Pacific Journal of Mathematics

DIRECTED GRAPHS AS UNIONS OF PARTIAL ORDERS

Peter C. Fishburn and Joel Spencer

Vol. 39, No. 1 May 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

DIRECTED GRAPHS AS UNIONS OF PARTIAL ORDERS

PETER C. FISHBURN AND JOEL H. SPENCER

The index of an irreflexive binary relation R is the smallest cardinal number $\sigma(R)$ such that R equals the union of $\sigma(R)$ partial orders. With $s(n)$ the largest index for an R defined on n points, it is shown that $s(n)/\log_2 n \to 1$ as $n \to \infty$. The index function is examined for symmetric R's and almost transitive R's, and a characterization for $\sigma(R) \leqq 2$ is presented. It is shown also that

$$\inf \{n: s(n) > 3\} \leqq 13 ,$$

but the exact value of $\inf\{n: s(n) > 3\}$ is presently unknown.

1. Introduction. A binary relation on a set X is a subset of ordered pairs xy in $X \times X$. A directed graph (hereafter *digraph*[1]) $G = (X, R)$ is a nonempty set X and an irreflexive ($xx \notin R$) binary relation R on X. If $\phi \subset Y \subseteqq X$ then $G \mid Y$ is the digraph obtained from $G = (X, R)$ by deleting all points in X-Y.

A *partial order* P on X is an irreflexive and transitive ($xy \in P$ & $yz \in P \Rightarrow xz \in P$) binary relation on X. A digraph $G = (X, R)$ is *resolved* by a set of partial orders on X if and only if R equals the *union* of the partial orders in the set. Since $\{xy\}$ is a partial order when $xy \in R$, every G is resolved by some set of partial orders.

The *index*[2] of a digraph $G = (X, R)$ is the smallest cardinal number $\sigma(R)$ such that R is resolved by $\sigma(R)$ partial orders on X. Clearly $\sigma(R) = 1$ if and only if R is a partial order. $\sigma(\{ab, ba\}) = 2$, and $\sigma(R) = 3$ for the cyclic triangle $R = \{ab, bc, ca\}$. The smallest X that we know of that admits an R with $\sigma(R) = 4$ has 13 points. (See Figure 1.) In connection with a later characterization of $\sigma \leqq 2$ we present an R with $\sigma(R) = 2$ where R cannot be the union of two *disjoint* partial orders.

Our definition of $\sigma(R)$ is motivated by Dushnik and Miller's definition [2] of the *dimension of a partial order* P on X as the smallest cardinal number $D(P)$ such that P equals the *intersection* of $D(P)$ linear orders on X. *A linear order* L on X is a complete ($x \neq y \Rightarrow xy \in L$ or $yx \in L$) partial order, and a *chain* in X is a linear

[1] We shall sometimes refer to a binary relation as a digraph, omitting explicit mention of the set on which the relation is defined.

[2] It is tempting to use "dimension" instead of "index," but since the former term is used for a number of other concepts in the theory of binary relations we favor the latter here. It would be proper to write $\sigma(G)$ instead of $\sigma(R)$, but since $\sigma(R) = \sigma(R')$ if R is isomorphic to R' the specific omission of X will cause no problems.

order on a subset of X. A number of facts about $D(P)$ are summarized in [1], which gives other references.

This paper examines the index function σ for digraphs. The next section focuses on large values for $\sigma(R)$. Our first theorem, based on a theorem in Folkman [4], shows that $\sigma(R)$ can be arbitrarily large for both *symmetric* ($xy \in R \Rightarrow yx \in R$) and asymmetric ($xy \in R \Rightarrow yx \notin R$) digraphs. The second theorem examines the behavior of σ in the following way. Let

$$s(n) = \sup\{\sigma(R)\colon R \text{ is an irreflexive binary relation on } n \text{ points}\}\ ,$$

the largest σ for a digraph with n points. When u is a real-valued function on $\{1, 2, \cdots\}$ and $u(n)$ remains bounded as n gets large, we write $u = 0(1)$ according to popular convention. Theorem 2 states that

$$\log_2 n - \frac{1}{2}\log_2\log_2 n + 0(1) \geqq s(n) \geqq \log_2 n - \frac{3}{2}\log_2\log_2 n - 0(1)\ .$$

This gives another proof that σ can be arbitrarily large, and shows that $s(n)/\log_2(n)$ approaches 1 as n gets large.

The rest of the paper is mostly concerned with small values of σ. Section 3 presents an (X, R) with $|X| = 13$ and $\sigma(R) = 4$. We do not presently know the smallest X that admits an R with $\sigma(R) = 4$.

Symmetric digraphs (X, S) are examined in § 4, where we give a necessary and sufficient condition for $\sigma(S) \leqq 2$. Suppose that P is a partial order on X and

$$S = \{xy\colon xy \in X \times X \ \&\ x \neq y \ \&\ xy \notin P \ \&\ yx \notin P\}\ .$$

Then S is a symmetric digraph. We note that when S is defined in this way, then $D(P) \leqq 2$ if and only if $\sigma(S) \leqq 2$, and

$$D(P) \leqq n \Rightarrow \sigma(S) \leqq 2(n-1)\ .$$

The question of whether $\sigma(S) \leqq n \Rightarrow D(P) \leqq f(n)$ for some function f is presently open.

A binary relation R is almost transitive[3] if and only if $(ab \in R \ \&\ bc \in R \ \&\ a \neq c) \Rightarrow ac \in R$. Section 5 proves that $\sigma(R) \leqq 2$ when R is an almost transitive digraph.

Section 6 then gives a general characterization of $\sigma(R) \leqq 2$ that is stated in terms of a partition of the subset of R whose elements

[3] Harary, Norman and Cartwright [**7**, p. 7] call this transitivity, but we use the modifier to distinguish it from the more common use of "transitivity" in which a, b and c do not have to be distinct.

are involved in nontransitive adjacent pairs such as xy, $yz \in R$ & $xz \notin R$.

2. Digraphs with large indices.

THEOREM 1. *If n is a positive integer then there are asymmetric and symmetric digraphs whose indices exceed n.*

Our proof is based on a specialization of Theorem 2 in Folkman [4]. A *graph* (X, E) is a nonempty set X and a set E of unordered pairs $\{x, y\}$ with $x, y \in X$ and $x \neq y$. A *triangle of* (X, E) is a set $\{\{a, b\}, \{b, c\}, \{a, c\}\} \subseteqq E$. A *partition* of X is a set of mutually disjoint subsets of X whose union equals X.

LEMMA 1 (Folkman). *Let m be a positive integer. Then there is a graph (X, E) that includes no triangles, and every partition $\{C_1, \cdots, C_k\}$ of X with $k \leqq m$ contains a C_i such that $a, b \in C_i$ for some $\{a, b\} \in E$.*

Proof of Theorem 1. Let (X, E) be such a graph for $m = 2^n$. Let (X, R) be any digraph for which $xy \in R$ or $yx \in R$ if and only if $\{x, y\} \in E$. Suppose that R is the union of partial orders $P_1, \cdots, P_n$ on X. Since E has no triangles, any subset of a P_i is a partial order and hence we can assume $P_i \cap P_j = \emptyset$ when $i \neq j$. Letting $A(x) = \{i\colon$ for some $y \in X, xy \in P_i\}$, partition X so that x and y are in the same element of the partition if and only if $A(x) = A(y)$. The number of elements in the partition does not exceed 2^n. Thus, by Lemma 1, the partition contains an element Y with $x, y \in Y$ and $\{x, y\} \in E$. Then $A(x) = A(y)$. Since $xy \in R$ or $yx \in R$, take $xy \in P_j$ for definiteness with $j \in A(x)$. Since $j \in A(y)$ also, there is a $z \in X$ such that $yz \in P_j$. Transitivity then implies that $xz \in P_j$ and hence that E includes a triangle, which contradicts our initial hypothesis. Therefore $\sigma(R) > n$. By the definition of R it can be taken to be either asymmetric or symmetric (or neither).

Henceforth in this section all logarithms are to base 2 unless indicated otherwise. $[r] =$ (largest integer $\leqq r$) and $\{r\} =$ (smallest integer $\geqq r$).

THEOREM 2. $\log n - 1/2 \log\log n + 0(1) \geqq s(n) \geqq \log n - 3/2 \log\log n - 0(1)$.

We show first the upper bound, using two preparatory lemmas.

LEMMA 2. *In any digraph $G = (H, R)$ with $|H| = m$ there exists $D \subseteq H$ such that $|D| \geqq \{\log_4 m\} = \{1/2 \log m)$ and $\sigma(G|D) \leqq 2$.*

Proof. We use induction on m, the lemma being obvious for small values of m. Fix $x \in H$. Split $H^* = H - \{x\}$ into four parts:

$$T_1 = \{y \in H^*: xy \notin R \ \&\ yx \notin R\} \qquad S_1 = \emptyset$$

$$T_2 = \{y \in H^*: xy \in R \ \&\ yx \notin R\} \qquad S_2 = \{x\} \times D_2$$

$$T_3 = \{y \in H^*: xy \notin R \ \&\ yx \in R\} \qquad S_3 = D_3 \times \{x\}$$

$$T_4 = \{y \in H^*: xy \in R \ \&\ yx \in R\} \qquad S_4' = \{x\} \times D_4\,,$$

$$S_4'' = D_4 \times \{x\}\,.$$

Some $|T_i| \geqq \{(m-1)/4\}$. By induction find $D_i \subseteq T_i$ with

$$|D_i| \geqq \{\log_4 |T_i|\} \geqq \{\log_4 \{(m-1)/4\}\} = \{\log_4 m\} - 1$$

and $G|D_i = P_1 \cup P_2$. Then set $D = D_i \cup \{x\}$. $G|D = (P_1 \cup S_i) \cup (P_2 \cup S_i)$ except for $i = 4$ when $G|D = (P_1 \cup S_4') \cup (P_2 \cup S_4'')$.

LEMMA 3. *In any digraph $G = (X, R)$ with $|X| = n$ there is a partition $\{D_1, \cdots, D_t\}$ of X such that $t < 3n/\log n$ and $\sigma(G|D_i) \leqq 2$ for each i.*

Proof. Given G, by Lemma 2 find D_1 such that

$$|D_1| = x_1 \geqq \{\log_4 n\}\,.$$

By induction find D_i such that

$$|D_i| = x_i \geqq \left\{\log_4\left(n - \sum_{j=1}^{i-1} x_j\right)\right\}\,.$$

From elementary calculus we can show $\sum_{i=1}^{t} x_i \geqq n$ for

$$t \leqq (2+\varepsilon)n/\log n\,.$$

We now show the upper bound for Theorem 2. Let $G = (X, R)$ with $|X| = n$. Take $D_1, \cdots, D_t$ as in Lemma 3. Let $\{A_i^*, B_i^*\}$ be a partition of $\{1, \cdots, t\}$ for $i = 1, \cdots, s$ such that for all $1 \leqq j \neq k \leqq t$ there exists i, $1 \leqq i \leqq s$, such that $j \in A_i^*$ & $k \in B_i^*$. By Spencer [12] we may take

$$s = \log t + 1/2 \log\log t + 0(1) \leqq \log n - 1/2 \log\log n + 0(1)\,.$$

$\{A_i^*, B_i^*\}$ induces a partition $\{A_i, B_i\}$ of X with

$$A_i = \bigcup_{j \in A_i^*} D_j\,, \qquad B_i = \bigcup_{j \in B_i^*} D_j\,.$$

Then set

$$P_i = \{xy\colon x \in A_i \ \& \ y \in B_i \ \& \ xy \in R\} \quad \text{for } i = 1, \cdots, s .$$

Since $\sigma(G \mid D_i) \leqq 2$, $G \mid D_i = P'_i \cup P''_i$. Set

$$P' = \bigcup_{i=1}^{s} P'_i,\ P'' = \bigcup_{i=1}^{s} P''_i .$$

Then $R = P' \cup P'' \cup P_1 \cup \cdots \cup P_s$, giving the upper bound of Theorem 2.

We turn to the lower bound of the theorem, again using two preliminary lemmas. A complete asymmetric digraph is a *tournament*.[4] We shall show that a "random" tournament $T = (X, R)$ with $|X| = n$ has $\sigma(T) \geqq \log n - 3/2 \log \log n - 0(1)$. Intuitively speaking, we show that all $P \subseteqq T$ are essentially bipartite.

Let T^n be the set of tournaments with $X = \{1, 2, \cdots, n\}$. We say that $T = (X, R) \in T^n$ has *property* α if and only if there are A, $B \subseteqq X$ with $|A| = |B| \geqq 3 \log n$ and $A \times B \subseteqq R$. T has *property* β if and only if there is an $A \subseteqq X$ and a linear order L on A such that $|A| \geqq (\log n)^2$ and

$$(*) \qquad |R \cap L| \leqq \frac{1}{3}\binom{|A|}{2} .$$

LEMMA 4. *For n sufficiently large there exists $T \in T^n$ satisfying neither property α nor property β.*

Proof. If $T \in T^n$ has property α, there are $A, B \subseteqq X$ with $|A| = |B| = [3 \log n]$ and $A \times B \subseteqq R$. Set $t = [3 \log n]$. For fixed A and B, 2^{-t^2} is the proportion of $T \in T^n$ that satisfy this condition. There are less than n^{2t} choices of A and B, so less than $n^{2t}2^{-t^2}$ of the $T \in T^n$ satisfy α. $n^{2t}2^{-t^2} \to 0$ as $n \to \infty$.

If $T \in T^n$ has property β, there exists $A \subseteqq X$ and L on A such that $|A| = [(\log n)^2]$ and $(*)$ holds. There are less than $n^{(\log n)^2}$ choices of A and then $[(\log n)^2]!$ choices of L. Given A and L, the proportion of $T \in T^n$ satisfying $(*)$ is the probability of at most $\binom{t}{2}/3$ heads in $\binom{t}{2}$ flips of a fair coin where $t = |A| \sim (\log n)^2$. This probability is approximately $p^{-\binom{t}{2}}$ where $p = 3^{1/3}\ (3/2)^{2/3} > 1$. Thus the proportion of $T \in T^n$ satisfying β is less than

$$n^{(\log n)^2}\,[(\log n)^2]!\ p^{-\binom{t}{2}},\ \text{which} \longrightarrow 0 \text{ as } n \longrightarrow \infty .$$

Thus for n sufficiently large some $T \in T^n$ can satisfy neither α nor β.

---

[4] See Moon [**9**] for extensive discussion of tournaments. See also [**3**, **10**, **11**] for resulted to the present paper.

LEMMA 5. *If* $T_1, \cdots, T_n \subseteqq \{1, \cdots, s\}$ *then there are* $n/\binom{s}{s/2}$ T_i *which are mutually comparable.*[6]

Proof. We use a technique due to Lubell [8]. There are $s!$ maximal chains of subsets of $\{1, \cdots, s\}$ under the ordering of $\subset$. If $|T_i| = a$ then T_i is in $a!\ (s-a)! \geqq (s/2)!^2 = s!/\binom{s}{s/2}$ maximal chains. Thus some maximal chain must contain $n[s!/\binom{s}{s/2}]/s!$ T_i.

In the following proof of the lower bound of Theorem 2 we use the fact that $1/\binom{s}{s/2} \sim \sqrt{\pi/2}\ \sqrt{s}\ 2^{-s}$.

Let $G = (X, R)$ be a tournament that satisfies neither α nor β (Lemma 4). Suppose that $R = P_1 \cup \cdots \cup P_s$. Define

$$\begin{aligned} W_i &= \{x \in X\colon |\{y \in X\colon xy \in P_i\}| > 3 \log n\} \\ L_i &= \{x \in X\colon |\{y \in X\colon yx \in P_i\}| > 3 \log n\} \\ R_i &= X - W_i - L_i \end{aligned}$$

for $1 \leqq i \leqq s$. (We split X into winners, losers, and the rest.) By Lemma 4, $W_i \cap L_i = \varnothing$. For $x \in X$ set

$$T_x = \{i\colon x \in W_i \cup R_i\} \subseteqq \{1, \cdots, s\}\ .$$

By Lemma 5 find $V \subseteqq X$ such that $|V| \geqq n\sqrt{\pi/2}\sqrt{s}\ 2^{-s}$ and $T_x \subseteqq T_y$ or $T_y \subseteqq T_x$ whenever $x, y \in V$. Induce a linear order L on V by setting $xy \in L$ if $T_x \subset T_y$: when $T_x = T_y$, L is defined in any fixed manner.

Now assume $s < \log n - 3/2 \log\log n - 7$. Then $|V| \geqq 2^7\sqrt{\pi/2}$ $(\log n)^2$. Set

$$Z_i = L \cap P_i \qquad\qquad 1 \leqq i \leqq s\ .$$

Given $xy \in Z_i$, $T_x \subseteqq T_y$ so that we cannot have $x \in W_i$ & $y \in L_i$. And since $W_i \cap L_i = \varnothing$ we cannot have $x \in L_i$ & $y \in W_i$. Therefore

$$Z_i = \{xy \in Z_i\colon x \text{ or } y \in R_i\} \cup \{xy \in Z_i\colon x, y \in W_i\} \cup \{xy \in Z_i\colon x, y \in L_i\}\ .$$

There are at most $6 \log n\ |V|$, $3 \log n\ |V|$ and $3 \log n\ |V|$ ordered pairs in the first, second and third parts respectively of this decomposition of Z_i. Thus $|Z_i| \leqq 12 \log n\ |V|$. Since G does not have property β it follows that

$$\frac{1}{3}\binom{|V|}{2} \leqq |R \cap L| \leqq \sum_{i=1}^{n} |Z_i| \leqq 12\,(\log n)^2 |V|$$

and hence that $|V| \leqq 72\,(\log n)^2 + 1$. Since this contradicts $|V| \geqq 2^7\sqrt{\pi/2}\,(\log n)^2$ it must be true that $s \geqq \log n - 3/2 \log\log n - 0(1)$.

[5] T_i and T_j are mutually comparable if and only if $T_i \subseteq T_j$ or $T_j \subseteq T_i$.

This completes the proof of Theorem 2.

If a sufficiently good bound could be placed on

$$\{xy \in P_i\colon\ x \text{ or } y \in R_i \text{ or } x,\, y \in W_i \text{ or } x,\, y \in L_i\}$$

then one could prove $s(n) = \log n - 1/2 \log\log n + o(\log\log n)$. One might even show that $s(n) = \log n - 1/2 \log\log n + 0(1)$.

3. A digraph with $\sigma = 4$ and $|X| = 13$. Although the theorems of the preceding section show that there are digraphs with large indices, they are of little use in attempting to discover the smallest X that admits an R for which $\sigma(R) = n$. Figure 1 shows the smallest X that we know of for which $\sigma(R) = 4$.

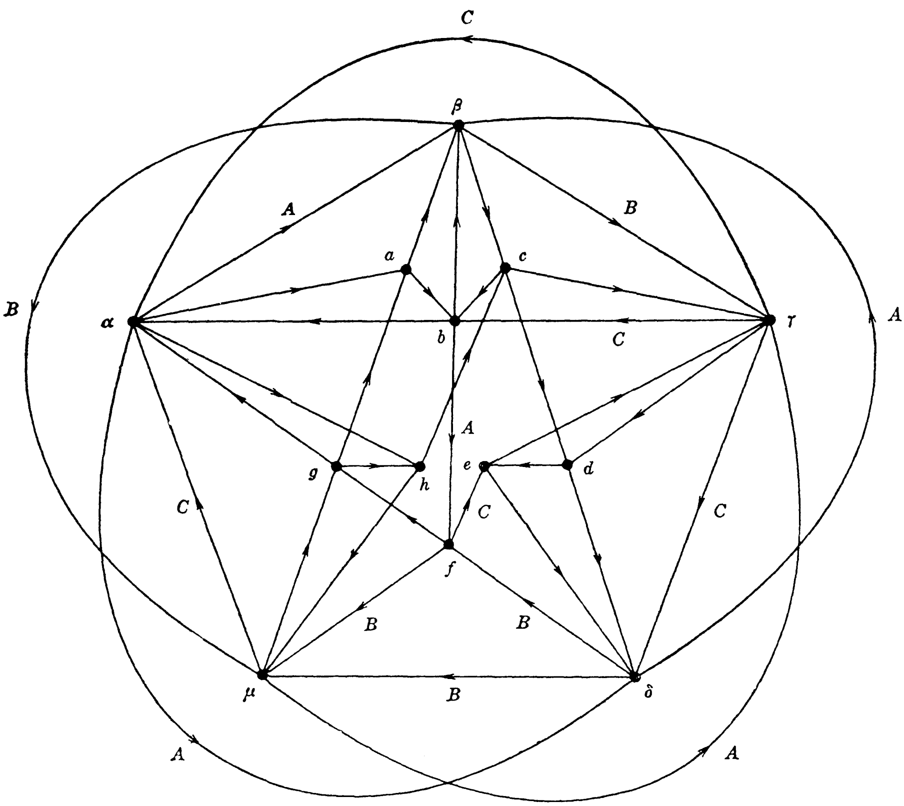

FIGURE 1

Assume that $\sigma(R) = 3$ for Figure 1, with A, B and C three partial orders whose union equals R. Then one of A, B and C must contain exactly one of $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\mu$ and $\mu\alpha$ and the other two must each contain exactly two of these ordered pairs in alternating fashion.

Suppose for example that $\alpha\beta \in A$, $\beta\gamma \in B$, $\gamma\delta \in C$, $\delta\mu \in B$, $\mu\alpha \in C$. Then $\gamma\alpha$, $\delta\beta$, $\mu\gamma$, $\alpha\delta$, and $\beta\mu$ must be respectively in C, A, A, A, and B. Then $\gamma b \in C$ and δf, $f\mu \in B$. Since $\gamma b \in C$ and $f\mu \in B$, $bf \in A$. Since $bf \in A$ and $\delta f \in B$, $fe \in C$. By the cyclic triangle $\{fe, e\delta, \delta f\}$, $e\delta$ must be in A. But since $\delta\beta \in A$ this implies $e\beta \in A$, which is false. A similar contradiction to $\sigma = 3$ is obtained when any alternative assignment is made for $\alpha\beta, \beta\gamma, \cdots, \mu\alpha$.

4. Indices of symmetric digraphs.

In this section we consider symmetric ($xy \in S \Rightarrow yx \in S$) digraphs (X, S). For any binary relation R, $R^* = \{xy\colon yx \in R\}$, the *converse* or *dual of* R.

A graph (X, E) is a *comparability graph* if and only if there is a partial order P on X such that $\{x, y\} \in E$ if and only if $xy \in P \cup P^*$. Ghouila-Houri [5] and Gilmore and Hoffman [6] provide characterizations of comparability graphs. When (X, S) is a symmetric digraph, $(X, E(S))$ will denote the graph in which $\{x, y\} \in E(S)$ if and only if $xy \in S$.

THEOREM 3. *Suppose that (X, S) is a symmetric digraph. Then $\sigma(S) \leqq 2$ if and only if $(X, E(S))$ is a comparability graph.*

Proof. If $(X, E(S))$ is a comparability graph then $S = P \cup P^*$ for a partial order P, and thus $\sigma(S) \leqq 2$. Conversely, if $S = P_1 \cup P_2$ with P_1 and P_2 partial orders, then $P_2 = P_1^*$.

In [1] it is shown that if (X, P) is a transitive digraph (so that P is a partial order) and if $S = \{xy\colon x \neq y \;\&\; xy \notin P \cup P^*\}$ then $D(P) \leqq 2$ if and only if $(X, E(S))$ is a comparability graph. Hence, as a corollary to Theorem 3 we have $D(P) \leqq 2$ if and only if $\sigma(S) \leqq 2$. Our next theorem extends this in one direction.

THEOREM 4. *Suppose that P on X is a partial order and let $S = \{xy\colon x \neq y \;\&\; xy \notin P \cup P^*\}$. Then $D(P) \leqq n \Rightarrow \sigma(S) \leqq 2(n-1)$ for $n > 1$.*

Proof. The theorem is true for $n = 2$. Using induction, assume it's true for all $n < m$ and suppose $D(P) = m$ with $P = \bigcap_1^m L_i$ where each L_i is a linear order. Let $P' = \bigcap_2^m L_i$ and

$$S' = \{xy\colon x \neq y \;\&\; xy \notin P' \cup (P')^*\} .$$

Since $D(P') \leqq m - 1$, the induction hypothesis gives $\sigma(S') \leqq 2(m-2)$. Clearly $S' \subseteqq S$ and $S - S' = (P' \cap L_1^*) \cup ((P')^* \cap L_1)$. Since $P' \cap L_1^*$ is a partial order (the intersection of two partial orders) and $(P')^* \cap L_1$ is a partial order, $\sigma(S) \leqq \sigma(S') + 2 \leqq 2(m-2) + 2 = 2(m-1)$.

5. Almost transitive digraphs. The proof of the next theorem has several similarities to Szpilrajn's proof [13] of the theorem that any partial order P on X can be extended to a linear order L with $P \subseteqq L$. We recall that R is almost transitive if and only if $(ab \in R$ & $bc \in R$ & $a \neq c) \Rightarrow ac \in R$.

THEOREM 5. *$\sigma(R) \leqq 2$ if (X, R) is an almost transitive digraph.*

Proof. Assume that (X, R) is an almost transitive digraph. Let $A = \{ab: ab \in R \text{ \& } ba \notin R\}$, the asymmetric part of R. Let $A^+ = \{ab: ab \in A \text{ or } \{aa_1, a_1a_2, \cdots, a_nb\} \subseteqq A$ for distinct $a_1, \cdots, a_n$ in X that are different from a and $b\}$, the almost transitive closure of A. Clearly $A^+ \subseteqq R$ and A^+ is almost transitive.

To show that A^+ is a partial order it suffices to show that it is asymmetric. To the contrary suppose that $xy \in A^+$ and $yx \in A^+$. Then from the definition of A^+ and almost transitivity for R it follows easily that there is a $c \in X$ for which $cx \in A$ and $xc \in R$, which contradicts the definition of A. Hence A^+ is a partial order.

Let $\mathscr{P} = \{P: P$ is a partial order on X & $A^+ \subseteqq P \subseteqq R\}$. It follows easily from Zorn's lemma that there is a $P^* \in \mathscr{P}$ such that $P^* \subset P$ for no $P \in \mathscr{P}$. Letting P^* be maximal in this sense we now prove that

$$ab,\ ba \in R \Rightarrow ab \in P^* \text{ or } ba \in P^* .$$

To the contrary suppose that each of ab and ba is in R and neither is in P^*. Then let

$$W = \{xy: x \neq y \text{ \& } (xa \in P^* \text{ or } x = a) \text{ \& } (by \in P^* \text{ or } y = b)\} ,$$

and let $V = P^* \cup W$, so that $P^* \subset V$. We show that V is a partial order (clearly $A^+ \subseteqq V \subseteqq R$), thus contradicting the maximality of P^*. V is irreflexive since P^* and W are irreflexive. For transitivity take $xy, yz \in V$. If both xy and yz are in P^* then $xz \in P^*$ by the transitivity of P^*.

Suppose next that $xy \in P^*$ and $yz \in W$. The latter gives $(ya \in P^*$ or $y = a)$, from which $xa \in P^*$ follows, and it gives also $(bz \in P^*$ or $z = b)$, from which $xz \in V$ follows unless $x = z$. But if $x = z$ we have $xa \in P^*$ and $(bx \in P^*$ or $x = b)$, which give $ba \in P^*$, contradicting the hypothesis that $ba \notin P^*$. Hence $xy \in P^*$ & $yz \in W \Rightarrow xz \in V$. Similarly, $xy \in W$ & $yz \in P^* \Rightarrow xz \in V$.

The final case for transitivity is $xy, yz \in W$. Then $(xa \in P^*$ or $x = a)$ and $(bz \in P^*$ or $z = b)$ so that $xz \in W$ unless $x = z$. But if $x = z$ then $[(xa \in P^* \text{ or } x = a) \text{ \& } (bx \in P^* \text{ } or \text{ } x = b)] \Rightarrow (ba \in P^* \text{ or } b = a)$, which is false. Hence V is a partial order, a contradiction to the maximality of P^*, and therefore

$$ab, ba \in R \Longrightarrow ab \in P^* \text{ or } ba \in P^* .$$

Finally, let $Q = R - P^*$ so that $R = P^* \cup Q$. Q is irreflexive since R is irreflexive. Suppose that $xy, yz \in Q$. Then, since both xy and yz are in R but not A, yx and zy are in R and must be in P^* by the preceding analysis. Therefore $zx \in P^*$ and $z \neq x$. Then, by almost transitivity of R, $xz \in R$ and thus $xz \in Q$ since P^* is asymmetric.

Thus $R = P^* \cup Q$, the union of two partial orders.

6. A partition characterization for $\sigma \leqq 2$. Given a digraph (X, R) let K be the set of all ordered pairs of pairs in R that deny transitivity, so that

$$xyKyz \text{ if and only if } xy \in R \;\&\; yz \in R \;\&\; xz \notin R ,$$

and let V be the subset of R involved in these intransitivities so that

$$V = \{xy\colon xyKyz \text{ or } zxKxy \text{ for some } z \in X\} .$$

Suppose that $\sigma(R) \leqq 2$. If $xyKyz$ then xy and yz must be in different resolving partial orders, so that the digraph (V, K) must be bipartite or 2-colorable. Moreover, if xy and yz are in V and in the same resolving partial order and if $xz \in V$ also, then transitivity requires that xz be in this partial order. These two necessary conditions for $\sigma(R) \leqq 2$ are reflected in A1 and A2 of Theorem 6. Their insufficiency for $\sigma(R) \leqq 2$ is noted later. (Note that $\sigma(R) = 1$ if and only if $V = \varnothing$.)

THEOREM 6. *Suppose that (X, R) is a digraph and $V \neq \varnothing$. Then $\sigma(R) = 2$ if and only if V can be partitioned into V_1 and V_2 so that*

A1. *$xyKyz \Rightarrow xy$ and yz are in different V_i ,*

A2. *$xy, yz \in V_i \;\&\; xz \in V \Rightarrow xz \in V_i$,*

A3. *$xy \in R - V \Rightarrow$ (1) and (2) do not hold simultaneously:*

(1) $(yz \in V_2 \;\&\; xz \in V_1)$ *or* $(zx \in V_2 \;\&\; zy \in V_1)$, *for some* $z \in X$,

(2) $(yw \in V_1 \;\&\; xw \in V_2)$ *or* $(wx \in V_1 \;\&\; wy \in V_2)$, *for some* $w \in X$.

If $R = P_1 \cup P_2$ then $V_i = P_i \cap V$ for $i = 1, 2$ are easily seen to satisfy A1 through A3, and $V_1 \cap V_2 = \varnothing$.

Before proving sufficiency we show that A1 and A2 are not sufficient for $\sigma = 2$. All directed edges in the 13-point asymmetric

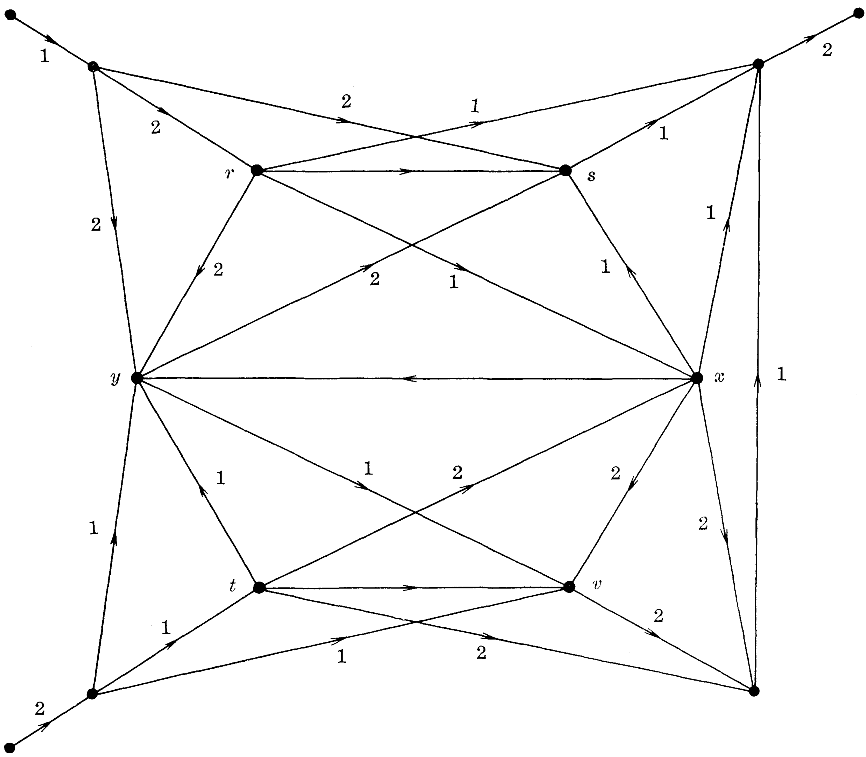

FIGURE 2

digraph of Figure 2 are in V except for xy, rs and tv, and A1 and A2 hold. Labels 1 and 2 for P_1 and P_2 are assigned to the edges in V in the only way consistent with A1 and A2, beginning with P_1 in the upper left corner. For $\sigma(R) = 2$ we require rs and tv in both P_1 and P_2, but xy violates A3 and cannot be assigned either

$$P_1\,[rx \in P_1 \;\&\; ry \notin P_1] \text{ or } P_2\,[tx \in P_2 \;\&\; ty \notin P_2]\;.$$

By deleting the edge xy from Figure 2 we obtain an R with $\sigma(R) = 2$ where R is not the union of two *disjoint* partial orders.

Sufficiency Proof for Theorem 6. With $V \neq \varnothing$ let A1, A2 and A3 hold. For $i = 1, 2$ let

$$S_i = \{xy\colon xy \in R - V \;\&\; \text{(i) holds}\}\;.$$

Let $R^0 = R - V - S_1 - S_2$ and for $i = 1, 2$ define P_i by

$$P_i = V_i \cup S_i \cup R^0\;.$$

Since $P_i \subseteqq R$, it is irreflexive. We now prove that P_1 is transitive. The proof for P_2 is similar.

Assume that $xy, yz \in P_1$. Then $xz \in R$, for if both xy and yz are in V_1 then $xz \in R$ by A1, and if one of xy and yz is in $S_1 \cup R^0$ then $xz \in R$ by the definitions. Thus $xz \in P_1$ unless $xz \in V_2 \cup S_2$. $xz \in V_2$ is contradicted in all cases:

1. $xy, yz \in V_1 \Rightarrow xz \notin V_2$, by A2;
2. $xy \in V_1$ & $yz \in S_1 \Rightarrow xz \notin V_2$, by A3;
3. $xy \in V_1$ & $yz \in R^0 \Rightarrow xz \notin V_2$, by A3;
4. $xy, yz \in S_1 \cup R^0$. Then $ax \in R \Rightarrow ay \in R \Rightarrow az \in R$ and $za \in R \Rightarrow ya \in R \Rightarrow xa \in R$. Hence neither $axKxz$ nor $xzKza$ can hold.

It remains to show that $xz \notin S_2$. Assume $xz \in S_2$ to the contrary and for definiteness take $zw \in V_1$ and $xw \in V_2$ (Figure 3). We note first

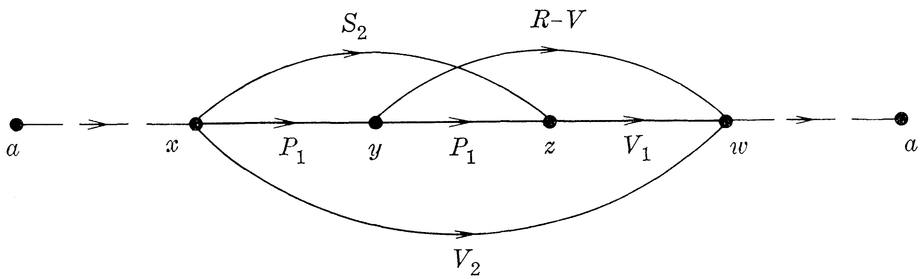

FIGURE 3

that $yw \notin V_2$, for $yw \in V_2 \Rightarrow yz \in S_2$. Moreover, $yw \notin V_1$, for $yw \in V_1$ & $xy \in V_1$ contradict A2, and $yw \in V_1$ & $xy \in S_1 \cup R^0$ contradict the definition of S_2 along with A3. Hence $yw \in R - V$. Now if $ax \in V_1$ then $ay \in R$ and hence (since $yw \in R - V$) $aw \in R$; and if $wa \in V_1$ then $za \in R$ and hence (since $xz \in R - V$) $xa \in R$. Since $xw \in V_2$ requires either $axKxw$ with $ax \in V_1$ or $xwKwa$ with $wa \in V_1$, and since $ax \in V_1$ contradicts $axKxw$ (since $aw \in R$) and $wa \in V_1$ contradicts $xwKwa$ (since $xa \in R$), the proof is complete.

REFERENCES

1. K. A. Baker, P. C. Fishburn, and F. S. Roberts, *Partial Orders of Dimension* 2, *Interval Orders, and Interval Graphs*, RAND Paper P-4376, The RAND Corporation, 1970.

2. B. Dushnik and E. W. Miller, *Partially Ordered Sets*, Amer. J. Math., **63** (1941), 600-610.

3. P. Erdös and L. Moser, *On the Representation of Directed Graphs as Unions of Orderings*, Math. Inst. Hung. Acad. Sci., **9** (1964), 125-132.

4. J. Folkman, *Graphs with Monochromatic Complete Subgraphs in Every Edge Coloring*, SIAM J. Appl. Math., **18** (1970), 19-24.

5. A. Ghouila-Houri, *Caractérisation des graphes nonorientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre*, C. R. Acad. Sci. Paris **254** (1962), 1370-1371.

6. P. C. Gilmore and A. J. Hoffman, *A Characterization of Comparability Graphs and of Interval Graphs*, Canad. J. Math., **16** (1964), 539-548.
7. F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models: An Introduction to the Theory of Directed Graphs*, Wiley, New York, 1965.
8. D. Lubell, *A Short Proof of Sperner's Lemma*, J. Combinatorial Theory, **1** (1966), 299.
9. J. W. Moon, *Topics on Tournaments*, Holt, New York, 1969.
10. K. B. Reid and E. T. Parker, *Disproof of a conjecture of Erdös and Moser on tournaments*, J. Combinatorial Theory, **9** (1970), 225-238..
11. J. Spencer, *Notes on Combinatorial Mathematics: Transitive Subtournaments*, RAND Memorandum RM-6156-PR, The RAND Corporation, 1969.
12. ———, *Minimal Completely Separating Systems*, J. Combinatorial Theory, **8** (1970), 446-447.
13. E. Szpilrajn, *Sur extension de l'ordre partiel.* Fundamenta Math., **16** (1930), 386-389.

Received July 22, 1970 and in revised form October 12, 1970.

THE INSTITUTE FOR ADVANCED STUDY AND THE RESEARCH ANALYSIS CORPORATION
AND
THE RAND CORPORATION, SANTA MONICA

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 39, No. 1 May, 1971