPRESENTING $F^*$-ALGEBRAS

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The purpose of this paper is to obtain a concrete representation for $F^*$-algebras with identity: a Frechet algebra with involution for which there exists a determining sequence of $B^*$-seminorms. The main result is Theorem 3.4 which is described here. Let $A$ be an $F^*$-algebra with identity. Let $\{(\tau_\lambda, H_\lambda): \lambda \in A\}$ be a complete family of irreducible Hilbert space representations of $A$. Let $H = \Sigma_\lambda \oplus H_\lambda$, define $E \subseteq A$ to be equicontinuous provided $\sup_{\lambda \in E} ||\tau_\lambda(a)|| < \infty (a \in A)$, and let $X = \{x \in H: \text{Supp}(x) \text{ is equicontinuous}\}$. The linear space $X$ is given the final topology $\tau_f$ determined by the family $\{H_E = \{x \in H: \text{Supp}(x) \subseteq E\}; E \text{ equicontinuous}\}$ of subspaces of $X$. Let $X_f$ be $(X, \tau_f)$ and let $L^*(X_f)$ be all operators on $X$ which have an adjoint relative to the inner product inherited from $H$ such that both the operator and its adjoint are $\tau_f$-continuous. This algebra will be endowed with the topology $\mathcal{I}_f$ of bounded convergence. Let $L^+(X)$ be all operators which have adjoints. It has a natural topology $\mathcal{I}_+$ described in §2. Define $\pi: A \to L^+_d(X)$ by $\pi(a)(x_\lambda) = \{\tau_\lambda(a)x_\lambda\}$ for $a \in A$ and $x = \{x_\lambda\} \in X$. Then $\pi(A) \subseteq L^+(X) = L^*(X_f)$, and (1) $\pi: A \to (L^*(X_f), \mathcal{I}_f)$ is a topological $^*$-isomorphism (into) and (2) $\pi: A \to (L^+(X), \mathcal{I}_+)$ is a topological $^*$-isomorphism (into).

In §1 we recall some results about Frechet $^*$-algebras with identity, their positive functionals and Hilbert space representations, and set the notation for the remainder of the paper.

In §2 we obtain the results about algebras of operators on certain inner product spaces necessary to prove the main representation theorem.

In §4 we define the concept of an enveloping algebra $E(A)$ for a Frechet $^*$-algebra with identity, $A$, and show that $E(A)$ can be realized either as the inverse limit of the enveloping algebras of the Banach $^*$-algebras in an inverse limit decomposition of $A$ or as an algebra of operators naturally constructed from the irreducible Hilbert space representations of $A$. Also we show that $E(A)$ has the property that every Hilbert space representation of $A$ factors through $E(A)$, but that there are representations of $A$ in algebras $L^+(X)$ which fail to factor through $E(A)$.

1. Preliminaries. A Frechet algebra is a complete metrizable topological algebra whose topology is determined by a (countable) family of seminorms (submultiplicative, convex, symmetric function-
als). We may assume that such a family \( \{ || \cdot ||_n \}_{n=1}^{\infty} \) for \( A \) is ascending: \( ||a||_n \leq ||a||_{n+1} (a \in A, n \in N) \), and that \( ||e||_n = 1 (n \in N) \) if \( A \) has an identity \( e \). A Fréchet *-algebra is a Fréchet algebra with a continuous involution. If \( A \) is a Fréchet *-algebra with identity we can choose a sequence \( \{ || \cdot ||_n \}_{n=1}^{\infty} \) of seminorms for \( A \) such that (1) \( \{ || \cdot ||_n \} \) determines the topology of \( A \), (2) \( \{ || \cdot ||_n \} \) is ascending, (3) \( ||e||_n = 1 (n \in N) \), and (4) \( ||a^*||_n = ||a||_n (a \in A, n \in N) \). Such a sequence we shall call a *-sequence of seminorms for \( A \). An \( F^* \)-algebra is a Fréchet *-algebra, for which there is an ascending determining sequence \( \{ || \cdot ||_n \} \) of seminorms for \( A \) each of which has the \( \ast \)-property: \( ||a^*a||_n = ||a||_n^2 (a \in A, n \in N) \). Such a sequence we shall call an \( F^* \)-sequence of seminorms. The usual constructions (see [5]) show that every Fréchet *-algebra (resp., \( F^* \)-algebra) is an inverse limit of Banach *-algebra (resp., \( B^* \)-algebras).

Let \( (A, \{ || \cdot ||_n \}) \) be a Fréchet *-algebra with identity \( e \). We denote by \( P(A) \) the set of all positive functionals on \( A \) and by \( K(A) \) those \( f \in P(A) \) for which \( f(e) = 1 \). For each \( n \in N \) we let \( P_n(A) \) (resp., \( K_n(A) \)) be the set of all \( f \in P(A) \) (resp., \( K(A) \)) such that \( |f(a)| \leq f(e)||a||_n (a \in A) \). If \( \{ \rho_n, \rho^*_n, N \} \) is the inverse system generated by \( \{ || \cdot ||_n \} \) with \( \rho_n: A \to A_n \) natural map of \( A \) onto the \( n \)th member \( A_n \), then for each \( n \) the dense homomorphism \( \rho_n \) induces a one-to-one map \( \rho^*_n \) of \( P(A_n) \) onto \( P_n(A) \). \((K(A))\) onto \( K_n(A) \)). Moreover, \( \rho^*_n \) preserves indecomposability. A theorem of Do-Shing [2] states every positive functional on \( A \) is continuous so we have \( P(A) = \cup_{n=1}^{\infty} P_n(A) \) and \( K(A) = \cup_{n=1}^{\infty} K_n(A) \). Also, \( K(A) \) is a weak* closed, convex subset of \( A^* \) and is the closed convex hull of its extreme points \( \text{ext} (K(A)) \) which is exactly \( \cup_{n=1}^{\infty} \text{ext} (K_n(A)) \).

A Hilbert space representation of a Fréchet *-algebra \( A \) with identity is a \( \ast \)-homomorphism \( \mu: A \to \mathcal{B}(H) \) for a Hilbert space \( H \). A consequence of Do-Shing’s theorem (see Lemma 3.1 below) is that every such representation is continuous. Moreover, there is a one-to-one correspondence between the members of \( K(A) \) and the equivalence classes of cyclic Hilbert space representations of \( A \) (with unit cyclic vectors). This correspondence matches elements of \( K_n(A) \) with those representations which can be factored through \( A_n \). Also, the indecomposable positive functionals on \( A \) correspond to classes of irreducible Hilbert space representations of \( A \).

The \( \ast \)-radical, \( R^*(A) \), of \( A \) is the set \( \{ a \in A: f(a^*a) = 0 \ (f \in P(A)) \} = \{ a \in A: f(a^*a) = 0 \ (f \in \text{ext} (K(A))) \} \cap \{ \ker \pi: \pi \text{ is an irreducible Hilbert space representation of } A \} \). If \( A \) is an \( F^* \)-algebra with identity, then \( R^*(A) = (0) \). Hence, if we let \( A \) be all equivalence classes of irreducible Hilbert space representations of \( A \) and for each \( \lambda \in \Lambda \) we choose \( \pi_\lambda \in \lambda \) with representation space \( H_\lambda \), then \( \{ (\pi_\lambda, H_\lambda): \lambda \in \Lambda \} \) is a
complete family of irreducible Hilbert space representations of $A$. A family constructed in this manner for a Fréchet $\star$-algebra $A$ will be called a standard family of irreducible Hilbert space representations of $A$. If $\{\pi_\lambda, H_\lambda\} : \lambda \in \Lambda$ is a standard family for $A$ and we let $E_n = \{\lambda : \pi_\lambda$ factors through $A_n\}$, then for each $\lambda \in E_n$ there exists a unique irreducible representation $\sigma_\lambda$ of $A_n$ on $H_\lambda$ so that $\sigma_\lambda \circ \rho_n = \pi_\lambda$. The family $\{\sigma_\lambda : \lambda \in E_n\}$ is a complete family of irreducible representations for $A_n$ (in case $A$ is $F^*$) and the direct sum $\sum_{\lambda \in E_n} \sigma_\lambda$ on $\bigoplus_{\lambda \in E_n} H_\lambda$ is an isometry and $\ast$-isomorphism of $A_n$ into $\mathcal{B}(\bigoplus_{\lambda \in E_n} H_\lambda)$.

We have included no proofs of the facts quoted above since those concerning Fréchet $\ast$-algebras are proved for the more general class of locally $m$-convex $\ast$-algebras in [1], and those relating to Banach $\ast$-algebras can be found in [6].

2. Certain operator algebras. In this section we obtain the results about special algebras of operators on direct sums and inductive limits of Hilbert spaces which we need in the proof of the main representation theorem in § 3. The concepts considered in the first part of this section are discussed in detail in G. Lassner’s work [4].

We first establish our notation. If $X$ is a complex vector space we denote by $\mathcal{L}_a(X)$ the algebra of all linear transformations on $X$. If $X$ has a locally convex topology $\tau$ we denote by $\mathcal{L}(X)$, or by $\mathcal{L}(X, \tau)$ if there are several topologies on $X$ in the discussion, the subalgebra of $\mathcal{L}_a(X)$ consisting of all $\tau$-continuous operators. For a locally convex TVS $(X, \tau)$ we denote by $\mathcal{S}$ the family of all $\tau$-bounded subsets of $X$ (with an appropriate subscript on $\mathcal{S}$ if there are several topologies on $X$). The topology of bounded convergence $\mathcal{B}$ is the topology on $\mathcal{L}(X)$ with base at $0$ $\{Nbd(M, U) : M \in \mathcal{S}, U \text{ an } \tau\text{-neighborhood of } 0 \text{ in } X\}$, where $Nbd(M, U) = \{T \in \mathcal{L}(X) : T(M) \subseteq U\}$.

**Definition.** Let $X$ be an inner product space with inner product $(\cdot, \cdot)$, and let $H$ be the completion of $X$. $\mathcal{L}^+(X)$ is the subset of $\mathcal{L}_a(X)$ which consists of all $T \in \mathcal{L}_a(X)$ which have an adjoint in $\mathcal{L}_a(X)$: there exists $S \in \mathcal{L}_a(X)$ such that $(Tx, y) = (x, Sy)(x, y \in X)$.

**Lemma 2.1.** (Lassner) $\mathcal{L}^+(X)$ is a $\ast$-algebra with involution $T \mapsto T^\ast$. Also, (1) each $T \in \mathcal{L}^+(X)$ is closed, (2) if $X = H$, then $\mathcal{L}^+(X) = \mathcal{B}(H)$, and (3) if there is a closed operator in $\mathcal{L}^+(X)$, then $X = H$.

**Proof.** This is merely a compilation of Lemmas 2.1 and 2.2 of [4].

**Definition.** An $Op^\ast$-algebra on $X$ is a $\ast$-subalgebra $\mathfrak{A}$ with
identity $I$ of $\mathcal{L}^+(X)$, (i.e., the identity of $\mathfrak{A}$ is the identity operator on $X$).

**Definition.** Let $\mathfrak{A}$ be an Op*-algebra on $X$. We define a locally convex topology $\tau_\mathfrak{A}$ by taking as a sub-basic family of seminorms $\{|| \cdot ||_T: T \in \mathfrak{A}\}$, where $||x||_T = ||Tx|| (x \in X)$. This is the coarsest topology on $X$ with respect to which each operator in $\mathfrak{A}$ is a continuous map into $H$.

Lassner shows [Lemma 3.1, 4] that each $T \in \mathfrak{A}$ is a continuous linear transformation on $(X, \tau_\mathfrak{A})$. Since $I \in \mathfrak{A}$ it follows that $\tau_\mathfrak{A}$ is finer than the norm topology of $H$ restricted to $X$.

**Definition.** Let $\mathfrak{A}$ be an Op*-algebra on $X$. We define two topologies $\mathcal{I}_\mathfrak{A}$ and $\mathcal{I}_\mathfrak{A}^*$ on $\mathfrak{A}$ by:

1. $\mathcal{I}_\mathfrak{A}$ is defined the family $\{|| \cdot ||_M: M \in \mathcal{S}_\mathfrak{A}\}$ of seminorms where, (a) $\mathcal{S}_\mathfrak{A}$ is the family of $\tau_\mathfrak{A}$-bounded subset, of $X$ and (b) $||T||_M = \sup \{|(Tx, y)|: x, y \in M\}$.
2. $\mathcal{I}_\mathfrak{A}^*$ is the restriction to $\mathfrak{A}$ of the topology $\mathcal{I}_\mathfrak{A}$ on $\mathcal{L}(X, \tau_\mathfrak{A})$.

**Lemma 2.2.** (Lassner) If $\mathfrak{A}$ is an Op*-algebra on $X$, then, 

1. $(\mathfrak{A}, \mathcal{I}_\mathfrak{A})$ is a locally convex algebra (separate continuity of multiplication), but the involution is not in general continuous.
2. $(\mathfrak{A}, \mathcal{I}_\mathfrak{A}^*)$ is a locally convex algebra with continuous involution.
3. $\mathcal{I}_\mathfrak{A} \leq \mathcal{I}_\mathfrak{A}^*$, and $\mathcal{I}_\mathfrak{A} = \mathcal{I}_\mathfrak{A}^*$ if, and only if, the multiplication in $(\mathfrak{A}, \mathcal{I}_\mathfrak{A})$ is (jointly) continuous.

**Proof.** This is a compilation of Theorems 4.1 and 4.2 and Example 5.1 of [4].

**Notation.** For the maximal Op*-algebra $\mathcal{L}^+(X)$ on $X$ we shall let $\tau_+$ and $\mathcal{I}_+$ replace the clumsier notation $(\tau_{\mathcal{L}^+(X)}, \mathcal{I}_{\mathcal{L}^+(X)})$ of the definitions above.

We now specialize to a particular class of inner product spaces. Let $\{H_\beta: \beta \in B\}$ be a family of Hilbert spaces, let $H = \Sigma_\beta \perp H_\beta$ and let $X = \Sigma_\beta H_\beta$ (the algebraic direct sum). For $\beta \in B$ we let $p_\beta: X \to H_\beta$ be the natural projection and let $q_\beta: H_\beta \to X$ be the natural injection. For $x \in X$ we define $\text{Supp}(x) = \{\beta: p_\beta(x) \neq 0\}$.

The locally convex direct sum topology, $\tau_f$, on $X$ is the final topology determined by the family $\{q_\beta: \beta \in B\}$. We shall abbreviate $(X, \tau_f)$ by $X_f$.

**Lemma 2.3.** If $X = \Sigma_\beta H_\beta$, then $\mathcal{L}^+(X)$ is isomorphic to the algebra of all $B^2$-matrices $(T_{\alpha \beta})_{\alpha, \beta \in B}$ such that
(1) \( T_{a\beta} \in \mathcal{L}(H_{\beta}, H_{\alpha})(\alpha, \beta \in B) \),
(2) for each \( \alpha \in B \) the set \( B_{\alpha} = \{ \beta : T_{a\beta} \neq 0 \} \) is finite, and
(3) for each \( \beta \in B \) the set \( B_{\beta} = \{ \alpha : T_{a\beta} \neq 0 \} \) is finite.

Proof. If \((T_{a\beta})\) is a matrix satisfying (1)—(3) we define \( T : X \to X \) by \( T(x_{\beta}) = \sum_{\alpha} T_{a\beta} x_{\beta} \) for each \( a \). Since \( \text{Supp}(x) \) is finite, \( \sum_{\alpha} T_{a\beta} x_{\beta} \) converges in \( H_{\alpha} \) for each \( a \in B \) and it is easily seen that the set of \( \alpha \) for which \( \sum_{\beta} T_{a\beta} x_{\beta} \) is nonzero is contained in \( \bigcup \{ B_{\beta} : \beta \in \text{Supp}(x) \} \), a finite set. Thus, \( T \in \mathcal{L}(X) \) and by considering the matrix \((T^{*}_{\beta\alpha}) = (T_{\beta\alpha}^{*}) : H_{\beta} \to H_{\alpha} \) we obtain an adjoint for \( T \) in \( \mathcal{L}^{*}(X) \); hence, \( T \in \mathcal{L}^{*}(X) \).

Fix \( \beta \in B \). If \( B_{\beta} \) is not finite, then there exists a sequence \( \{x_{\beta}^{j}\} \) in \( H_{\beta} \) such that \( T_{\alpha\beta} x_{\beta} = 0 \) for \( j \geq n(x_{\beta}) \). We choose sequences \( \{n_{j}\} \) in \( N \) and \( \{x_{\beta}^{j}\} \) in \( H_{\beta} \) by the following procedure. Let \( n_{1} = 1 \) and choose \( x_{\beta}^{1} \in H_{\beta} \) so that \( ||x_{\beta}^{1}|| < 2^{-1} \) and \( T_{x_{\beta}^{1}} x_{\beta} = 0 \) (hereafter \( T_{j} \) will be used for \( T_{x_{\beta}^{j}} \)). There exists \( n_{2} > n_{1} \) such that \( T_{j} x_{\beta} = 0 \) for \( j \geq n_{2} \). Choose \( x_{\beta}^{2} \in H_{\beta} \) such that \( T_{n_{2}} x_{\beta} = 0 \) and \( ||x_{\beta}^{2}|| < \min(2^{-2}, 2^{-2} ||T_{n_{1}} x_{\beta}|| ||T_{n_{1}}||) \). Continuing inductively we obtain sequences \( \{n_{j}\} \) and \( \{x_{\beta}^{j}\} \) so that:

(1) \( 1 = n_{1} < n_{2} < \ldots \).
(2) \( ||x_{\beta}^{j}|| < \min(2^{-j}, 2^{-j} ||T_{n_{j+1}} x_{\beta}|| ||T_{n_{j+1}}||, \ldots, 2^{-j} ||T_{n_{j-1}} x_{\beta}|| ||T_{n_{j-1}}||) \)
(3) \( T_{n_{j}} x_{\beta} = 0 \)
(4) \( T_{n_{j}} x_{\beta} = 0 \) (\( i > j \)).

We let \( x = \sum_{j=1}^{\infty} x_{\beta}^{j} \in H_{\beta} \). We claim that \( T_{n_{k}} x = 0 \) (\( k \geq 1 \)). Fix \( k \in N \), then \( T_{n_{k}} x = \sum_{j=k}^{\infty} T_{n_{k}} x_{\beta}^{j} + \sum_{j=k+1}^{\infty} T_{n_{k}} x_{\beta}^{j} \). For \( j \leq k - 1 \) we have \( T_{n_{k}} x_{\beta}^{j} = 0 \), and for \( j > k + 1 \) we have \( ||T_{n_{k}} x_{\beta}^{j}|| \leq ||T_{n_{k}}|| ||x_{\beta}^{j}|| \). If \( T_{n_{k}} x = 0 \), then, \( ||T_{n_{k}} x_{\beta}^{j}|| \leq ||T_{n_{k}}|| ||x_{\beta}^{j}|| \leq \sum_{j=k+1}^{\infty} 2^{-j} \sum_{j=k+1}^{\infty} 2^{-j} ||T_{n_{k}} x_{\beta}^{j}|| < ||T_{n_{k}} x_{\beta}^{j}|| \), a contradiction.

That \( B_{\alpha} \) is finite for each \( \alpha \in B \) follows by applying the same argument to \( T^{*} \).

Lemma 2.4. Let \( \{X_{\beta}\} \) be a family of Banach spaces and let \( X = \sum_{\beta} X_{\beta} \). For \( c \in R^{+} \) define \( p_{c} : X \to R^{+} \) by \( p_{c}(x) = \sum_{\beta} c_{\beta} ||x_{\beta}|| \). Then
\[ \{p_c : c \in \mathbb{R}_+^\beta \} \text{ defines the locally convex direct sum topology } \tau_f \text{ on } X. \]

**Proof.** Clearly, \( \{p_c \} \) defines a separated locally convex topology \( \tau' \) on \( X \). Since \( \tau_f \) is the final topology generated by the injections \( q_\beta : X_\beta \to X(\beta \in B) \), it suffices to show (1) if \( p \) is a \( \tau_f \)-continuous seminorm on \( X \), then there exists \( c \in \mathbb{R}_+^\beta \) so that \( p \leq p_c \) (hence, \( \tau_f \leq \tau' \)), and (2), for each \( \beta \in B \) the map \( q_\beta : X_\beta \to (X, \tau') \) is continuous.

(1) Fix a \( \tau_f \)-continuous seminorm \( p \) on \( X \). For each \( \beta \in B \) the map \( p \circ q_\beta : X_\beta \to \mathbb{R} \) is a continuous seminorm on \( X_\beta \). Hence, there exists \( c_\beta \in \mathbb{R}_+ \) so that \( p \circ q_\beta(x_\beta) \leq c_\beta \|x_\beta\| \) \( (x_\beta \in X_\beta) \). This defines the function \( c \in \mathbb{R}_+^\beta \). If \( x = \{x_\beta\} \in X \), then,

\[
    p(x) = p(\sum_\beta q_\beta(x_\beta)) \leq \sum_\beta p \circ q_\beta(x_\beta) \\
    \leq \sum_\beta c_\beta \|x_\beta\| = p_c(x) .
\]

(2) Fix \( \beta \in B, c \in \mathbb{R}_+^\beta \). Then \( p_c(q_\beta x_\beta) = c_\beta \|x_\beta\| \) and \( q_\beta : X_\beta \to (X, \tau') \) is continuous.

**Lemma 2.5.** Let \( X = \sum_\beta H_\beta \). For each \( c \in \mathbb{R}_+^\beta \) we define \( \| \cdot \|_c : X \to \mathbb{R} \) by \( \|x\|_c = (\sum_\beta c_\beta \|x_\beta\|)^{1/2} \) for \( x = \{x_\beta\} \in X \). If \( B \) is countable then \( \tau_f \) is defined by the seminorms \( \{\| \cdot \|_c : c \in \mathbb{R}_+^\beta \} \).

**Proof.** Suppose \( B = N \). We have for each \( c \in \mathbb{R}_+^\beta \) that \( \| \cdot \|_c \leq p_c \), so \( \tau_{\{\| \cdot \|_c\}} \leq \tau_f \). We fix \( c \in \mathbb{R}_+^\beta \)

For \( x \in X \) we have:

\[
    p_c(x) = \sum_\alpha n^{-1}(nc_n \|x_n\|) \\
    \leq (\sum_\alpha n^{-1/2})(\sum_\alpha (nc_n^2 \|x_n\|^2)^{1/2} \\
    = (\text{constant}). \|x\|_{\| \cdot \|_c} .
\]

**Theorem 2.6.** If \( X = \sum_\beta H_\beta \), then \( \mathcal{L}^+(X) \subseteq \mathcal{L}(X_f) \); hence, \( \mathcal{L}^+(X) = \mathcal{L}^*(X_f) \), the algebra of continuous operators on \( X_f \) with continuous adjoints.

**Proof.** It suffices to show that for each \( T \in \mathcal{L}^+(X) \) and \( \beta \in B \) the operator \( T \circ q_\beta : H_\beta \to X_f \) is continuous (see [Prop. 6.1, p. 54, 7]). Fix \( T \in \mathcal{L}^+(X) \), \( \beta \in B \) and a seminorm \( p_c, c \in \mathbb{R}_+^\beta \), for \( \tau_f \). The set \( B^\beta = \{ \alpha : T_{\alpha \beta} \neq 0 \} \) is finite, so for \( x_\beta \in H_\beta \) we have:

\[
    p_c(T \circ q_\beta(x_\beta)) = \sum_\alpha c_\alpha \| (T \circ q_\beta(x_\beta))_\alpha \| \\
    = \sum_{\alpha \in B^\beta} c_\alpha \| T_{\alpha \beta} x_\beta \| .
\]

Hence, \( T \circ q_\beta \) is continuous.

We now turn from direct sums to inductive limits. Let \( \{H_\lambda : \lambda \in \Lambda \} \) be a family of Hilbert spaces. Let \( H = \bigoplus H_\lambda \). We fix a
family \( \mathcal{E} \) of subsets of \( A \) which satisfies (a) \( \mathcal{E} \) is closed under finite unions, and (b) all subsets of a member of \( \mathcal{E} \) belong to \( \mathcal{E} \) (i.e., \( \mathcal{E} \) is an ideal in the lattice \( 2^A \)). We let \( X = \{ x \in H : \text{Supp}(x) \in \mathcal{F} \} \), and for \( E \in \mathcal{E} \) we let \( H_E = \{ x \in H : \text{Supp}(x) \subseteq E \} \), a Hilbert space, and let \( i_{E_n} \) be the identity injection of \( H_E \) into \( X \). Finally, let \( \tau_f \) be the final topology on \( X \) determined by the family \( \{ i_E : E \in \mathcal{E} \} \) of injections.

Since \( X \) has an inner product the \( \mathcal{O}_f^* \)-algebra \( \mathcal{L}^+(X) \) is defined. A subalgebra of \( \mathcal{L}^+(X) \) of importance here is \( \mathcal{L}_f(X) = \{ T \in \mathcal{L}^+(X) : \text{for each } E \in \mathcal{E}, T(H_E) \subseteq H_E \text{ and } T^*(H_E) \subseteq H_E \} \); i.e., \( \mathcal{L}_f(X) \) consists of all elements of \( \mathcal{L}^+(X) \) which are reduced by each \( H_E(E \in \mathcal{E}) \).

We denote the topologies on \( X \) determined by \( \mathcal{L}^+(X) \) and \( \mathcal{L}_f(X) \) by \( \tau_+ \) and \( \tau_f \) (respectively) and the corresponding families of bounded sets by \( \mathcal{L} \) and \( \mathcal{L}_f \).

**ASSUMPTION.** Throughout the remainder of this section we assume the existence of an ascending countable cofinal (with respect to the partial order \( \subseteq \) on \( \mathcal{E} \)) subfamily \( \mathcal{E}_n = \{ E_n \}_{n=1}^\infty \). We let the corresponding Hilbert spaces \( H_{E_n} \) and injections \( i_{E_n} \) be denoted \( H_n \) and \( i_n(n \in \mathbb{N}) \).

**Lemma 2.7.** The final topology on \( X \) generated by the family \( \{ i_n \}_{n=1}^\infty \) is \( \tau_f \). Hence, \( (X, \tau_f) \) is a strict inductive limit of the sequence of Hilbert spaces \( \{ H_n \} \) and;

1. \( \tau_f | H_n \) is the norm topology on \( H_n \).
2. \( M \subseteq X \) is \( \tau_f \)-bounded if, and only if, there exists \( n \in \mathbb{N} \) such that \( M \) is a (norm) bounded subset of \( H_n \).

**Proof.** That the final topologies are the same can either be easily proved directly or deduced from Proposition 3, p. 159 of [3]. That we have a strict inductive limit and (1) follow from the fact that \( \tau_{n+1} | H_n = \tau_n \) (trivial if one writes out the norm of an element of \( H_n \) considered as an element of \( H_{n+1} \)) and a theorem of Dieudonné-Schwartz (see [pp. 159-160, 3]). Claim (2) is another theorem of Dieudonné-Schwartz (see [p. 161, 3]).

**Theorem 2.8.** \( \mathcal{L}_f = \mathcal{L}_f \)

**Proof.** We show first that \( \mathcal{L}_f(X) \) is a subalgebra of \( \mathcal{L}(X_f) \). Fix \( T \in \mathcal{L}_f(X), n \in \mathbb{N} \). We must show that \( T \circ i_n = T | H_n : H_n \rightarrow X_f \) is continuous. But \( T(H_n) \subseteq H_n \) and \( \tau_f | T_n = \tau_n \). Thus, we must show that \( T | H_n : H_n \rightarrow H_n \) is continuous. Since \( T \in \mathcal{L}_f(X) \) we have that \( T^*(H_n) \subseteq H_n \), so \( (T | H_n)^* = T^* | H_n \) and \( T | H_n \) has an everywhere defined adjoint on \( H_n \), hence is continuous.

Let \( M \in \mathcal{L}_f \). Then \( M \) is a bounded subset of \( H_n \) for some \( n \in \mathbb{N} \).
For each \( T \in \mathcal{L}(X) \), \( T(M) \) is a bounded subset of \( H^* \) since \( T \mid H^* \) is continuous on \( H^* \). Hence, \( T(M) \) is bounded in \( H \). Since \( T \) was arbitrary, \( M \in \mathcal{I}_r \).

Suppose \( M \in \mathcal{I}_r \). Case (a). There exist sequences \( \{n_j\} \) in \( N \) and \( \{x_j\} \) in \( M \) so that:

1. \( 1 = n_1 < n_2 < \cdots \)
2. \( x_j \in H_{n_{j+2}} \setminus H_{n_j} (j = 1, 2, \ldots) \).

Choose \( x_i \in M \setminus H_{n_i} (n_i = 1) \) let \( n_2 \) be sufficiently large that \( \text{Supp} (x_i) \subseteq E_{n_2} \), choose \( x_2 \in M \setminus H_{n_2} \), etc. Let \( D_j = E_{n_j} \setminus E_{n_{j-1}} (n_0 = 0, E_0 = \emptyset) \), and set \( C_j = (\sum_{\lambda \in D_j} ||x_{j,\lambda}||)^{1/2} \). Define \( T: X \to X \) by \( (Tx)_j = jC_j^{-1} x_j \) if \( \lambda \in D_j \). It is easily verified that \( T \in \mathcal{L}_a(X), \ T^* = T, \) and \( \text{Supp} (T) \subseteq \text{Supp} (x) \)

for \( x \in X \). Hence, \( T \in \mathcal{I}_r(X) \). Also,

\[
||Tx||^2 = \sum_{j=1}^n \sum_{\lambda \in D_j} (jC_j^{-1})^2 ||x_{n,\lambda}||^2 
\geq (nC_n^{-1})^2 \sum_{\lambda \in D_n} ||x_{n,\lambda}||^2 = n^2 .
\]

Hence, \( \sup_{x \in M} ||Tx|| = \infty \), and \( M \in \mathcal{I}_r \). Case (b). \( M \subseteq H^*_w \), for some \( n \), but is unbounded. Easy to show that \( M \in \mathcal{I}_r \).

**Theorem 2.9.** \( \mathcal{I}_r = \mathcal{I}_b \mid \mathcal{L}_r(X) \), where \( \mathcal{I}_b \) is the topology of bounded convergence on the algebra \( \mathcal{L}(X) \).

**Proof.** (1) \( \mathcal{I}_r \preceq \mathcal{I}_b \) on \( \mathcal{L}_r(X) \): Fix a \( \mathcal{I}_r \)-neighborhood of 0 in \( \mathcal{L}_r(X) \), \( \text{Nbd} (M, \varepsilon) = \{ T: ||T||_M < \varepsilon \} \), where \( M \in \mathcal{I}_r, \varepsilon > 0 \). Since \( \mathcal{I}_r = \mathcal{I}_f \) there exists \( n \in N \) so that \( M \subseteq H_n \) and \( ||M|| = \sup_{x \in M} ||x|| < \infty \).

Let \( U \) be a \( \tau_r \)-neighborhood of 0 in \( X \) so that \( U \cap H_n \subseteq S_n(0, (2 ||M||)^{-1} \varepsilon) \), the \( (2 ||M||)^{-1} \varepsilon \)-ball about 0 in \( H_n \). If \( T \in \mathcal{L}_r(X) \cap \text{Nbd} (M, U) = \{ S \in \mathcal{L}_r(X): S(M) \subseteq U \} \), then \( T(M) \subseteq U \cap H_n \) and \( ||Tx|| < (2 ||M||)^{-1} \varepsilon \) for \( x \in M \). Now

\[
||T||_M = \sup \{ \|(Tx, y)\|: x, y \in M \} 
\leq \sup \{ \sup_{x \in M} ||(Tx, y)|| \mid \varepsilon \leq ||M|| \} 
\leq \sup \{ ||M|| \cdot ||Tx|| \} \leq \varepsilon/2 < \varepsilon .
\]

This shows that \( \mathcal{I}_r \preceq \mathcal{I}_b \).

(2) \( \mathcal{I}_b \preceq \mathcal{I}_r \) on \( \mathcal{L}_r(X) \): Fix a \( \mathcal{I}_b \)-neighborhood of 0 in \( \mathcal{L}_r(X) \), \( \text{Nbd} (M, U) \) where \( M \in \mathcal{I}_f \), \( U \) is a \( \tau_r \)-neighborhood of 0 in \( X \). There exists \( n \in N \) so that \( M \) is a bounded subset of \( H_n \). Let \( M_i = M \cup S_n(0, 1) \), bounded subset of \( H^*_w \); hence, a \( \tau_r \)-bounded subset of \( X \). Choose \( \varepsilon > 0 \) so that \( S_n(0, \varepsilon) \subseteq U \cap H_n \). Suppose \( T \in \text{Nbd} (M_i, \varepsilon) \). If \( x \in M \), then

\[
||Tx|| = \sup_{||y|| \leq 1} |(Tx, y)| \leq \sup \{ ||(Tx, y)||: z, y \in M \} = ||T||_{M_i} < \varepsilon .
\]

So \( T(M) \subseteq S_n(0, \varepsilon) \subseteq U \), and \( T \in \text{Nbd} (M, U) \). Hence \( \mathcal{I}_b \preceq \mathcal{I}_r \).
We now show that $\mathcal{L}^+(X)$ is a subalgebra of $\mathcal{L}(X_f)$. The problem here is that we do not have an obvious characterization of elements of $\mathcal{L}^+(X)$. We have a fixed cofinal sequence $\{E_n\}$ in $\mathcal{S}$. We let $D_n = E_n \setminus E_{n-1}$. Let $K_n = \sum_{i \in D_n} \oplus H_i$ and let $Y = \sum_n K_n$.

**Lemma 2.10.** The map $u: X \to Y$ defined by $u([x_\lambda]_{\lambda \in \Lambda}) = \{[x_\lambda]_{\lambda \in D_n}\}_{n=1}^\infty$ has the following properties:

1. $u$ is a linear isomorphism (onto)
2. $u$ is unitary: $(u(x), u(x')) = (x, x')$ for $x, x' \in X$.
3. $u: (X, \tau_f) \to (Y, \tau_f)$ is topological.

**Proof.** It is easily verified that $u$ is a linear isomorphism (into). If $y \in Y$, then there exists $n \in N$ so that $y_j = 0$ for $j > n$. Then $y_j = 0$ if $\lambda \in D_j, j > n$. Set $x = \{y_{k_\lambda}: \lambda \in D_k, k = 1, 2, \ldots \}$. Then $x \in H$ and $\text{Supp}(x) \subseteq \bigcup_{n=1}^\infty D_j = E_n$. Clearly $u(x) = y$, and (1) is proved. That $u$ is unitary depends only on the fact that the series obtained by taking inner products is absolutely convergent so can be rearranged at will.

(3) We show first that $u$ is continuous. We recall Lemma 2.5 and fix a seminorm $|| \cdot ||_c, c \in \mathbb{R}^+$. If $x \in H_n$, then:

$$|| u(x) ||_c^2 = \sum_{j=1}^n c_j^2 || u(x)_j ||^2$$

$$= \sum_{j=1}^n c_j^2 \{ \sum_{i \in D_j} || x_i ||^2 \}$$

$$= \sum_{i \in D_j} \{ \sum_{i \in D_j} || x_i ||^2 \}$$

$$\leq (\max_{1 \leq j \leq n} c_j^2) || x ||^2 .$$

Thus, $|| u(x) ||_c \leq C(n) || x ||$ for $x \in H_n$ and $u \circ i_n: H_n \to Y_f$ is continuous for arbitrary $n \in N$. Hence, $u$ is continuous $(X_f \to Y_f)$.

Since $X_f$ and $Y_f$ are $(LF)$ spaces (each is a strict inductive limit of Hilbert spaces) and $u$ is a continuous surjection of $X_f$ to $Y_f$ it follows that $u$ is an open map (see [Prop 2.2, p. 78, 7]).

**Theorem 2.11.** $\mathcal{L}^+(X) \subseteq \mathcal{L}(X_f)$; hence, $\mathcal{L}^+(X) = \mathcal{L}^*(X_f)$, the subalgebra of $L(X_f)$ which consists of all operators $T \in \mathcal{L}(X_f)$ whose adjoint $T^*$ exists and belongs to $\mathcal{L}(X_f)$.

**Proof.** Let $u$ and $Y$ be as in Lemma 2.10. Since $u$ is a unitary linear isomorphism the induced map $u^*: \mathcal{L}_a(X) \to \mathcal{L}_a(Y)$ defined by $u^*(T) = u \circ T \circ u^{-1}$, which maps $\mathcal{L}_a(X)$ isomorphically onto $\mathcal{L}_a(Y)$, maps $\mathcal{L}^+(X)$ onto $\mathcal{L}^+(Y)$. Also, since $u: X_f \to Y_f$ is topological the same map $u^*$ maps $\mathcal{L}(X_f)$ onto $\mathcal{L}(Y_f)$. Since $\mathcal{L}^+(Y) \subseteq \mathcal{L}(Y_f)$ (Theorem 2.6) we must have that $\mathcal{L}^+(X) \subseteq \mathcal{L}(X_f)$. 

Theorem 2.12. $\mathcal{L}_+(X) = \mathcal{L}_b(X)$.

Proof. We show first that $\tau_+ \leq \tau_f$. It suffices to show that for each $n \in \mathbb{N}$ the injection $i_n : H_n \rightarrow (X, \tau_+)$ is continuous. Fix a seminorm $\| \cdot \|_{\tau_+}$ for $\tau_+$, where $T \in \mathcal{L}(X)$. Since $T \in \mathcal{L}(X)$ we have $T \circ i_n = T | H_n : H_n \rightarrow X_f$ is continuous; hence, $T | H_n : H_n \rightarrow H$ is continuous ($\tau_{\text{norm}} \leq \tau_f$). But then there exists $C_T > 0$ so that $\| T | H_n(x) \| \leq C_T \| x \|$ for $x \in H_n$; i.e., $\| x \|_{\tau_+} \leq C_T \| x \|$ ($x \in H_n$).

Note also $\| T \|_{H_n} = \| T \|_{\operatorname{B}(H_n)}$. Thus, $\tau_+ \leq \tau_f$.

Theorem 2.13. $(\mathcal{L}_r(X), \mathcal{T}_r)$ is complete.

Proof. Let $\{ T_\alpha \}$ be a $\mathcal{T}_r$-Cauchy set in $\mathcal{L}_r(X)$. For each $E \in \mathcal{E}$ we let $M_E$ be the unit ball in $H_E$. Then, $\{ M_E \} \subseteq \mathcal{L}_r$ and if we fix $E \in \mathcal{E}$, then $\{ T_\alpha | H_E \}$ is a Cauchy net in $\mathcal{B}(H_E)$:

$$\| T_\alpha | H_E - T_\beta | H_E \| = \sup \{ \| T_\alpha - T_\beta \| x \| : x \in H_E, \| x \| \leq 1 \}$$

$$= \sup \{ \| ((T_\alpha - T_\beta)x, y) : x, y \in M_E \}$$

$$= \| T_\alpha - T_\beta \|_{M_E}.$$ Note also $\| T_\alpha^* | H_E - T_\beta^* | H_E \| = \| T_\alpha - T_\beta \|_{M_E}$. Thus, $T_\alpha | H_E \rightarrow T_\beta | H_E$ in $\mathcal{B}(H_E)$ and $T_\alpha^* | H_E \rightarrow S_\beta \in \mathcal{B}(H_E)$. We define $T$ and $S$ on $X$ by $Tx = T_\beta x$ if $x \in H_E$ and $Sx = S_\beta x$ if $x \in H_E$. If $E \subseteq F$, then $T_\beta | H_E = T_\alpha | H_E$, $S_\beta | H_E = S_\alpha$. Hence, $T$ and $S$ are well-defined linear transformations on $X$. Clearly, both $T$ and $S$ leave each $H_E$ invariant and $S = T^*$. Thus, $T \in \mathcal{L}_r(X)$. That $\mathcal{T}_r$-lim $T_\alpha = T$ is easily checked.

Theorem 2.14. $(\mathcal{L}_r(X), \mathcal{T}_r)$ is an $F^*$-algebra with identity. In fact, $(\mathcal{L}_r(X), \mathcal{T}_r) \cong \lim_{\alpha \in \mathcal{N}} \operatorname{inv} \mathcal{B}(H_n)$.

Proof. For each $n \in \mathbb{N}$ we let $M_n$ be the unit ball in $H_n$. Then $\{ M_n \} \subseteq \mathcal{L}_r$ and is “essentially” cofinal: if $M \in \mathcal{L}_r$, then $M$ is a bounded subset of some $H_n$, hence there exists $k \in R_+$ so that $M \subseteq kM_n$. But then $\| T \|_{M_n} \leq k^2 \| T \| _{\operatorname{B}(H_n)}$. Thus, the topology $\mathcal{T}_r$ is determined by the ascending family $\{ \| \cdot \|_{M_n} \}$ of (linear) seminorms, and $(\mathcal{L}_r(X), \mathcal{T}_r)$ is a complete metrizable algebra. As we saw in Theorem 2.13 for $T \in \mathcal{L}_r(X)$ and $n \in \mathbb{N}$ we have $\| T \|_{M_n} = \| T \|_{M_n} = \| T \| _{H_n}$. Thus, each $\| \cdot \|_{M_n}$ is a $B^*$-seminorm, and $(\mathcal{L}_r(X), \mathcal{T}_r)$ is an $F^*$-algebra with identity. The last part of the conclusion was essentially proved in Theorem 2.13. The map $\rho_n : \mathcal{L}_r(X) \rightarrow \mathcal{B}(H_n)$ is just the restriction map; as is the bonding map $\rho^* : \mathcal{B}(H_n) \rightarrow \mathcal{B}(H_{n-1})(n \in \mathbb{N})$. 
3. A representation theorem for $F^*$-algebras. In this section we give three concrete faithful (topologically and algebraically) representations for an abstract $F^*$-algebra with identity as an algebra of operators on a vector space formed from the irreducible Hilbert space representations of the algebra.

We let $A$ be an $F^*$-algebra with identity and let $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ be a standard family of irreducible Hilbert space representations of $A$. Since $A$ is an $F^*$-algebra the family is complete.

If $A$ is a $B^*$-algebra and $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ is a standard family for $A$, then $\pi: A \to \mathcal{B}(\sum_i H_i)$ defined by $\pi(a) = \{\pi_\lambda(a)x_\lambda\}$ is an isometry and $^*$-isomorphism. It is easily seen that for non-$B^*$-algebras (but still $F^*$-) this is impossible. In fact, one cannot even define $\pi(a)$ on $\sum_i H_i$ for all $a \in A$, unless every $a \in A$ has bounded norm: $\sup_{\lambda} ||a||_\lambda < \infty$ for some determining family of seminorms. If one moves to the other extreme and defines $\pi(a)$ by the same formula on $X = \sum_i H_i$ (algebraic sum), then $\pi(a)$ makes sense and $\pi: A \to (L(X_f), \mathcal{I}_0)$ is a continuous $^*$-isomorphism but fails to be topological. This is the case because the final topology on $X$, hence the topology $\mathcal{T}_0$ on $L(X_f)$, depends on finite subsets of $A$ whereas that of $A$ depends on much larger subsets of $A$. Thus, we must seek a middle ground in order to achieve a faithful representation of $A$ in this manner. Before we introduce the basic concept we first prove a crucial fact about Hilbert space representations of Fréchet $^*$-algebras.

**Lemma 3.1.** Let $A$ be a Fréchet $^*$-algebra with identity, and let $\mu: A \to \mathcal{B}(H)$ be a representation of $A$ on the Hilbert space $H$. Then $\mu$ is continuous.

**Proof.** Fix $\varepsilon > 0$. Let $V = \{a \in A: ||\mu(a)|| \leq \varepsilon\} = \bigcap \{V_{x,y}: ||x||, ||y|| \leq 1\}$, where $V_{x,y} = \{x \in A: |(\mu(a)x, y)| \leq \varepsilon\}$. For each pair $x, y \in H$ such that $||x||, ||y|| \leq 1$ the set $V_{x,y}$ is convex and balanced. Since for each $z \in H$ the map $a \to (\mu(a)x, z)$ is continuous (Do-Shing’s Theorem [2]), we have that $a \to (\mu(a)x, y)$ is continuous (polarization formula). Thus, each $V_{x,y}$ is closed. So $V$ is closed, convex, and balanced. It is easily verified that $V$ is absorbing; hence is a neighborhood of 0 in $A$.

**Definition.** Let $A$ be a Fréchet $^*$-algebra with identity and let $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ be a standard family of irreducible Hilbert space representations of $A$. A subset $E$ of $A$ will be called equicontinuous if, and only if $\sup_{\lambda \in E} ||\pi_\lambda(a)|| < \infty$ for each $a \in A$. The family of all equicontinuous subsets of $A$ will be denoted $\mathcal{E}(A)$.

**Lemma 3.2.** If $A$ is a Fréchet $^*$-algebra with identity and $\{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\}$ be a standard family of irreducible Hilbert space representations of $A$. A subset $E$ of $A$ will be called equicontinuous if, and only if $\sup_{\lambda \in E} ||\pi_\lambda(a)|| < \infty$ for each $a \in A$. The family of all equicontinuous subsets of $A$ will be denoted $\mathcal{E}(A)$.
\(H_{\lambda} \): \(\lambda \in \Lambda\) is a standard family for \(A\), then \(E \subseteq \Lambda\) is equicontinuous if, and only if, \(\sum_{\lambda \in E} \pi_{\lambda}\) defines a continuous representation of \(A\) on \(\sum_{\lambda \in E} \oplus H_{\lambda}\).

**Proof.** Suppose \(E \subseteq \Lambda\) is equicontinuous. For \(a \in A\) we let \(C_{a} = \sup_{\lambda \in E} ||\pi_{\lambda}(a)||\) and define \(\pi: A \to \mathfrak{B}(\sum_{\lambda \in E} \oplus H_{\lambda})\) by \(\pi(a)x_{\lambda} = \{\pi_{\lambda}(a)x_{\lambda}\}\). Now \(||\{\pi_{\lambda}(a)x_{\lambda}\}||^{2} = \sum_{\lambda} ||\pi_{\lambda}(a)x_{\lambda}||^{2} \leq C_{a}^{2} ||x||^{2}\). So \(\pi(a)\) maps \(\sum_{\lambda \in E} \oplus H_{\lambda}\) into itself, and \(\pi\) is a representation of \(A\) on \(\sum_{\lambda \in E} \oplus H_{\lambda}\).

Conversely, suppose we can define a representation of \(A\) on \(\sum_{\lambda \in E} \oplus H_{\lambda}\) by the direct sum formula. It is clear that \(||\pi_{\lambda}(a)|| \leq ||\pi(a)||\) for each \(\lambda \in E\) and \(a \in A\).

**Lemma 3.3.** Let \(A, \{(\pi_{\lambda}, H_{\lambda}) : \lambda \in \Lambda\}\) be as above. Let \(\{|| \cdot ||_{n}\}\) be a \(*\)-sequence of seminorms for \(A\). For \(n \in \mathbb{N}\) we set \(E_{n} = \{\lambda \in \Lambda : ||\pi_{\lambda}(a)|| \leq ||a||_{n} (a \in A)\}\). Then \(E \subseteq \Lambda\) is equicontinuous if, and only if, \(E\) is contained in some \(E_{n}\). In particular, the increasing sequence \(\{E_{n}\}\) is cofinal in \(\mathfrak{E}(A)\).

**Proof.** If \(E \subseteq E_{n}\) for some \(n\), then clearly \(E \in \mathfrak{E}(A)\). Conversely, if \(E \in \mathfrak{E}(A)\) then \(\pi: A \to \mathfrak{B}(\sum_{\lambda \in E} \oplus H_{\lambda})\) defined as in Lemma 3.2 is a continuous representation of \(A\). Hence, there exists \(C > 0, n \in \mathbb{N}\) so that \(||\pi(a)|| \leq C||a||_{n} (a \in A)\). It is easily verified that we can take \(C = 1\), and the condition is satisfied.

We set \(H = \sum_{\lambda \in \Lambda} \oplus H_{\lambda}\) and let \(X = \{x \in H : \text{Supp}(x) \in \mathfrak{E}(A)\}\). We are now in the situation of the second part of § 2 with \(H_{E} = \{x : \text{Supp}(x) \subseteq E\}\), \(i_{E}: H_{E} \to X\) the natural injection, \(\tau_{f}\) the final topology determined by the family \(\{i_{E} : E \in \mathfrak{E}(A)\}\). If \(\{|| \cdot ||_{n}\}\) is any \(F^{*}\)sequence of seminorms for our \(F^{*}\)-algebra \(A\) with identity, then we let \(H_{n} = H_{E_{n}}\) and \(\pi_{n}: A \to \mathfrak{B}(H_{n})\) the induced representation of \(A\) on \(H_{n}\).

**Lemma 3.4.** With the definitions given immediately above for each \(n \in \mathbb{N}\) and \(a \in A\) it is the case that \(||a||_{n} = ||\pi_{n}(a)||\).

**Proof.** In § 1 we indicated that \(\{\pi_{\lambda} : \lambda \in E_{n}\}\) induces a complete standard family \(\{\sigma_{\lambda} : \lambda \in E_{n}\}\) of irreducible Hilbert space representations of the \(B^{*}\)-algebra \(A_{n}\), the completion of \(A/[a : ||a||_{n} = 0]\) with respect to the induced norm, and if \(\rho_{n}\) is the natural projection of \(A\) into \(A_{n}\) we have \(\sigma_{\lambda} \circ \rho_{n} = \pi_{\lambda}(\lambda \in E_{n})\). If we let \(\sigma_{n} = \sum_{\lambda \in E_{n}} \sigma_{\lambda} : A_{n} \to \mathfrak{B}(H_{n})\), then \(||\sigma_{n}(a_{n})|| = ||a_{n}||\) for each \(a_{n} \in A_{n}\). But \(||\sigma_{n}(a_{n})|| = \sup_{\lambda \in E_{n}} ||\sigma_{\lambda}(a_{n})||\). Thus, if \(a \in A\), then \(||a||_{n} = ||\rho_{n}a|| = \sup_{\lambda \in E_{n}} ||\sigma_{\lambda}(\rho_{n}a)|| = \sup_{\lambda \in E_{n}} ||\pi_{\lambda}(a)|| = ||\pi_{n}(a)||\).

From the above construction we can infer more. For each \(a \in A\) there exists \(\lambda_{0} \in E_{n}\) such that \(||a||_{n} = ||\pi_{\lambda_{0}}(a)||\). This can be proved...
for \( A \) by reducing the problem to that for a hermitian elements, then showing that it holds on the algebra generated by the element and extending to the full algebra.

**Theorem 3.5.** Let \( A \) be an \( F^* \)-algebra with identity, \( \{(\pi_\lambda, H_\lambda): \lambda \in \Lambda\} \) a standard family of irreducible Hilbert representations of \( A \), \( H = \sum_{\lambda \in \Lambda} \oplus H_\lambda \), and \( X = \{x \in H: \text{Supp}(x) \text{ is equicontinuous}\} \). Let \( \pi: A \to \mathcal{L}_b(X) \) be defined by \( \pi(a)_{\lambda} = \{\pi_\lambda(a)x\}_{a \in A, \ x = \{x_\lambda\} \in X} \). Then

1. For each \( a \in A \) the function \( \pi(a) \) defined above is indeed in \( \mathcal{L}_0(X) \); in fact, \( \pi(a) \in \mathcal{L}_r(X) \).
2. \( \pi: A \to (\mathcal{L}^+(X), \mathcal{I}_+) \) is a topological *-isomorphism (into).
3. \( \pi: A \to (\mathcal{L}^+(X), \mathcal{I}^+) \) is a topological *-isomorphism (into).
4. \( \pi: A \to (\pi(A), \mathcal{I}^+) \) is a topological *-isomorphism.

**Proof.** (1) Fix \( a \in A, x \in X \). Then

\[
\sum_{\lambda} ||\pi_\lambda(a)x_\lambda||^2 \leq \sum_{\lambda} ||\pi_\lambda(a)||^2 ||x_\lambda||^2: \lambda \in \text{Supp}(x)\]

\[
\leq \sup \{||\pi_\lambda(a)||^2: \lambda \in \text{Supp}(x)\} \cdot ||x||^2.
\]

Thus, \( \pi(a)x \in H \) and \( \text{Supp}(\pi(a)x) \subseteq \text{Supp}(x) \in \mathcal{E}(A) \), so \( \pi(a) \) maps \( X \) into itself. Moreover, if \( a \in A \) and \( x, y \in X \) we have \( \langle \pi(a)x, y \rangle = \langle x, \pi(a^*)y \rangle \); so \( \pi(a) \in \mathcal{L}^+(X) \). It is clear that \( \pi(a) \in \mathcal{L}_r(X) \).

(2) and (3) It is clear that \( \pi \) is a *-isomorphism. Since \( \mathcal{I}_+ | \mathcal{L}_r(X) = \mathcal{I}_b | \mathcal{L}_r(X) = \mathcal{I}_r \) (Theorem 2.12) and \( \pi(A) \subseteq \mathcal{L}_r(X) \) it is necessary and sufficient that we show \( \pi: A \to (\mathcal{L}_r(X), \mathcal{I}_r) \) is topological. We fix an \( F^* \)-sequence \( \{|\cdot|_{M_n}\} \) of seminorms for \( A \), let \( \{E_n\}, \{H_n\} \), and \( \{\pi_n\} \) be the corresponding cofinal sequence in \( \mathcal{E}(A) \), Hilbert space sequence in \( X \), and sequence of representations of \( A \) (respectively). We note that \( \pi_n(a) = \pi(a)|H_n \) for \( n \in \mathbb{N}, a \in A \). We recall from Theorem 2.14 that \( (\mathcal{L}_r(X), \mathcal{I}_r) \) is an \( F^* \)-algebra with identity and that \( \{|\cdot|_{M_n}\} \) is an \( F^* \)-sequence of seminorms for \( \mathcal{L}_r(X) \), where \( M_n \) is the closed unit ball in \( H_n(n \in \mathbb{N}) \). Moreover, for each \( n \in \mathbb{N} \) and \( T \in \mathcal{L}_r(X) \) we have \( ||T||_{M_n} = ||T|_{H_n}|| \), the norm of the (bounded) restriction of \( T \) to \( H_n \). If \( a \in A \), then \( ||a||_{M} = ||\pi_n(a)|| \) (Lemma 3.4) and the latter is \( ||\pi(a)||_{M_n} = ||\pi(a)||_{M_n} \).

We again fix an \( F^* \)-sequence \( \{|\cdot|_{M}\} \) of seminorms for \( A \). Let \( \{E_n\} \), etc. be as in "(2) and (3)" above. We let \( \mathcal{A} = \pi(A) \), an \( \text{Op^*} \)-algebra on \( X \) with corresponding family \( \mathcal{I}_n \) of bounded subsets of \( X \). The topology \( \mathcal{I}_n \) on \( \mathcal{A} \) is defined by the seminorms \( \{|\cdot|_{M}: M \in \mathcal{I}_n\} \), where \( M \subseteq X \) belongs to \( \mathcal{I}_n \) if, and only if, \( \sup_{x \in M} ||Tx|| < \infty \) for each \( T \in \mathcal{A} \). Lassner's Lemma 5.2 [4] says that \( \pi: A \to (\mathcal{A}, \mathcal{I}_n) \) is continuous. Fix \( n \in \mathbb{N} \) and let \( M_n \) be the closed unit ball in \( H_n \) as above. Since \( \mathcal{A} \subseteq \mathcal{L}^+(X), \mathcal{I}_n \subseteq \mathcal{I}_n \) and \( \{M_n\} \subseteq \mathcal{I}_n \); so \( \{M_n\} \subseteq \mathcal{I}_n \). We know from above that \( ||a||_n = ||\pi(a)||_{M_n} \) (\( a \in A, n \in \mathbb{N} \)). This es-
establishes the openness of $\pi: A \to (\mathbb{A}, \mathcal{T}_a)$. 

**Remarks.** (1) Do-Shing [2] obtains a representation theorem for $LMC^*$-algebras (the same as $F^*$-algebra without the metrizability restriction) which uses essentially the same Hilbert space, but he maps $A$ onto an algebra of unbounded operators with special properties. Also, he does not consider topological properties of the map.

(2) The main problem in studying non-commutative Fréchet $*$-algebras is the lack of models against which to compare the abstract algebras. A corollary to Do-Shing’s theorem on positive functionals on Fréchet $*$-algebras is that every one induces a cyclic Hilbert space representation, but as we have seen we cannot represent these algebras faithfully on Hilbert spaces. The examples discussed above, the algebras $\mathcal{L}_r(X)$, are quite similar to those considered by E. A. Michael in Appendix A of his memoir [5], where in our case the underlying locally convex space is an inductive limit of Banach (Hilbert) spaces. It seems that the class he defined in [5] might include most examples of noncommutative $F$-algebras, except those built from a commutative $F$-algebra and a noncommutative Banach algebra by tensor products, e.g., $C(X, B)$ where $X$ is an appropriate topological space and $B$ is a Banach algebra.

4. Enveloping algebras. In this section we define the enveloping algebra of a Fréchet $*$-algebra with identity, relate it to inverse limit decompositions of the algebra, and realize it as an algebra of operators naturally constructed from $A$.

We fix a Fréchet $*$-algebra with identity, $A$, and also fix a standard family $\{\{\pi_\lambda, H_\lambda\}: \lambda \in \Lambda\}$ of irreducible Hilbert space representations of $A$. We recall that $K(A) = \{f: f$ is a positive functional on $A, f(e) = 1\}$. 

**Lemma 4.1.** If $E \subseteq K(A)$ and $\{|| \cdot ||_n\}$ is a $*$-sequence of seminorms for $A$, then the following statements are equivalent.

1. $E$ is equicontinuous.
2. $\sup_{f \in E} f(a^*a) < \infty (a \in A)$.
3. There $n \in N$ such that $E \subseteq K_n(A)$. 

**Proof.** (1) and (2) are clearly equivalent by the uniform boundedness principle for Fréchet spaces: if $E \subseteq A^*$ and $E$ is pointwise bounded ($\sigma(A^*, A)$-bounded), then $E$ is equicontinuous (see [Theorem 4.2, p. 83, 7]). It is also clear that (1) and (3) are equivalent, since $K_n(A)$ is the intersection with $K(A)$ of the polar of the neighborhood $\{a \in A: ||a||_n \leq 1\}$. 

DEFINITION. Let $\mathcal{E}(K)$ be all equicontinuous subsets of $K(A)$. For $E \in \mathcal{E}(K)$ we define

$$|a|_E = \left(\sup \{f(a*a)\colon f \in E\}\right)^{1/2} \quad (a \in A).$$

**Theorem 4.2.** If $A$ is a Fréchet $*$-algebra, $\{\cdot |_E \colon E \in \mathcal{E}(K)\}$ is the family of maps defined above, then

1. Each $| \cdot |_E$ is a linear seminorm on $A$.
2. $R^*(A) = \{a \in A\colon |a|_E = 0 \text{ for each } E \in \mathcal{E}(K)\}$.
3. If $\{|n\}$ is any $*$-sequence of seminorms for $A$, then the topology of $(A/R^*(A), \{|n|\})$ is determined by the $B^*$-seminorms $\{|n|\}$, where $|n| = |\cdot|_{K_n(A)}$. Hence,
4. The completion $E(A)$ of $(A/R^*(A), \{|n|\})$ is an $F^*$-algebra with identity.

**Proof.** (1) and (2) are trivial to verify and (4) follows from (3), which we now prove. Fix a $*$-sequence $\{|n|\}$ of seminorms for $A$. For each $n \in N$ we set $E_n = \{x \in A\colon \|x\|_n \leq |a|_n (a \in A)\}$ and define $\pi_n : A \rightarrow \mathcal{B}(\sum_{\lambda \in E_n} H_\lambda)$ by $\pi_n(a)(\xi_\lambda) = \{\pi_n(a)\xi_\lambda\}_{\lambda \in E_n}$ for each $a \in A$. We shall show that for each $n \in N$ and $a \in A$ we have $\|\pi_n(a)\| = |a|_n$. Fix $n \in N$. For $\lambda \in A$ we choose a unit vector $\xi_\lambda \in H_\lambda$, define $f_\lambda : A \rightarrow C$ by $f_\lambda(a) = (\pi_\lambda(a)\xi_\lambda, \xi_\lambda)$, let $K_\lambda$ be the completion of $A/[a\colon f_\lambda(a*a) = 0]$ with respect to the induced inner product $\langle [a], [b]\rangle = f_\lambda(b^*a)$, where $[a]$ is the coset containing $a$. Finally, define $\psi_\lambda : A \rightarrow \mathcal{B}(K_\lambda)$ by $\psi_\lambda([b]) = [ab]$, on $A/[a\colon f_\lambda(a*a) = 0]$, and extending these norm-continuous operators to $K_\lambda$. There exists an isomorphism $U : H_\lambda K_\lambda$ so that $U\pi_\lambda = \psi_\lambda U$. Hence, for $a \in A$ we have $\|\pi_n(a)\|^2 = \|\psi_n(a)\|^2 = \sup \{f_\lambda(b^*a*b) : f_\lambda(b^*b) = 1\} \geq f_\lambda(a)$. If $f_\lambda(b^*b) = 1$, then $f_{\lambda,b} : c \rightarrow f_\lambda(b^*cb)$ also belongs to $K_n(A)$ (that $f_{\lambda,b}$ does is clear) and $f_{\lambda,b}(a^*a) \leq |a|_n$. Hence, $\|\pi_n(a)\| \leq |a|_n$ for each $\lambda \in E_n$, and $\|\pi_n(a)\| = \sup |\pi_n(a)| : \lambda \in E_n \leq |a|_n$. Then reverse inequality follows from the fact that $|a|_n = \sup \{f(a*a) : f \in K_n(A), f \text{ is extreme}\}$.

**Definition.** We shall call the algebra $E(A)$ in Theorem 4.2 (4) the enveloping algebra of $A$.

**Theorem 4.3.** If $(A, \{|n|\})$ is a Fréchet $*$-algebra with identity and if $\{A_n\}$ is the corresponding inverse limit system of Banach $*$-algebras with identity, then $E(A) = \lim_n \text{inv} \{E(A_n)\}$.

**Proof.** We let $\rho_n$ be the natural map of $A$ into $A_n$ and $\rho^n : A_n \rightarrow A_{n-1}$ for $n \geq 2$ the induced bonding map. For $n \in N$ we let $E_n$ be the enveloping algebra of $A_n$ and let $\Phi_n$ be the natural map of $A_n$ into $E_n$. Finally, we let $\varphi$ be the map of $A$ into $E(A)$.
For \( n \in \mathbb{N} \) we have the diagram

\[
\begin{array}{ccc}
A_{n-1} & \xleftarrow{\rho^n} & A_n \\
\psi_{n-1} \downarrow & & \downarrow \psi_n \\
E_{n-1} & \to & E_n
\end{array}
\]

Now \( \ker (\psi_n) = R^*(A_n) \) and \( \rho^n(R^*(A_n)) \subseteq R^*(A_{n-1}) \). Thus, we have an induced map \( \sigma^n: E_n \to E_{n-1} \). It is easily verified that \( \{E_n, \sigma^n, N\} \) is a dense inverse limit system of \( B^* \)-algebras (i.e., the bonding maps have dense range and are norm-decreasing). We let \( E = \lim_{\text{inv}} \{E_n, \sigma^n, N\} \) and consider \( E \) a subset of \( \prod_n E_n \) with the relative product topology.

We define \( \tau: E(A) \to E \) by first defining \( \tau \) on \( A/R^*(A) \) by the formula \( \tau(\varphi a) = \{\psi_n \rho_n a\} \). If \( a \in A \), then \( \varphi(a) = 0 \) if, and only if, \( a \in R^*(A) \) if, and only if, \( \rho_n(a) \in R^*(A_n) (n \in N) \) if, and only if, \( \psi_n \rho_n(A) = 0 \) (\( n \in N \)). Thus, \( \tau \) is well-defined and one-to-one \( A/R^*(A) \). Also, since all the maps involved have dense range it follows that \( \tau(A/R^*(A)) \) is dense in \( E \). Finally,

\[
|\tau(\varphi a)|_n^2 = |\psi_n \rho_n(a)|_n^2 = |\rho_n(a)|_n^2
\]

\[
= \sup \{f_*(\rho_n(a^*a)) : f \in K(A_n)\}
\]

\[
= \sup \{f(a^*a) : f \in K_n(A)\}
\]

\[
= |a|_n^2 \quad (n \in N, a \in A).
\]

Thus, \( \tau \) is an isometry in each seminorm; hence, extends to a topological map of \( E \) onto \( E(A) \). It is clear that the map is a \(*\)-isomorphism.

We now realize \( E(A) \) as an algebra of operators on \( X = \{x \in \sum_{i \in A} \bigoplus H_i : \text{Supp} (x) \text{ is equicontinuous}\} \). We use the same notation as in § 3.

**Theorem 4.4.** Let \( (A, \{\| \cdot \|_n\}) \) be a Fréchet \(*\)-algebra with identity and let \( (E(A), \{\| \cdot \|_n\}) \) be its enveloping algebra, with natural map \( \varphi: A \to E(A) \). For \( a \in A \) we define \( \pi(a) \) on \( X \) by \( \pi(a)\{x_i\} = \{\pi_i(a)x_i\} \). Then \( \pi: A \to \mathcal{L}_r(X) \) induces a topological \(*\)-isomorphism \( \bar{\sigma} \) of \( E(A) \) onto \( \overline{\pi(A)} \), where "topological" refers to any of the (equal on \( \mathcal{L}_r(X) \)) topologies \( \mathcal{T}_r, \mathcal{T}_+ \), or \( \mathcal{T}_6 \) on \( \mathcal{L}_r(X) \) and the closure of \( \pi(A) \) is with respect to these topologies.

**Proof.** Since for each \( a \in A \) and \( n \in N \) we have \( |a|_n = \|\pi_n(a)\| \) we have that \( \ker \pi = R^*(A) \), so there is an induced map \( \sigma: A/R^*(A) \to \mathcal{L}_r(X) \) so that the following diagram commutes:
We have shown in Theorem 2.13 that the topologies $\mathcal{S}_r = \mathcal{S}_b = \mathcal{S}_+$ on $\mathcal{L}_r(X)$ are defined by the sequence $\{|| \cdot ||_n\}$ of seminorms and that $||T||_n = ||T||_{H_*}$. Also, we know that $\pi_n(a)$ is $\pi(a)|H_n$. So from Theorem 4.2 we have $|a|_n = ||\pi_n(a)||$, and hence, $|a|_n = ||\pi(a)||_{M_n}$. It follows that $\sigma: A/R^*(A) \rightarrow \mathcal{S}_r(X)$ is topological, and since $(\mathcal{L}_r(X), \mathcal{S}_r)$ is complete $\sigma$ extends to a topological $^*$-isomorphism of $E(A)$ into $\mathcal{L}_r(X)$.

If $A$ is a Banach $^*$-algebra with identity and $E(A)$ its enveloping algebra, then every Hilbert space representation of $A$ factors through $E(A)$. We conclude our discussion of enveloping algebras by examining this problem for Fréchet $^*$-algebras. We consider only representations in $\mathcal{L}^+(X)$, since this is enough to illustrate the problems involved.

**Lemma 4.5.** If $(A, \{|| \cdot ||_n\})$ is a Fréchet $^*$-algebra with identity, $\{|| \cdot ||_n\}$ the corresponding sequence of $B^*$-seminorms on $A$ used to define the topology of $E(A)$, and if $\mu: A \rightarrow \mathcal{L}^+(X)$ is an essential representation of $A$ on $X$ ($\mu(e) = I$) then, for each $M \in \mathcal{S}_+$ there exists $n \in \mathbb{N}$ and $C > 0$ such that $||\mu(a)||_M \leq C|a|_n(a \in A)$.

**Proof.** Fix $M \in \mathcal{S}_+$. We let $||M|| = \sup \{||x||: x \in M\}$ (||M|| < $\infty$, since $M$ is bounded in the Hilbert space completion of $X$). Since $\mu$ is continuous there exist $n \in \mathbb{N}$ and $C > 0$ such that $||\mu(a)||_M \leq C|a|_n(a \in A)$.

Fix $x \in M$. Then $f_x: a \rightarrow (\mu(a)x, x)$ is a positive functional on $A$. Also, $|f_x(a)| = ||(\mu(a)x, x)|| \leq ||\mu(a)||_M \leq C|a|_n$. Hence, $f_x \in P_n(A)$ for each $x \in M$. Therefore, if $x \in M$ and $x \neq 0$, the positive functional $f_x(e)^{-1}f_x$ belongs to $K_n(A)$ and $f_x(e)^{-1}f_x(a^*a) \leq |a|_n^2(a \in A)$. So we have $f_x(a^*a) \leq f_x(e)|a|_n^2(x \in M, a \in A)$. But $f_x(e) = (\mu(e)x, x) = ||x||^2 \leq ||M||^2$. Hence, $f_x(a^*a) \leq ||M||^2|a|_n^2(x \in M, a \in A)$.

For $x, y \in M, a \in A$ we have

$$(\mu(a)x, y) = ||\mu(a)x|| \cdot ||y||$$

$$\leq ||M|| \cdot (||\mu(a)x||^2)^{1/2}$$

$$\leq ||M|| \cdot (||a^*a||_x, x)^{1/2}$$

$$\leq ||M||^2 |a|_n.$$
Thus, $||\mu(a)||_M \leq ||M||^2 |a|_n(a \in A)$.

**Theorem 4.6.** If $(A, \{||.||_n\})$ is a Fréchet $*$-algebra with identity, $(E(A), (|.|_n))$ its enveloping algebra, $\varphi$ the natural map of $A$ into $E(A)$, and if $\mu: A \to \mathcal{L}^+(X)$ is an essential representation of $A$ on $X$, then there exists a continuous representation $\sigma$ of $A/R^*(A)$ on $X$ so that $\sigma \varphi = \mu$. If $\pi(A)$ is contained in a $\mathcal{T}_+$-complete subalgebra of $\mathcal{L}^+(X)$, then $\sigma$ extends to a representation of $E(A)$ on $X$. In particular, this is the case if $X$ is Hilbert space. Hence, all Hilbert space representations of $A$ factor through $E(A)$.

**Proof.** We need only show that $\sigma$ can be defined on $A/R^*(A)$ so that $\sigma \varphi = \mu$. The other claims follow from Lemma 4.5. It is sufficient to show that $\ker \varphi \subset \ker \mu$. If $a \in \ker \varphi = R^*(A)$ and if $x \in X$, then $b \to (\mu(b)x, x)$ is a positive functional on $A$; hence, $(\mu(a^*a)x, x) = ||\mu(a)x||^2 = 0$. Thus, $\mu(a) = 0$ and $a \in \ker \mu$.

**Example 4.7.** We show here that some representations $\mu: A \to \mathcal{L}^+(X)$ fail to factor through $E(A)$.

Let $A = C^\infty (R)$ with the topology determined by the seminorms $||a||_n = \sum_{k=0}^n (k!)^{-1} ||a^{(k)}||_{n,\infty}$, where $a^{(k)}$ is the $k$th derivative of $a$ and $||.||_{n,\infty}$ is the supremum on $[-n, n]$. Then $A$ is a commutative Fréchet $*$-algebra with identity (involution is conjugation), and

1. $||a||_n = ||a||_{n,\infty}(a \in A, n \in N)$
2. $R^*(A) = 0$, hence
3. $A/R^*(A) = (A, \{|.|_n\})$ and $E(A)$ is just $C(R)$ with the compact-open topology. We use hereafter $|.|_n$ for $||.||_{n,\infty}$.

Let $X = C_0^\infty (R)$, the compactly-supported $C^\infty$ functions on $R$, considered as a dense subspace of $L^2 (R)$. We note that if $a \in C(R)$ and $f \in C_0^\infty (R)$, then there exists $n \in N$ such that $||af|| \leq ||a||_n ||f||$ ($n$ depends on $f$, $n$ is any positive integer so that $\text{Supp} (f) \subseteq [-n, n]$), and $||.||_n$ is the norm in $L^2 (R)$.

Define $\mu: A \to \mathcal{L}^+(X)$ by $\mu(a)f = af$. It is clear that this formula actually does define a linear transformation on $X$ and that (1) $\mu(a)^* = \mu(a^*)$, (2) $\mu$ is a representation of $A$ in $\mathcal{L}^+(X)$, hence, (3) $\mu: (A, \{|.|_n\}) \to \mathcal{L}^+(X)$ is continuous (by Theorem 4.6). We now show that $\mu$ cannot be extended continuously to $C(R)$. We prove (4): if $\tilde{\mu}$ is the extension to $C(R)$ of $\mu$ and if $f \in X$, then, we must have $\tilde{\mu}(a)f = af$ ($a \in C(R)$). We know that there exists $n \in N$ and $C > 0$ so that $||\tilde{\mu}(a)f|| \leq C |a|_n$ for each $a \in C(R)$. Fix $a \in C(R)$ and choose $\{a_j\} \subseteq C^\infty (R)$ so that $C(R) - \lim_j a_j = a$. Choose $C > 0$ and $n \in N$ as above (for $f \in X$) and such that $\text{Supp} (f) \subseteq [-n, n]$. Then $||\tilde{\mu}(a)f - \tilde{\mu}(a_j)f|| \leq C |a - a_j|_n$. Hence, $\{\tilde{\mu}(a_j)f\}$ converges in $L^2 (R)$ to $\tilde{\mu}(a)f$. But $\tilde{\mu}(a_j)f = a_jf$ and by our earlier estimate $||a_jf - af|| \leq ||f|| |a_j - a|_n$. So
\( \hat{\mu}(a_j)f = a_jf \) converges to \( \hat{\mu}(a)f \) and to \( af \). Thus \( \hat{\mu}(a)f = af \) for each \( f \in X \). But \( C(R) \cdot C_0^\infty(R) \nsubseteq C_0^\infty(R) \), so \( \mu \) fails to extend to \( C(R) \).

**REMARKS.** In the last example we could have considered \( \mu \) a representation of \( C^\infty(R) \) in \( \mathcal{L}(C^\infty_0(R), \mathcal{I}_b) \). It is not too difficult to show that \( \mu \) is continuous when thought of this way. It clearly still fails to extend.

Finally, we do not know whether representations of \( A \) in \( (\mathcal{L}^*(X), \mathcal{I}_b) \) where \( X \) is a locally convex TVS with a continuous inner product are necessarily continuous, in contrast to representations in \( \mathcal{L}^+(X) \). It probably is possible to find an example of a discontinuous representation, since the topology \( \mathcal{I}_b \) need not be related to the inner product.

**REFERENCES**


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