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A DUALITY BETWEEN TRANSPOTENCE ELEMENTS AND MASSEY PRODUCTS

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The purpose of this note is to show that if v is an element whose suspension is nonzero, and if u is dual to v, then the transpotence $\varphi_k(v)$ is defined and nonzero if and only if the k-Massey product $\langle u\rangle^k$ is defined and nonzero.

We wish to thank Dr. Samuel Gitler for a helpful conversation on this material.

1. Preliminaries.

1.1. *The Cobar Construction*: (Adams [1]). Let C be a simply connected DGA coalgebra over K with co-associative diagonal map where K is a commutative ring with unit. The Cobar Construction $\bar{F}(C)$ is the direct sum of the n-fold tensor products of the desuspension of $\bar{C} = \text{Ker}\ (\varepsilon)$ where $\varepsilon\colon C \to K$ is the augmentation. Suppose C has a differential $\{d_n\colon C_n \to C_{n-1}\}$. A typical element is a linear combination of elements of the form

$$x = s^{-1}(c_1) \otimes \cdots \otimes s^{-1}(c_n) = [c_1 \mid \cdots \mid c_n]$$

where x has bidegree $(-n, m)$ and $m = \sum_{i=1}^{n}$ degree (c_i). The differential in $\bar{F}(C)$ is defined on elements of bidegree $(-1, *)$ by

$$d[c] = [-dc] + \sum_i (-1)^{\deg c_i'}[c_i' \mid c_i'']$$

where

$$\Delta(c) = c \otimes 1 + 1 \otimes c + \sum_i c_i' \otimes c_i''$$

$\Delta\colon C \to C \otimes C$ being the diagonal mapping of C. The differential is extended to all of $\bar{F}(C)$ by the requirement that $\bar{F}(C)$ be a DGA-algebra.

If C has a differential of degree $+1$ instead of -1, we no longer ask that C be a simply connected but only connected, and the element $[c_1 \mid \cdots \mid c_n]$ is assigned bidegree (n, m).

1.2. *The Bar Construction*. Let A be a connected associative DGA algebra over K. Let $\varepsilon\colon A \to K$ be the augmentation. Let $\bar{A} = \ker \varepsilon$. Then the Bar Construction $\bar{B}(A)$ is the direct sum of the n-fold tensor products of the suspension of $\bar{A}$. Let

$$\{d_n\colon A_n \to A_{n-1}\}$$

be the differential in A. $\bar{B}(A)$ is bigraded by assigning the element $[a_1 | \cdots | a_n]$ degree (n, m) where $m = \sum_{i=1}^{n} \deg a_i$. $\bar{B}(A)$ has a differential $d = d_E + d_I$ where

$$d_E([a_1 | \cdots | a_n]) = \sum_{i=1}^{n-1} (-1)^{u(i)} [a_1 | \cdots | a_i a_{i+1} | \cdots | a_n]$$

$$d_I([a_1 | \cdots | a_n]) = \sum_{i=1}^{n} (-1)^{u(i-1)} [a_1 | \cdots | \partial a_i | \cdots | a_n]$$

where

$$u(i) = i + \sum_{k-1}^{i} \deg a_k \ .$$

We also mention that $[a | \cdots (k) \cdots | a]$ is $\gamma_k[a]$, the kth divided power of $[a]$.

If instead of the above the differential of A has degree $+1$, we put the bidegree of $[a_1 | \cdots | a_n]$ to be $(-n, m)$. In this case we will always assume A is simply connected.

1.3. *The Suspension Map.* In the case of the Bar Construction the suspension map σ: $H_*(A) \to H_*(\bar{B}(A))$ is represented by $a \to [a]$. In the case of the Cobar Construction, σ: $H_*(PA) \to H_*(\bar{F}(A))$ is represented by $a \to [a]$ where PA is the subcomplex of primitive chains.

DEFINITION 1. The Massey Product $\langle u \rangle^k$. (Kraines [6]).

Let A be a DGA algebra over K. Suppose $a_1, \cdots, a_{k-1}$ are given in A such that a_1 is a cycle (or cocycle) and that

$$\partial a_n = \sum_{r=1}^{n-1} (-1)^{\deg a_r} a_r a_{n-r} \text{ for } n = 2, \cdots, k-1 \ .$$

Suppose u is represented by a_1. Then the Massey Product $\langle u \rangle^k$ is represented by the cycle

$$\sum_{r=1}^{k-1} (-1)^{\deg a_r} a_r \cdot a_{k-r}$$

THEOREM 1. (Kraines, [6]). *The operation $\langle u \rangle^k$ depends only on the class $\{a_1\} \in H(A)$.*

DEFINITION 2. (Gitler, [5]). Suppose that A is an associative DGA algebra. Suppose $x \in H(A)$ is such that $v^k = 0$. The transpotence $\varphi_k(v) \in H(\bar{B}(A))_{/Im\sigma}$ is defined as follows: If $b \in A$ represents v then there exists $M \in A$ such that $\partial M = -b^k$. $\varphi_k(v)$ is represented by

$$(-1)^w [b^{k-1} | b] + [M] \text{ where } w = (1)^{\deg b^{k-1}} + 1 \ .$$

2. Main Theorem.

THEOREM 2. *Let C be a co-associative DGA coalgebra over K and let A be the dual associative DGA algebra over K. Suppose $H(A; K)$ and $H(\bar{B}(A); K)$ are free and of finite type over K. Let v in $H(A)$ and v in $H(\bar{F}(C); K)$ be such that the Kronecker index $\langle \sigma(v), u\rangle$ is 1. Then $\varphi_k(v)$ is defined and is not zero in $H(\bar{B}(A); K)$ if and only if $\langle u\rangle^k$ is defined and not zero in $H(\bar{F}(C); K)$. In this case*

$$\langle \varphi_k(v), \langle u\rangle^k\rangle = 1 .$$

In order to prove this theorem we shall consider the Eilenberg-Moore Spectral Sequences with

$E^2 = \mathrm{Cotor}^{H(\bar{B}(A);K)}\ (K, K)$

$E^r \Rightarrow E^\circ H(\bar{F}(\bar{B}(A)); K) \approx H(A; K)$ as algebras, and dually,

$(E')^2 = \mathrm{Tor}^{H(\bar{F}(C);K)}\ (K, K)$

$(E')^r \Rightarrow E^\circ H(B(\bar{F}(\bar{C}); K) \approx H(C; K)$ as coalgebras.

We also note that the Kronecker Index $\langle\,,\rangle$: $C \otimes A \rightarrow K$ induces a pairing

$$\langle\ ,\ \rangle\colon\ \bar{F}(C) \otimes \bar{B}(A) \rightarrow K$$

LEMMA 1. *Let $b \in A$ represent $v \in H(A)$. Suppose $v^k = 0$. Then*

$$d_k[\varphi_k(v)] = [\sigma b]^k \text{ in } E^k .$$

Proof. Let

$$V = \sum_{i=1}^{k-1} P(i)\, [[b^i \mid b]]\ ([[b]])^{k-i-1} \text{ where } P(i) = (-1)^{\deg b^i+1}$$

and the outside bars refer to the Cobar Construction and the inside bars refer to the Bar Construction.

Taking ∂V gives a telescoping series and so

$$\partial V = [\sigma b]^k + (-1)^w[\sigma(b^k)]. \text{ Here } (-1)^w = P(k-1) .$$

In E^1, V represents the class $(-1)^w[[b^{k-1} \mid b]] + [[M]] = [\varphi_k(v)]$.

The Lemma follows from the definition of a spectral sequence of a bi complex.

LEMMA 2. *Let $a \in \bar{F}(C)$ represent u. Then, by definition,*

$$\gamma_k[a] = [a \mid \cdots (k) \cdots \mid a] \in \bar{B}(\bar{F}(C)) .$$

If $\gamma_k[a]$ lives to E^{k-1} then $\langle u\rangle^k$ is defined and

$$d_k(\gamma_k[a]) = \langle u\rangle^k \text{ in } (E')^k .$$

Proof. We first make an observation: Suppose $\langle u\rangle^t$ is defined. Let (a_i) be a defining system for $\langle u\rangle^t$. Let

$$W = \sum_{r=2}^{t} \sum_{i_1+\cdots+i_r=t} [a_{i_1} \mid \cdots \mid a_{i_r}] \in \bar{B}(\bar{F}(C)) .$$

Then

$$\partial W = \sum_{i=1}^{t-1} (-1)^{\deg a_i+1} [a_i a_{t-i}] .$$

Now to prove Lemma 2, we use induction on k. Suppose the lemma is true for $k-1$. Suppose $\gamma_k[a]$ lives to E_{k-1}. Since E is a spectral sequence of DGA coalgebras, and $d_{k-1}(\gamma_k[a]) = 0$, we have

$$\Delta\, d_{k-1}\gamma_k[a] = d_{k-1}^{\otimes}\, \Delta\, \gamma_k[a] = d_{k-1}^{\otimes} \sum_{i=0}^{k} \gamma_i[a] \otimes \gamma_{k-i}[a] = 0$$

where $d^{\otimes}$ is the differential in $E' \otimes E'$. That is, in particular when $i = k-1$ in the above, we see

$$d_{k-1}\gamma_{k-1}[a] \otimes [a] = 0 \text{ so } d_{k-1}\, \gamma_{k-1}[a] = 0 .$$

Now by inductive hypothesis, $\langle u\rangle^{k-1}$ is defined so there is a defining system $(a_1, \cdots, a_{k-1})$ for $\langle u\rangle^{k-1}$ and a cochain a_k such that

$$\delta a_k = \sum_{i=2}^{k-2} (-1)^{\deg a_{i-1}} a_{i-1} a_{k-i}$$

since $\langle u\rangle^{k-1} = d_{k-1}\, \gamma_{k-1}[a] = 0$.

The observation at the beginning of this lemma shows that

$$d_k\gamma_k[a] = \langle u\rangle^k .$$

We now give the proof of Theorem 2:

Assume $\varphi_k(v)$ is defined and nonzero. We are assuming $= 1 = \langle \sigma v, u\rangle$. Hence

$$\begin{aligned} 1 &= \langle \sigma v, u\rangle = \langle \sigma b, a\rangle = \langle [\sigma b]^k, \gamma_k[a]\rangle = \langle d_k \varphi_k(v), \gamma_k[a]\rangle \\ &= \langle \varphi_k(v), d_k\gamma_k[a]\rangle = \langle \varphi_k(v), \langle u\rangle^k\rangle \end{aligned}$$

by the duality of the two spectral sequences and Lemma 2.

It remains to be shown that if $\langle u\rangle^k$ is defined and nonzero, then so is $\varphi_k(v)$. Consider the map

$A \to \bar{F}(\bar{B}(A))$ defined by

$b \to [[b]]$.

This map is homotopy multiplicative (in fact is a *SHM* map) and is an equivalence. Hence $[[b^k]]$ differs from $[\sigma b]^k$ by a boundary. But $[\sigma b]^k = [\sigma b \mid \cdots (k) \cdots \mid \sigma b]$ is dual to $\gamma_k[a] = [a \mid \cdots (k) \cdots \mid a]$ in $\bar{B}\bar{F}(C)$, and so $d_k\gamma_k[a] = \langle u\rangle^k$ is not zero in E^k (Lemma 2) and so does not survive to E^∞, i.e., represents 0 in E^∞. The dual element $[\sigma b]^k$ represents 0 in E^∞, i.e., $[[b^k]] \sim [\sigma b]^k \sim 0$. Therefore $b^k \sim 0$ and so $\varphi_k(v)$ is defined.

We wish to mention two applications:

Al: Let $K = Z_p$ and let X be a $K(\pi, n)$ space (p an odd prime). Let $C = C^*(X; Z_p)$ and $A = C_*(X; Z_p)$ be cochain and chain complexes for X of finite type. In the notation of Cartain, $A = A_*(\pi, n; Z_p)$ ([2]). Cartan proved that $\langle \varphi_p(v), \beta P^m(u) \rangle = \langle \sigma v, u \rangle$. Now by Theorem 2, if

$$\langle \sigma v, u \rangle = 1$$

then $\langle \varphi_p(v), \langle u \rangle^p \rangle = 1$. Hence $\langle \varphi_p(v), \beta P^m u + \langle u \rangle^p \rangle = 0$. By Lemma 18 ([5]), $\langle u \rangle^p = c\beta P^m u$. This gives an easy proof of the fact that $c = -1$. (Compare Theorem 19 [5]).

A2: Now let $x = CP^{k-1}$. Then in $H^*(CP^{k-1}; Z) = P(v)_{/(v^k)}$ we have $v^k = 0$. Then $\varphi_k(v)$ is defined in $H^*(\Omega\ CP^{k-1}; Z)$ and by the Theorem 2, so is $\langle u \rangle^k$ in $H_{3k-2}(\Omega\ CP^{k-1}; Z)$ where $u \in H_2(\Omega\, CP^{k-1}, Z)$ and $\langle \varphi(v), \langle u \rangle^k \rangle = 1$. This gives another proof of the results of Stasheff ([7]).

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