GENERATORS OF THE MAXIMAL IDEALS OF $A(\bar{D})$

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Let $A = A(\overline{D})$ be the sup norm algebra of functions continuous in $\overline{D}$ and holomorphic in $D$, where $D$ is a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$. This paper gives necessary and sufficient local conditions that a subfamily of $A$ generates the maximal ideal $\mathcal{M}_w(\overline{D})$ of functions in $A$ vanishing at $w \in \overline{D}$. In particular, it shows that $\mathcal{M}_w(\overline{D})$ is generated by $z_1 - w_1, \ldots, z_n - w_n$ when $w \in D$.

In [3], Gleason shows that if $m$ is an (algebraically) finitely generated maximal ideal of a commutative Banach algebra $A$, the maximal ideal space $\mathcal{M}_A$ can be given an analytic structure near $m$, in terms of which the Gelfand transforms of the elements of $A$ are holomorphic functions.

In a sense, the results of this paper go in the opposite direction. We consider a bounded domain $D$ in $\mathbb{C}^n$, with $C^2$ strictly pseudoconvex boundary, and study the algebra $A = A(\overline{D})$ of functions continuous on $D$ and holomorphic in $D$. By a recent result, Henkin [4], Kerzman [7], Lieb [9], $A$ equals the closure in $C(\overline{D})$ of the algebra $O(\overline{D})$ of functions holomorphic in some neighbourhood of $\overline{D}$, from which it follows that $\mathcal{M}_A \approx \overline{D}$.

We first fix the notation. If $w \in \overline{D}$, $\mathcal{M}_w$ denotes the maximal ideal of the ring $O_w$ of germs of holomorphic functions at $w$, while $\mathcal{M}_w(\overline{D})$ is the maximal ideal in $A$ of functions vanishing at $w$. If $f$ is a function on some neighbourhood of $w$, $f_w$ denotes the germ of $f$ at $w$.

**Theorem 1.** Let $w \in D$, and $f_1, \ldots, f_N \in A$. Then $f_1, \ldots, f_N$ generate $\mathcal{M}_w(\overline{D})$ if and only if

1. $f_{1,w}, \ldots, f_{N,w}$ generate $\mathcal{M}_w$, and
2. $w$ is the only common zero of $f_1, \ldots, f_N$ in $\overline{D}$.

**Corollary.** If $w \in D$, $z_1 - w_1, \ldots, z_n - w_n$ generate $\mathcal{M}_w(\overline{D})$.

Below we give the more general theorem 2, which also gives a similar characterization of generators of $\mathcal{M}_w(\overline{D})$ when $w \in \partial D$. When $n = 2$, Kerzman and Nagel [8] have shown that $z_1 - w_1$ and $z_2 - w_2$ generate $\mathcal{M}_w(\overline{D})$ when $w \in D$, as well as similar results for algebras with Hölder norms. I want to thank Dr. Kerzman for sending me a copy of his thesis [7], where these results are stated.

The main tool in the proof is the following result, which is proved in [11]:
**Lemma 1.** Suppose $u \in C^{(0,q)}(D)$ is bounded, with $\bar{\partial}u = 0$, $q \geq 1$. Then there exists a $v \in C^{(0,q-1)}(D)$ with $\bar{\partial}v = u$, such that $v$ has a continuous extension to $\bar{D}$.

A closely related result is given in Lieb [10], while a stronger result for $(0,1)$-forms, involving Hölder estimates, is given in Kerzman [7].

It is convenient to prove first a more general result. If $U$ is open in $\bar{D}$, let $H(U)$ denote functions in $C(U)$ that are holomorphic in $D \cap U$. When $w \in \bar{D}$, we define $H_w = \lim_{U \ni w} H(U)$, so $H_w$ is the space of germs at $w$ of continuous functions on $\bar{D}$ that are holomorphic in $D$. It is easy to see that $H$ is the sheaf of $A$-holomorphic functions in the sense of [2].

**Proposition 1.** Let $D$ be as above, $w \in \bar{D}$, and suppose $f_1, \ldots, f_N$ have $w$ as their only common zero. We let $I$ denote the ideal in $A$ generated by $f_1, \ldots, f_N$, and $I_w$ the ideal in $H_w$ generated by $f_1w, \ldots, f_Nw$. If $f \in A$ and $f_w \in I_w$, then $f \in I$.

**Proof.** By assumption, we may write $f = \sum_{i=1}^N g_i \cdot f_i$ on a neighbourhood $U$ of $w$ in $\bar{D}$, with $g_i, \ldots, g_N \in H(U)$. We want to write $f = \sum_{i=1}^N h_i \cdot f_i$, with $h_i, \ldots, h_N \in A$, and shall first solve the problem differentiably. As the sets $N_i = \{z \in \bar{D} \setminus \{w\}: f_i(z) = 0\}$, $i = 1, \ldots, N$, are closed in $\mathbb{C}^n \setminus \{w\}$, it is well known how to construct $\bar{\varphi}_1, \ldots, \bar{\varphi}_N$ with $\bar{\varphi}_i = 0$ on a neighbourhood of $N_i$, $i = 1, \ldots, N$, that form a $C^\infty$ partition of unity on $\mathbb{C}^n \setminus \{w\}$. Choose $\varphi_0 \in C^\infty_0(U')$, where $U' \cap \bar{D} = U$, with $\varphi_0 = 1$ on a neighbourhood $U_i$ of $w$, and define $\varphi_i = (1 - \varphi_0) \varphi_i$, $i = 1, \ldots, N$.

If we define

$$g'_i = \varphi_0 \cdot g_i + \frac{\varphi_i \cdot f}{f_i}, \text{ clearly } \sum_{i=1}^N g'_i \cdot f_i = f \text{ on } \bar{D}.$$  

The $g'_i \in C^\infty(D) \cap C(\bar{D})$, and are holomorphic in $U_i \cap D$.

We want to use Lemma 1 to modify the $g'_i$s to get $h_i$s in $A$. To handle the combinatorial difficulties, we apply the homological argument of [6].

**Notation.** $L_r = \{u \in C^{(0,r)}(D), u \text{ and } \bar{\partial}u \text{ have bounded coefficients}\}$, while $L^*_r = L_r \otimes \Lambda^r \mathbb{C}^N$, $0 \leq r, s$.

If we choose a basis $e_1, \ldots, e_N$ in $\mathbb{C}^N$, the elements in $L^*_r$ may be written uniquely as $\sum_{|I|=r} u_I \otimes e^I$, where $u_I \in L_r$, $e^I = e_{i_1} \wedge \cdots \wedge e_{i_r}$, and we sum over strictly increasing sequences $I = (i_1, \ldots, i_r)$. We define $\bar{\partial}$ on $L^*_r$ by $\bar{\partial}(u \otimes \omega) = (\bar{\partial}u) \otimes \omega$ and linearity. Clearly
\[ \delta L_r^* \subset L_{r+1}^*, \text{and lemma 1 gives:} \]

**Lemma 1'.** If \( k \in L_r^* \) and \( \delta k = 0 \), \( r \geq 1 \), there exists a \( k' \in L_{r-1}^* \), such that \( \delta k' = k \), and \( k' \) has a continuous extension to \( \bar{D} \).

The product determined by \( (u \otimes \omega) \cdot (u' \otimes \omega') = (u \wedge u') \otimes (\omega \wedge \omega') \) is clearly a bilinear map \( L_r^* \times L_r^* \rightarrow L_{r+s}^* \).

Let \( e_1^*, \ldots, e_N^* \) be the reciprocal basis to \( e_1, \ldots, e_N \), so \( \langle e_i^*, e_j \rangle = \delta_{ij} \). We define \( P_f : L_r^* \rightarrow L_r^{*-1} \) by

\[
P_f(d \otimes \omega) = \sum_{i=1}^N (f_i \cdot u) \otimes (e_i^* \cdot \omega), \text{and linearity.}
\]

(For the definition of \( \int \), see [12] Ch. 1.)

\[ P_f : L_r^* \rightarrow L_r^* \] maps \( \sum_{i=1}^N u_i \otimes e_i \) to \( \sum_{i=1}^N f_i \cdot u_i \); in particular, \( P_f g' = f \), when \( g' = \sum_{i=1}^N g_i \otimes u_i \).

A simple computation gives \( P_f^2 = 0 \), while the derivation property of \( \int \) gives

(\( i \)) \[ P_f(k \cdot k') = (P_f k) \cdot k' + (-1)^r k \cdot P_f k' \]

when \( k \in L_r^* \).

Let \( M_r^* = \{ k \in L_r^* : k|_{U_1} = 0 \} \).

**Lemma 2.** The complex \( 0 \rightarrow M_r^* \xrightarrow{P_f} M_{r-1}^* \xrightarrow{P_f} \cdots \xrightarrow{P_f} M_0^* \rightarrow 0 \) is exact.

**Proof.** Let \( \varphi \in C^\infty(C^N) \) be zero near \( w \) and one outside \( U_1 \). We put \( k_0 = \sum_{i=1}^N (\varphi \cdot \bar{\varphi}_i) / f_i \otimes e_i \). Clearly \( k_0 \in L_0^* \), and \( P_f k_0 \in L_0^* \) is identically one in \( D \setminus U_1 \). If \( k \in M_r^* \) and \( P_f k = 0 \), \( k \cdot k \in M_{r-1}^* \), and by (\( i \)), \( P_f(k_0 \cdot k) = (P_f k_0) \cdot k = k \).

As \( f_1, \ldots, f_N \) are holomorphic in \( D \), \( P_f \) and \( \bar{\delta} \) commute.

**Lemma 3.** If \( k \in M_r^* \) and \( P_f k = \bar{\delta} k = 0 \), there exists a \( k' \in L_{r+1}^* \), with \( P_f k' = k \) and \( \bar{\delta} k' = 0 \).

This is trivially true when \( r > n \), and the proof goes by downward induction on \( r \). Suppose the lemma is valid for \( r+1 \). By Lemma 2, there exists a \( k_1 \in M_{r+1}^* \) with \( P_f k_1 = k \). Clearly \( \bar{\delta} M_{r+1}^* \subset M_{r+1}^* \), while \( P_f \bar{\delta} k_1 = \bar{\delta} P_f k_1 = 0 \). Using the induction hypothesis, we can find \( k_2 \in L_{r+2}^* \) with \( P_f k_2 = \bar{\delta} k_1 \) and \( \bar{\delta} k_2 = 0 \). By Lemma 1', \( k_2 = \bar{\delta} k_3 \), with \( k_3 \in L_{r+2}^* \). If we put \( k' = k_1 - P_f k_3 \), we get \( k' \in L_{r+1}^* \), with \( \bar{\delta} k' = \bar{\delta} k_1 - P_f \bar{\delta} k_3 = 0 \), and \( P_f k' = P_f k_1 - P_f k_3 = k \). This completes the induction step.
**Proof of Proposition 1.** As the $g_i$'s are holomorphic in $U_i \cap D$, $\bar{\partial}g' \in M_1$. Applying Lemma 1' and Lemma 3, we find a $k \in L_2^0$, with $\bar{\partial}P'k = P'\bar{\partial}k = \bar{\partial}g'$, such that $k$ is continuous on $\bar{D}$. If $h = g' - P'k$, $\bar{\partial}h = 0$. Writing $h = \sum_{i=1}^N h_i \otimes e_i$, this means that $h_1, \ldots, h_N \in A$, and $\sum_{i=1}^N h_i \cdot f_i = f$.

**THEOREM 2.** Let $w \in \bar{D}$, and let $M_w$ denote the unique maximal ideal of $H_w$. The family $(f_i)_{i \in I}$ in $A$ generates $\mathcal{M}_w(\bar{D})$ if and only if

1. $(f_i)_i$ generates $M_w$, and
2. $w$ is the only common zero of functions $f_i$ in $\bar{D}$

**Proof.** I. The sufficiency of (1) and (2): If $f \in \mathcal{M}_w(\bar{D})$, we have $f_w \in M_w$, and by (1) $f_w$ belongs to some ideal $[f_{i_1,w}, \ldots, f_{i_n,w}]$. As $(z_i - w_i)_w, \ldots, f(z_n - w_n)_w$ belong to $M_w$, the functions $z_i - w_i$, $i = 1, \ldots, n$, may be expressed as linear combinations of functions $f_{i_{M+1}}, \ldots, f_{i_p}$ in the family on some open neighbourhood $V$ of $w$ in $\bar{D}$. Then $f_{i_{M+1}}, \ldots, f_{i_p}$ have $w$ as their only common zero in $V$. By condition (2) and the compactness of $D \setminus V$, there exist $f_{i_{p+1}}, \ldots, f_{i_N}$ in the family with no common zeroes outside $V$. Now proposition 1 implies that $f \in [f_{i_1}, \ldots, f_{i_N}]$.

II. The necessity of (1) and (2): If $(f_i)_{i \in I}$ generate $\mathcal{M}_w(\bar{D})$, condition (2) follows from the fact that $A$ separates points in $\bar{D}$. Condition (1) follows from

**PROPOSITION 2.** The germs at $w$ of elements in $\mathcal{M}_w(\bar{D})$ generate $M_w$.

The following proof of Proposition 2 was kindly communicated to me by Dr. R. M. Range, and replaces a more complicated argument of my own:

When $w \in D$, $z_1 - w_1, \ldots, z_n - w_n$ generate $\mathcal{M}_w = M_w$. Thus we may assume $w \in \partial D$, and consider an $f \in H(U \cap \bar{D})$ with $f(w) = 0$, where $U$ is some neighbourhood of $w$ in $C^n$. We choose $\varphi \in C_0^\infty(U)$ such that $\varphi \equiv 1$ on a smaller neighbourhood $V$ of $w$. As $D$ is strictly pseudoconvex, we may extend it inside $V$ to a strictly pseudoconvex domain $D'$ containing $w$. As $\bar{\partial}(\varphi \cdot f)$ vanishes on $V \cap D$, it may be extended by zero to a smooth, bounded, $\bar{\partial}$-closed $(0, 1)$-form $\omega$ on $D'$. By Lemma 1, the equation $\bar{\partial}g = \omega$ has a solution in $C^\infty(D') \cap C(\bar{D}')$, and we may assume $g(w) = 0$. As $g$ is holomorphic in $D' \cap V$, we may write it near $w$ as $g = \sum_{i=1}^n g_i(z_i - w_i)$, with $g_1, \ldots, g_n$ holomorphic. Thus $f_w = (\varphi \cdot f - g)_w + \sum_{i=1}^n g_i w(z_i - w_i)_w$, and $\varphi \cdot f - g|_{\bar{D}} \in \mathcal{M}_w(\bar{D})$. 
When \( w \in D \) and \( I \) is finite, Theorem 2 reduces to theorem 1. If \( w \in \partial D \), it follows from Gleason’s result that \( \mathcal{M}_w(\overline{D}) \) is not finitely generated. If \( M_w \) were finitely generated, it would by Proposition 2 be generated by finitely many elements of \( A \), which implies by the argument of I that \( \mathcal{M}_w(\overline{D}) \) must be finitely generated. Thus \( M_w \) is not finitely generated when \( w \in \partial D \). (This may also be proved in a more direct fashion).


**References**


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