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ON SOLVABLE O^* -GROUPS

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ON SOLVABLE O*-GROUPS

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The purpose of this paper is to prove the existence of O*-groups of arbitrary solvable length, as well as of non-solvable O*-groups.

By a *partial order* for a group G we mean a reflexive, antisymmetric and transitive relation, \leq , on G such that if g and h are elements of G and $g \leq h$, then $xgy \leq xhy$ for all x and y in G . If also any two elements g and h of G are *comparable* (i.e., either $g \leq h$ or $h \leq g$), then the partial order for G is called a *total* (or *full*, or *linear*) *order*. The group G is an *O*-group* if any partial order for G is included in some total order for G .

A group G is *solvable of length* n , where n is a positive integer, if the derived chain of G reaches the unit subgroup, E , in n steps:

$$G = G^1 \supseteq G^2 \supseteq \cdots \supseteq G^n \supseteq G^{n+1} = E,$$

where G^{i+1} is the derived group of G^i (denoted below by $G^{i+1} = [G^i, G^i]$).

It has been shown that non-abelian free groups are not O*-groups ([1], [2], [3], [4], [6]). Further, Kargapolov [5] and Kargapolov, Kokorin and Kopytov [6] have produced solvable groups which are not O*-groups even though they admit a full order: these are the free r -step solvable groups on k generators for $r \geq 3$ and $k \geq 2$. In view of these results one may ask if there exist solvable O*-groups of arbitrary length, and nonsolvable O*-groups. The answers are affirmative.

THEOREM. *For every positive integer m there exists an O*-group G that is solvable of length m .*

Proof. Let F be the free group on k generators for some fixed $k \geq 2$. Let F_i be the i th term in the lower central series for F , where $F_1 = F$, and let F^i be the i th derived group for F , where $F^1 = F$. Consider F/F_i , the free nilpotent group of class i with k generators. By varying i we shall obtain the desired groups G of the theorem.

We first claim that F/F_i is torsion-free for every positive integer i . If not, then for some i there exists an element $a \in F$ and a positive integer p such that $a \notin F_i$, but $a^p \in F_i$. Now $a \in F_h - F_{h+1}$ for some positive integer $h \leq i - 1$. Thus $a^p \in F_i \subseteq F_{h+1}$, and so F_h/F_{h+1} is not torsion-free. On the other hand, Witt's theorem (see, e.g., [8, p. 41]) states that F_h/F_{h+1} is a free abelian group (and hence torsion-

free), a contradiction. Thus F/F_i is torsion-free, as claimed.

Malcev [9] has shown that a torsion-free nilpotent group is an O^* -group. Hence F/F_i is an O^* -group for every positive integer i . Now for every such i the solvable length of F/F_i is finite, since F/F_i is nilpotent. Thus we shall complete our proof by establishing the following lemma.

LEMMA. *For every positive integer m , there exists an integer n such that solvable length of F/F_n is m .*

Proof. We first note that for every positive integer i , there exists an integer j such that $F_j \not\cong F^i$. This follows from the fact that $F^i \neq E$ for each i (hence F is not solvable), together with the theorem of Magnus (cf. [8, p. 38]) which asserts that $\bigcap_{i=1}^{\infty} F^i = E$. We next show that for each i and j ,

$$(1) \quad (F/F_j)^i = F^i F_j / F_j.$$

Indeed, it is readily seen that if A and B are subgroups of a group G and if B is invariant under conjugation by elements of A , then $(AB/B)^2 = A^2 B/B$. From this, an induction on i shows that for a normal subgroup N of a group G it is true that $(G/N)^i = G^i N/N$ for all i , which implies the desired result.

Note that for each i there exists J such that for $j \geq J$, the solvable length of F/F_j exceeds i . This follows from (1) and the fact that, by our first assertion, we can choose J such that $F_j \not\cong F^i$. In particular, then, the solvable length of F/F_j is unbounded with increasing j . Note also that the solvable length of F/F_{j+1} exceeds the solvable length of F/F_j by at most 1. For if F/F_j is solvable of length $r - 1$, then $(F/F_j)^r = E$. Thus, by (1) we have $F^r F_j / F_j = E$, which implies $F^r \subseteq F_j$. On the other hand, F/F_{j+1} has solvable length $\leq r$ since (again using (1))

$$\begin{aligned} (F/F_{j+1})^{r+1} &= [(F/F_{j+1})^r, (F/F_{j+1})^r] \\ &= [F^r F_{j+1} / F_{j+1}, F^r F_{j+1} / F_{j+1}] \subseteq [F_j / F_{j+1}, F_j / F_{j+1}] = E, \end{aligned}$$

where \subseteq holds since both F^r and F_{j+1} are subsets of F_j , and the final equality derives from the fact that F_j / F_{j+1} is abelian by Witt's theorem (above). The lemma follows at once from these results and the fact that $F/F_2 = F_1/F_2$ has solvable length 1 by Witt's theorem.

The proof of the theorem is now complete.

COROLLARY. *There exist nonsolvable O^* -groups.*

Proof. Kargapolov [5] and Kokorin [7] have shown that the re-

stricted direct product of O*-groups is an O*-group. Thus the restricted direct product, $G = \prod_{i=1}^{\infty} F/F_i$, of the groups F/F_i is an O*-group. If G were solvable of length m , then each F/F_i would have solvable length $\leq m$; for if a subgroup $H \subseteq G$, then $H^k \subseteq G^k$ for every k . Since this contradicts the fact noted above that the solvable length of F/F_j is unbounded with increasing j , G is a non-solvable O*-group.

Note. The mapping φ of F into the *unrestricted* (or *complete*) direct product, $\prod_{i=1}^{\infty} F/F_i$, of the groups F/F_i given by

$$\varphi(a) = (aF_1, \dots, aF_n, \dots) \text{ for every } a \in F$$

is a monomorphism by Magnus' theorem, above. Since F is not an O*-group (see [1], [4], or [6]), we have an immediate example of a subdirect product of O*-groups which is not itself an O*-group. (In [5], Kargapolov uses some of the groups F/F_i to show that the class of O*-groups is not closed under formation of unrestricted direct products.)

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REFERENCES

1. L. Fuchs, *On orderable groups*, Proc. Internat. Conf. Theory of Groups (Canberra, 1965), 89-98, Gordon and Breach, New York, 1967.
2. L. Fuchs and E. Sasiada, *Note on orderable groups*, Annales Universitatis Scientiarum Budapestinensis de Rolando Eotvos Nominatae, Sectio Mathematica (1964), 13-17.
3. W. C. Holland, *Extensions of Ordered Algebraic Structures*, Doctoral Thesis, Tulane University, 1961.
4. H. Hollister, *Groups in which every maximal partial order is isolated*, Proc. Amer. Math. Soc., **19** (1968), 467-469.
5. M. I. Kargapolov, *Fully orderable groups* (Russian), Algebra i Logika Sem., (6) **2** (1963), 5-14.
6. M. I. Kargapolov, A. I. Kokorin, and V. M. Kopytov, *On the theory of orderable groups* (Russian), Algebra i Logika Sem., (6) **4** (1965), 21-27.
7. A. I. Kokorin, *Ordering a direct product of ordered groups* (Russian), Ural. Gos. Univ. Mat. Zap., (3) **4** (1963), 95-96.
8. A. G. Kurosh, *The Theory of Groups*, vol. ii, Chelsea, New York, 1956.
9. A. I. Malcev, *On the completion of group order* (Russian), Trudy Mat. Inst. Steklov, **38** (1951), 173-175.

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