

# Pacific Journal of Mathematics

**ON SOLVABLE  $O^*$ -GROUPS**

DONALD PAUL MINASSIAN

## ON SOLVABLE $O^*$ -GROUPS

D. P. MINASSIAN

**The purpose of this paper is to prove the existence of  $O^*$ -groups of arbitrary solvable length, as well as of non-solvable  $O^*$ -groups.**

By a *partial order* for a group  $G$  we mean a reflexive, antisymmetric and transitive relation,  $\leq$ , on  $G$  such that if  $g$  and  $h$  are elements of  $G$  and  $g \leq h$ , then  $xyg \leq xhy$  for all  $x$  and  $y$  in  $G$ . If also any two elements  $g$  and  $h$  of  $G$  are *comparable* (i.e., either  $g \leq h$  or  $h \leq g$ ), then the partial order for  $G$  is called a *total* (or *full*, or *linear*) *order*. The group  $G$  is an  $O^*$ -group if any partial order for  $G$  is included in some total order for  $G$ .

A group  $G$  is *solvable of length  $n$* , where  $n$  is a positive integer, if the derived chain of  $G$  reaches the unit subgroup,  $E$ , in  $n$  steps:

$$G = G^1 \supseteq G^2 \supseteq \cdots \supseteq G^n \supseteq G^{n+1} = E,$$

where  $G^{i+1}$  is the derived group of  $G^i$  (denoted below by  $G^{i+1} = [G^i, G^i]$ ).

It has been shown that non-abelian free groups are not  $O^*$ -groups ([1], [2], [3], [4], [6]). Further, Kargapolov [5] and Kargapolov, Kokorin and Kopytov [6] have produced solvable groups which are not  $O^*$ -groups even though they admit a full order: these are the free  $r$ -step solvable groups on  $k$  generators for  $r \geq 3$  and  $k \geq 2$ . In view of these results one may ask if there exist solvable  $O^*$ -groups of arbitrary length, and nonsolvable  $O^*$ -groups. The answers are affirmative.

**THEOREM.** *For every positive integer  $m$  there exists an  $O^*$ -group  $G$  that is solvable of length  $m$ .*

*Proof.* Let  $F$  be the free group on  $k$  generators for some fixed  $k \geq 2$ . Let  $F_i$  be the  $i$ th term in the lower central series for  $F$ , where  $F_1 = F$ , and let  $F^i$  be the  $i$ th derived group for  $F$ , where  $F^1 = F$ . Consider  $F/F_i$ , the free nilpotent group of class  $i$  with  $k$  generators. By varying  $i$  we shall obtain the desired groups  $G$  of the theorem.

We first claim that  $F/F_i$  is torsion-free for every positive integer  $i$ . If not, then for some  $i$  there exists an element  $a \in F$  and a positive integer  $p$  such that  $a \notin F_i$ , but  $a^p \in F_i$ . Now  $a \in F_h - F_{h+1}$  for some positive integer  $h \leq i - 1$ . Thus  $a^p \in F_i \subseteq F_{h+1}$ , and so  $F_h/F_{h+1}$  is not torsion-free. On the other hand, Witt's theorem (see, e.g., [8, p. 41]) states that  $F_h/F_{h+1}$  is a free abelian group (and hence torsion-

free), a contradiction. Thus  $F/F_i$  is torsion-free, as claimed.

Malcev [9] has shown that a torsion-free nilpotent group is an  $O^*$ -group. Hence  $F/F_i$  is an  $O^*$ -group for every positive integer  $i$ . Now for every such  $i$  the solvable length of  $F/F_i$  is finite, since  $F/F_i$  is nilpotent. Thus we shall complete our proof by establishing the following lemma.

LEMMA. *For every positive integer  $m$ , there exists an integer  $n$  such that solvable length of  $F/F_n$  is  $m$ .*

*Proof.* We first note that for every positive integer  $i$ , there exists an integer  $j$  such that  $F_j \not\cong F^i$ . This follows from the fact that  $F^i \neq E$  for each  $i$  (hence  $F$  is not solvable), together with the theorem of Magnus (cf. [8, p. 38]) which asserts that  $\bigcap_{i=1}^{\infty} F_i = E$ . We next show that for each  $i$  and  $j$ ,

$$(1) \quad (F/F_j)^i = F^i F_j / F_j.$$

Indeed, it is readily seen that if  $A$  and  $B$  are subgroups of a group  $G$  and if  $B$  is invariant under conjugation by elements of  $A$ , then  $(AB/B)^2 = A^2 B/B$ . From this, an induction on  $i$  shows that for a normal subgroup  $N$  of a group  $G$  it is true that  $(G/N)^i = G^i N/N$  for all  $i$ , which implies the desired result.

Note that for each  $i$  there exists  $J$  such that for  $j \geq J$ , the solvable length of  $F/F_j$  exceeds  $i$ . This follows from (1) and the fact that, by our first assertion, we can choose  $J$  such that  $F_J \not\cong F^i$ . In particular, then, the solvable length of  $F/F_j$  is unbounded with increasing  $j$ . Note also that the solvable length of  $F/F_{j+1}$  exceeds the solvable length of  $F/F_j$  by at most 1. For if  $F/F_j$  is solvable of length  $r - 1$ , then  $(F/F_j)^r = E$ . Thus, by (1) we have  $F^r F_j / F_j = E$ , which implies  $F^r \subseteq F_j$ . On the other hand,  $F/F_{j+1}$  has solvable length  $\leq r$  since (again using (1))

$$\begin{aligned} (F/F_{j+1})^{r+1} &= [(F/F_{j+1})^r, (F/F_{j+1})^r] \\ &= [F^r F_{j+1} / F_{j+1}, F^r F_{j+1} / F_{j+1}] \subseteq [F_j / F_{j+1}, F_j / F_{j+1}] = E, \end{aligned}$$

where  $\subseteq$  holds since both  $F^r$  and  $F_{j+1}$  are subsets of  $F_j$ , and the final equality derives from the fact that  $F_j / F_{j+1}$  is abelian by Witt's theorem (above). The lemma follows at once from these results and the fact that  $F/F_2 = F_1 / F_2$  has solvable length 1 by Witt's theorem.

The proof of the theorem is now complete.

COROLLARY. *There exist nonsolvable  $O^*$ -groups.*

*Proof.* Kargaplov [5] and Kokorin [7] have shown that the re-

stricted direct product of O\*-groups is an O\*-group. Thus the restricted direct product,  $G = \prod_{i=1}^{\infty} F/F_i$ , of the groups  $F/F_i$  is an O\*-group. If  $G$  were solvable of length  $m$ , then each  $F/F_i$  would have solvable length  $\leq m$ ; for if a subgroup  $H \subseteq G$ , then  $H^k \subseteq G^k$  for every  $k$ . Since this contradicts the fact noted above that the solvable length of  $F/F_j$  is unbounded with increasing  $j$ ,  $G$  is a non-solvable O\*-group.

*Note.* The mapping  $\varphi$  of  $F$  into the *unrestricted* (or *complete*) direct product,  $\prod_{i=1}^{\infty} F/F_i$ , of the groups  $F/F_i$  given by

$$\varphi(a) = (aF_1, \dots, aF_n, \dots) \text{ for every } a \in F$$

is a monomorphism by Magnus' theorem, above. Since  $F$  is not an O\*-group (see [1], [4], or [6]), we have an immediate example of a subdirect product of O\*-groups which is not itself an O\*-group. (In [5], Kargapolov uses some of the groups  $F/F_i$  to show that the class of O\*-groups is not closed under formation of unrestricted direct products.)

*Acknowledgement.* I acknowledge the generously donated help of Professor E. F. Krause (University of Michigan), my adviser for the doctoral dissertation on which this paper is based.

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Received August 25, 1970 and in revised form April 12, 1971.

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