

# Pacific Journal of Mathematics

## **A SEMILATTICE DECOMPOSITION INTO SEMIGROUPS HAVING AT MOST ONE IDEMPOTENT**

MOHAN S. PUTCHA AND JULIAN WEISSGLASS

## A SEMILATTICE DECOMPOSITION INTO SEMIGROUPS HAVING AT MOST ONE IDEMPOTENT

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**A semigroup  $S$  is said to be viable if  $ab = ba$  whenever  $ab$  and  $ba$  are idempotents. The main theorem of this article proves in part that  $S$  is a viable semigroup if and only if  $S$  is a semi-lattice of  $\mathcal{S}$ -indecomposable semigroups having at most one idempotent.**

Furthermore, each semigroup appearing in the decomposition has a group ideal whenever it has an idempotent. Also included as part of the main theorem is the more elementary result that  $S$  is viable if and only if every  $\mathcal{J}$ -class contains at most one idempotent.

Throughout  $S$  will denote a semigroup and  $E = E(S)$  the set of idempotents of  $S$ .

**DEFINITION.** Let  $a, b \in S$ . We say  $a|b$  if there exist  $x, y \in S$  such that  $ax = ya = b$ . The set-valued function  $\mathfrak{M}$  on  $S$  is defined by  $\mathfrak{M}(a) = \{e|e \in E, a|e\}$ . The relation  $\delta$  on  $S$  is defined by  $a \delta b$  if  $\mathfrak{M}(a) = \mathfrak{M}(b)$ .

Our first goal is to show that if  $S$  is viable then  $\delta$  is a congruence on  $S$  and  $S/\delta$  is the semilattice described above.

**LEMMA 1.** *Let  $S$  be viable. If  $ab = e \in E$ , then  $bea = e$ .*

*Proof.*  $(bea)^2 = beabea = bea$ . Hence  $bea \in E$ . But clearly  $abe = e \in E$ . Hence  $bea = abe = e$ .

**LEMMA 2.** *Let  $S$  be viable. Suppose  $a \in S$  and  $e \in E$ . Then  $a|e$  if and only if  $e \in S^1aS^1$ .*

*Proof.* If  $a|e$ , then  $e \in S^1aS^1$  by definition. Conversely assume  $e = sat$  with  $s, t \in S^1$ . By (1),  $ates = e$  and  $tesa = e$ . Therefore  $a|e$ .

**THEOREM 3.** *Let  $S$  be viable. Then*

- (i)  $\delta$  is a congruence relation on  $S$  containing Green's relation  $\mathcal{H}$ .
- (ii)  $S/\delta$  is a semilattice and
- (iii) each  $\delta$ -class contains at most one idempotent and a group ideal whenever it contains an idempotent.

*Proof.* (i) Clearly  $\delta$  is an equivalence relation. We will show that  $\delta$  is right compatible. Assume  $a \delta b$ . If  $ac|e \in E$ , then

$acx = e$  for some  $x \in S$ . By (1),  $cxea = e$ . Hence  $a|e$ . Thus  $b|e$ , so  $yb = e$  for some  $y \in S$ . Therefore  $ybcxea = e$ , so  $bc|e$  by (2). Hence  $\mathfrak{M}(ac) \subseteq \mathfrak{M}(bc)$ . Similarly  $\mathfrak{M}(bc) \subseteq \mathfrak{M}(ac)$  and hence  $ac \delta bc$ . That  $\delta$  is left compatible follows analogously. Consequently,  $\delta$  is a congruence. It is immediate that  $\mathcal{H} \subseteq \delta$ .

(ii) To show  $S/\delta$  is a band, let  $a \in S$ . If  $a^2|e \in E$  then by (2),  $a|e$ . Hence  $\mathfrak{M}(a^2) \subseteq \mathfrak{M}(a)$ . Suppose  $a|e \in E$ , say  $ax = ya = e$ ,  $x, y \in S$ . Then  $ya^2x = e$ . Again using (2),  $a^2|e$ . Thus,  $\mathfrak{M}(a^2) = \mathfrak{M}(a)$  and  $a \delta a^2$ . So  $S/\delta$  is a band. Now let  $a, b \in S$ . If  $e \in \mathfrak{M}(ab)$ , then there exist  $x, y \in S$  such that  $abx = yab = e$ . Hence  $ya(ba)bx = e$ , and by (2),  $e \in \mathfrak{M}(ba)$ . Therefore  $\mathfrak{M}(ab) \subseteq \mathfrak{M}(ba)$ . By symmetry,  $\mathfrak{M}(ba) \subseteq \mathfrak{M}(ab)$ . Hence  $ab \delta ba$  and  $S/\delta$  is a semilattice.

(iii) Suppose,  $e_1 \delta e_2$  with  $e_1, e_2 \in E$ . Then  $e_1 \in \mathfrak{M}(e_1) = \mathfrak{M}(e_2)$ , so  $e_2|e_1$ . Similarly  $e_1|e_2$ . Hence  $e_1 \mathcal{H} e_2$  and by [2], Lemma 2.15,  $e_1 = e_2$ . Thus each  $\delta$ -class contains at most one idempotent. Now suppose  $A$  is a  $\delta$ -class containing an idempotent  $e$ . Let  $a \in A$ . Since  $e \in \mathfrak{M}(e) = \mathfrak{M}(a) = \mathfrak{M}(a^2)$ , there exists  $x \in S$  such that  $a^2x = e$ . Now  $a \delta a^2$  implies  $ax \delta a^2x$ , so  $ax \delta e \delta a$ . Hence  $ax \in A$  and  $a(ax) = e$  implies  $e$  is a right zero of  $A$ . Similarly  $e$  is a left zero and by [2], §2.5, Exercise 6,  $A$  has a group ideal.

A semigroup is said to be  $\mathcal{S}$ -indecomposable if it has no proper semilattice decomposition.

**COROLLARY 4.** *If the viable semigroup  $S$  is  $\mathcal{S}$ -indecomposable then  $S/\delta = 1$  and is either idempotent-free or has a group ideal and exactly one idempotent.*

**LEMMA 5.** *Assume  $I$  is an idempotent-free ideal of  $S$ . Then  $S$  is viable if and only if the Rees factor semigroup  $S/I$  is viable.*

*Proof.* Assume  $S$  is viable and that  $ab, ba \in E(S/I)$ . If  $ab \in I$ , then  $ba = b(ab)a \in I$ , so  $ab = ba$  in  $S/I$ . So we may assume  $ab$  and  $ba$  are not in  $I$ . But then  $ab, ba \in E(S)$ . Hence  $ab = ba$  in  $S$  and so in  $S/I$ . Therefore  $S/I$  is viable. Conversely, let  $ab, ba \in E(S)$ . Since  $S/I$  is viable  $ab = ba$  in  $S/I$ . But  $ab, ba \notin I$  since  $I$  is idempotent-free. Hence  $ab = ba$  in  $S$  and  $S$  is viable.

A semigroup  $S$  is said to be  $E$ -inversive if for every  $a \in S$  there exists  $x \in S$  such that  $ax \in E$ .

**THEOREM 6.** *The following are equivalent.*

- (i) *Every  $\mathcal{J}$ -class of  $S$  contains at most one idempotent*
- (ii)  *$S$  is viable.*
- (iii)  *$S$  is a semilattice of  $\mathcal{S}$ -indecomposable semigroups each of*

which contains at most one idempotent and a group ideal whenever it contains an idempotent.

(iv)  $S$  is a semilattice of semigroups having at most one idempotent.

(v)  $S$  is viable and  $E$ -inversive or an ideal extension of an idempotent-free semigroup by a viable  $E$ -inversive semigroup.

*Proof.* (i)  $\Rightarrow$  (ii) If  $ab$  and  $ba$  are idempotents then  $ab = a(ba)b \in S^1baS^1$ . Similarly  $ba \in S^1abS^1$ . Hence  $ab \mathcal{J} ba$ , so  $ab = ba$ .

(ii)  $\Rightarrow$  (iii) By Tamura [3],  $S$  is a semilattice of  $\mathcal{S}$ -indecomposable semigroups. Since subsemigroups of viable semigroups are viable, each component is viable. The result follows from (4).

(iii)  $\Rightarrow$  (iv) a fortiori

(iv)  $\Rightarrow$  (i) Suppose  $e, f \in E$  with  $e \mathcal{J} f$ . Then  $e$  and  $f$  are in the same component of the given semilattice decomposition. Hence  $e = f$ .

(ii)  $\Rightarrow$  (v) Let  $I = \{a \in S \mid \mathfrak{M}(a) = \emptyset\}$ . If  $I$  is empty then  $S$  is  $E$ -inversive. Otherwise,  $I$  is obviously an idempotent-free  $\delta$ -class of  $S$ . Moreover if  $ax|e$  or  $xa|e$ ,  $e \in E$ , then by (2),  $a|e$ . Hence,  $a \in I$  implies  $ax, xa \in I$  so that  $I$  is an ideal of  $S$ . By (5),  $S/I$  is viable. Since  $S/I$  has a zero, it is  $E$ -inversive. In fact, every nonzero element of  $S/I$  divides a nonzero idempotent of  $S/I$ .

(v)  $\Rightarrow$  (ii) Follows from (5).

REMARK. Observe that the semilattice decomposition of (iii) in general will not be isomorphic to  $S/\delta$  since in fact  $S$  may be idempotent free. Also,  $\mathcal{J}$  may be replaced  $\mathcal{D}$  in the theorem.

LEMMA 7.  $S$  is an ideal extension of a group by a nil semigroup if and only if  $S$  is a subdirect product of a group and a nil semigroup.

*Proof.* Suppose  $S$  is an ideal extension of a group  $G$  by a nil semigroup  $N$ . Let  $e$  be the identity of  $G$ . It is easy to see that  $e$  is central in  $S$ . It is well known that  $S$  is a subdirect product of subdirectly irreducible semigroups  $S_\alpha$  ( $\alpha \in \Omega$ ). Let  $\sigma_\alpha: S \rightarrow S_\alpha$  be the natural map. Let  $e_\alpha = e\sigma_\alpha$ . Then  $e_\alpha$  is a central idempotent in  $S_\alpha$  and hence is zero or 1 (cf. [1]). If  $e_\alpha = 0$ , then  $\sigma_\alpha(G) = 0$  and hence  $S_\alpha = \sigma_\alpha(S)$  is a nil semigroup. If  $e_\alpha = 1$ , then all of  $S_\alpha$  is contained in  $\sigma_\alpha(G)$  and hence  $S_\alpha$  is a group. Consequently each  $S_\alpha$  is a nil semigroup or a group. Let  $\Omega_1 = \{\alpha \mid \alpha \in \Omega, S_\alpha \text{ is nil}\}$  and let  $\Omega_2 = \{\alpha \mid \alpha \in \Omega, S_\alpha \text{ is a group}\}$ . Let  $\psi_i = \prod_{\alpha \in \Omega_i} \sigma_\alpha: S \rightarrow \prod_{\alpha \in \Omega_i} S_\alpha$  be defined for  $i = 1, 2$ . One can check that  $S$  is a subdirect product of  $S_{\psi_1}$  and  $S_{\psi_2}$  with  $S_{\psi_1}$  a nil semigroup and  $S_{\psi_2}$  a group.

Conversely, suppose  $S$  is a subdirect of a group  $G$  and a nil

semigroup  $N$ . Consider  $S$  embedded in  $G \times N$ . Let  $e$  be the identity of  $G$ . There exists  $a \in N$  such that  $(e, a) \in S$ . There exists a positive integer  $k$  such that  $a^k = 0$ . Hence  $(e, 0) = (e, a^k) = (e, a)^k \in S$ . If  $g \in G$ , there exists  $b \in N$  such that  $(g, b) \in S$ . Thus  $(g, 0) = (e, 0)(g, b) \in S$ . Hence  $G \times \{0\} \subseteq S$  and  $G \times \{0\}$  is an ideal of  $S$ . Let  $(g, a) \in S$ . Since  $a \in N$ , there exists a positive integer  $k$  such that  $a^k = 0$ . Hence  $(g, a)^k = (g^k, a^k) = (g^k, 0) \in G \times \{0\}$ . Therefore  $S$  is an ideal extension of the group  $G \times \{0\}$  by a nil semigroup.

**COROLLARY 8.** *The following are equivalent.*

- (i)  $S$  is viable and a power of each element lies in a subgroup.
- (ii)  $S$  is a semilattice of semigroups which are ideal extensions of groups by nil semigroups.
- (iii)  $S$  is a semilattice of semigroups each of which is a subdirect product of a nil semigroup.

Moreover the decompositions in (ii) and (iii) are the same and coincide with the  $\delta$ -decomposition as specified in Theorem 3.

A semigroup  $S$  is separative if  $x^2 = xy = y^2$  ( $x, y \in S$ ) implies  $x = y$ .

**COROLLARY 9.** *The following are equivalent.*

- (i)  $S$  is viable, separative and a power of each element of  $S$  lies in a subgroup.
- (ii)  $S$  is a semilattice of groups.

*Proof.* (i)  $\Rightarrow$  (ii) By (8), it suffices to show that if  $T$  is separative and an ideal extension of a group  $G$  by a nil semigroup, then  $T = G$ . Let  $e$  be the identity of  $G$ . Then  $e$  is central in  $T$ . If  $T \neq G$ , then there exists  $a \in T$ ,  $a \notin G$  with  $a^2 \in G$ . Then  $a^2 = (ae)^2 = a(ae)$ . Thus  $a = ae \in G$ , a contradiction. Hence  $T = G$ .

(ii)  $\Rightarrow$  (i) Obvious.

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