Pacific Journal of Mathematics

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Vol. 39, No. 1 May 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

INFINITE MATRICES SUMMING EVERY ALMOST PERIODIC SEQUENCE

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Necessary and sufficient conditions are given for infinite matrices to sum every almost periodic sequence and their basic properties as summability matrices are studied. It is then shown that these matrices enter naturally in the problem of the determination of the jump or total quadratic jump of normalized functions of bounded variation on the circle in terms of the limits of matrix transforms of certain functions of their Fourier-Stieltjes coefficients. The results obtained generalize the classical theorems of Fejér and Wiener as also the extensions of theorems of Wiener given by Lozinskiǐ, Keogh, Petersen and Matveev. Applications are *made to the study of coefficient properties of holomorphic* functions in the unit disk with positive real part.

1. R. H. C. Newton [11] proved that a regular matrix $A = (a_{n,k})$ sums every periodic sequence if and only if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ exists for each rational t. Vermes [15] generalized this result by proving that an arbitrary matrix $A = (a_{n,k})$ sums every periodic sequence if and only if for every rational t, (1) $\sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ converges and (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ exists.

The set P of all periodic sequences of complex numbers is a linear subspace of l_∞ that is not closed in the usual norm topology of the Banach space l_∞ since P is meager in l_∞. Berg and Wilansky [3] proved that the closure Q of P in l_∞ is the set of all semi-periodic sequences. (A sequence $x = \{x_k\}$ is called semi-periodic if for any $\varepsilon > 0$, there exists an integer r such that $|x_k - x_{k+rn}| < \varepsilon$ for every n and k). Berg [2], gave a characterization of infinite matrices summing every semi-periodic sequence which is rather involved. We first show that these matrices can be characterized simply as follows:

THEOREM 1. *An infinite matrix $A = (a_{n,k})$ sums every semi-periodic sequence if and only if* (1) $\|A\| = \sup_{n\geq 0} \sum_{k=0}^{\infty} |a_{n,k}| < \infty$ *and* (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ *exists for all rational t.*

Proof. If $x \in Q$, then for any $\varepsilon > 0$, there exists a $y \in P$ such that $\|x - y\|_\infty < \varepsilon$. If y is of period r, there exist constants $\lambda_1, \cdots, \lambda_r$ such that

$$\sum_{\nu=1}^{r} \exp(2\pi ik\nu/r) \cdot \lambda_\nu = y_k, \quad k = 0, 1, \cdots, r-1$$

so that

$$\left| \sum_{k=0}^{\infty} a_{m,k} x_k - \sum_{k=0}^{\infty} a_{n,k} x_k \right| \leqq \left| \sum_{k=0}^{\infty} (a_{m,k} - a_{n,k}) (x_k - y_k) \right|$$
$$+ \left| \sum_{k=0}^{\infty} (a_{m,k} - a_{n,k}) \Big(\sum_{\nu=1}^{r} \exp (2\pi i k\nu/r) \lambda_\nu \Big) \right|$$
$$\leqq 2 \| A \| \varepsilon + \varepsilon$$

for n and m sufficiently large. Hence $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} x_k$ exists. Thus (1) and (2) are sufficient. The necessity of (1) can be established as in Berg and Wilansky [2] (see also the proof of Theorem 2 below where similar arguments have been given to prove the necessity of Theorem 2 (1) independently of the use of Theorem 1), and that of (2) is immediate since $\{\exp(2\pi ikt)\}$ is periodic when t is rational.

2. A sequence $x = \{x_k\}$ of complex numbers is called almost periodic if to any $\varepsilon > 0$, there corresponds an integer $N = N(\varepsilon) > 0$ such that among any N consecutive integers there exists an integer r with the property $| x_k - x_{k+r} | < \varepsilon$ for all k. If we denote by AP the set of all almost periodic sequences of complex numbers, then clearly AP is a linear subspace of l_∞ and $P \subset \bar{P} = Q \subset AP \subset l_\infty$. Also AP is a closed subspace of l_∞. For if $\{x^{(n)}\}$ is a Cauchy sequence in AP, there exists an $x = \{x_k\} \in l_\infty$ such that $\lim_{n\to\infty} \| x^{(n)} - x \|_\infty = 0$. Given any $\varepsilon > 0$, we can choose an n such that $| x_k^{(n)} - x_k | < \varepsilon/3$ for every k. Since $x^{(n)} \in AP$, there exists an integer $N = N(\varepsilon)$ such that among N consecutive integers there is an integer r such that $| x_k^{(n)} - x_{k+r}^{(n)} | < \varepsilon/3$ for every k so that

$$| x_k - x_{k+r} | \leqq | x_k - x_k^{(n)} | + | x_k^{(n)} - x_{k+r}^{(n)} | + | x_{k+r}^{(n)} - x_{k+r} |$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for every k. Thus AP is a Banach space. We note that $Q \subsetneqq AP$ since if t is irrational, then $\{\exp(2\pi ikt)\}$ is almost periodic but not semi-periodic.

Infinite matrices summing every almost periodic sequence in AP can be characterized as follows:

THEOREM 2. *An infinite matrix* $A = (a_{n,k})$ *sums every almost periodic sequence if and only if* (1) $\| A \| = \sup_{n\geqq 0} \sum_{k=0}^{\infty} | a_{n,k} | < \infty$ *and* (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$ *exists for all* t.

Proof. Suppose that A sums every almost periodic sequence. Since for each t, $\{\exp(2\pi ikt)\} \in AP$, (2) holds. To prove the necessity of (1), we first observe that if $y \in l_1$, its norm $\| y \|_{(AP)^*}$ is identical with its l_1-norm. For if $y = \{y_k\} \in l_1$, we define a sequence $\tilde{x}$ of period n by the rule: $\tilde{x}_k = \operatorname{sgn} y_k$ for $k \leqq n$ so that

$$\| y \|_{(AP)^*} = \sup_{\substack{x \in AP \\ \| x \| = 1}} | y(x) | \geqq | y(\tilde{x}) | \geqq \sum_1^n | y_k | + \sum_{n+1}^\infty \tilde{x}_k y_k ,$$

where

$$\left| \sum_{n+1}^\infty \tilde{x}_k y_k \right| \leqq \sum_{n+1}^\infty | y_k | \longrightarrow 0$$

as $n \to \infty$. Thus $\| y \|_{(AP)^*} \geqq \| y \|_{l_1}$. Clearly $\| y \|_{(AP)^*} \leqq \| y \|_{l_1}$ so that $\| y \|_{(AP)^*} = \| y \|_{l_1}$.

For each fixed n, put

$$y_N(x) = \sum_{k=0}^N a_{n,k}\, x_k, \text{ where } x \in AP .$$

$y_N \in (AP)^*$ and $\lim_{N\to\infty} y_N(x)$ exists for each $x \in AP$. By the uniform boundedness principle,

$$\| y_N \|_{(AP)^*} = \| y_N \|_{l_1} = \sum_1^N | a_{n,k} | \leqq M_n < \infty$$

for each N so that $\sum_{k=0}^\infty | a_{n,k} | < \infty$ for each n. If we put

$$z_n(x) = \sum_{k=0}^\infty a_{n,k}\, x_k,\ x \in AP ,$$

then $z_n \in (AP)^*$ and $\lim_{n\to\infty} z_n(x)$ exists for each $x \in AP$. Applying once more the uniform boundedness principle, we get

$$\| A \| = \sup_{n \geqq 0} \sum_{k=0}^\infty | a_{n,k} | < \infty .$$

Thus (1) holds.

To prove the sufficiency of conditions (1) and (2), we note that if $x = \{x_k\} \in AP$, there exists a sequence $\{\sum_0^N b_j \exp(2\pi i \lambda_j k)\} \in AP$ such that for all k,

$$\left| x_k - \sum_0^N b_j \exp(2\pi i \lambda_j k) \right| < \varepsilon .$$

Now

$$\begin{aligned} \left| \sum_{k=0}^\infty a_{m,k}\, x_k - \sum_{k=0}^\infty a_{n,k}\, x_k \right| &\leqq \left| \sum_{k=0}^\infty (a_{m,k} - a_{n,k}) \Big(x_k - \sum_0^N b_j \exp(2\pi i \lambda_j k) \Big) \right| \\ &\quad + \left| \sum_{k=0}^\infty (a_{m,k} - a_{n,k}) \sum_0^N b_j \exp(2\pi i \lambda_j k) \right| \\ &\leqq 2 \| A \| \varepsilon + \varepsilon \end{aligned}$$

for m and n sufficiently large. Thus $\lim_{n\to\infty} \sum_{k=0}^\infty a_{n,k}\, x_k$ exists.

We call a matrix $A = (a_{n,k})$ satisfying the conditions (1) and (2) of Theorem 2, an almost periodic matrix. We now establish a few properties of these matrices. We recall that the set of all sequences summable by a given matrix $A = (a_{n,k})$ is called its convergence field and is denoted by (A). If (A) contains all convergent sequences then A is called conservative. It is known that A is conservative if and only if (1) $\|A\| < \infty$, (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = \alpha$ exists and (3) $\lim_{n\to\infty} a_{n,k} = \alpha_k$ exists for each fixed k. We have:

PROPOSITION 1. *An almost periodic matrix is always conservative.*

Proof. It is sufficient to show that $\lim_{n\to\infty} a_{n,k} = \alpha_k$ exists. If we put $K_n(t) = \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$, then $\{K_n\}$ is a sequence of continuous functions on $[0, 1]$ such that $\lim_{n\to\infty} K_n(t) = K(t)$ exists for each t and $|K_n(t)| \leqq \|A\| < \infty$ for all n and all t. By bounded convergence theorem,

$$\lim_{n\to\infty} a_{n,k} = \lim_{n\to\infty} \int_0^1 K_n(t) e^{-2\pi ikt}\, dt = \int_0^1 K(t)\, e^{-2\pi ikt}\, dt$$

exists for each k.

The converse is easily seen to be false.

A conservative matrix $A = (a_{n,k})$ is called multiplicative if there exists an $m > 0$ such that $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} A_n(x) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} x_k = mx$ and then A is called m-multiplicative. Since

$$\lim_{n\to\infty} A_n(x) = \alpha x + \sum_{k=0}^{\infty} \alpha_k (x_k - x) ,$$

it follows that a conservative matrix $A = (a_{n,k})$ is multiplicative if and only if $\lim_{n\to\infty} a_{n,k} = 0$ for each k. An examination of the proof of Proposition 1 shows that an almost periodic matrix $A = (a_{n,k})$ is multiplicative if and only if

$$\int_0^1 K(t)\, e^{-2\pi ikt}\, dt = 0 \qquad \textit{for all } k = 0, \pm 1, \pm 2, \cdots$$

so that, by the uniqueness of Fourier expansion, if and only if $K(t) = 0$ a.e. Thus we have:

PROPOSITION 2. *An almost periodic matrix $A = (a_{n,k})$ is multiplicative if and only if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt) = 0$ a.e. in $(0, 1)$.*

It may be remarked that there exist multiplicative almost periodic matrices for which the above limit is not zero for all $t \in (0, 1)$. The positive matrix $A = (a_{n,k})$ where $a_{n,2k} = 0$, $a_{n,2k+1} = n^k/(n+1)^{k+1}$ for $k = 0, 1, 2, \cdots$ is one such matrix. We also have:

PROPOSITION 3. *An almost periodic matrix $A = (a_{n,k})$ is regular if and only if* (1) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$ *and* (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp (2\pi ikt) = 0$ *a. e. in* $(0, 1)$.

We call an almost periodic matrix normal if (1) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$ and (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp (2\pi ikt) = 0$ for all $t \in (0, 1)$. Clearly a normal almost periodic matrix is regular.

3. A sequence $x = \{x_k\}$ is said to be $\mathscr{F}_A$ summable where $A = (a_{n,k})$ if $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\, x_{k+p}$ exists uniformly in $p = 0, 1, 2, \cdots$. An obvious modification of the reasoning used in the proof of Theorem 2 yields the following:

THEOREM 3. *Let A be a given matrix. Then every almost periodic sequence is summable $\mathscr{F}_A$ if and only if A is an almost periodic matrix.*

In particular, a sequence $x = \{x_k\}$ of complex numbers is called almost convergent if $\lim_{n\to\infty}(n + 1)^{-1} \sum_{k=0}^{n} x_{k+p}$ exists uniformly in $p = 0, 1, \cdots$ i.e., if it is summable $\mathscr{F}_A$ where A is the matrix of the arithmetic mean. Every almost periodic sequence is almost convergent (cf. Theorem 3) but not conversely. Lorentz [8] has proved that a matrix $A = (a_{n,k})$ sums every almost convergent sequence to its almost convergence limit if and only if (1) A is regular and (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} | \Delta a_{n,k} | = 0$ where $\Delta a_{n,k} = a_{n,k} - a_{n,k+1}$ for $k = 0, 1, \cdots$. He calls matrices $A = (a_{n,k})$ satisfying (1) and (2) strongly regular. A simple modification of his proof of the above characterization yields the following:

THEOREM 4. *A matrix $A = (a_{n,k})$ sums every almost convergent sequence if and only if* (1) *A is conservative and* (2) $\lim_{n\to\infty} \sum_{k=0}^{\infty} | \Delta(a_{n,k} - \alpha_k)| = 0$, *where* $\alpha_k = \lim_{n\to\infty} a_{n,k}$.

A natural problem in this connection is to determine whether there exist matrices that sum every almost periodic sequence without necessarily summing every almost convergent sequence. The fact that there exist almost convergent sequences that are not almost periodic does not resolve the problem since, a priori, it is not clear that the convergence field of an almost periodic matrix does not contain all almost convergent sequences. This is settled by the following:

THEOREM 5. *There exists a normal almost periodic matrix $A = (a_{n,k})$ such that $| A | = (| a_{n,k} |)$ is also almost periodic but A is not strongly regular.*

Proof. Let $A = (a_{n,k})$ be defined as follows:

$$a_{n,0} = 0\ ,$$
$$a_{n,k} = \frac{1}{n} \quad \text{for } 1 \leqq k \leqq n\ ,$$
$$a_{n,k} = \exp\{i\pi(k-n)\log(k-n)\}/n \quad \text{for } n < k \leqq 2n\ ,$$
$$a_{n,k} = 0 \qquad \text{for } k > 2n.$$

Clearly $\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{n,k} = 1$, since $\sum_{k=0}^{\infty} a_{n,k} = 1 + (1/n)\sum_{n=1}^{2n} \exp\{i\pi(k-n)\log(k-n)\}$ and the partial sums $s_n(x)$ of the series $\Sigma \exp\{i\pi k\log k + ikx\}$ are $O((n)^{1/2})$ uniformly in x (cf. Zygmund [17] p. 199). Also $\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{n,k}\exp(2\pi ikt) = 0$ for all $t \in (0, 1)$ since, in view of the above cited result,

$$\sum_{k=0}^{\infty} a_{n,k}\exp(2\pi ikt) = \frac{1}{n}\,\frac{e^{i2\pi t}(1-e^{i2\pi nt})}{1-e^{i2\pi t}} + O\left(\frac{\sqrt{n}}{n}\right)(n\to\infty)\ .$$

Also since $\|A\| = 2$, it follows that A is normal almost periodic. For $t \in (0, 1)$

$$\sum_{k=0}^{\infty} |a_{n,k}|\exp(2\pi ikt) = \frac{1}{n}\,e^{2\pi it}\,\frac{(1-e^{4\pi int})}{1-e^{2\pi it}} = o(1)\ (n\to\infty)$$

and

$$\sum_{k=0}^{\infty} |a_{n,k}| \to 2\ (n\to\infty)\ ,$$

so that $|A|$ is also almost periodic. However A is not strongly regular. In fact,

$$\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| = \frac{2}{n}\sum_{1}^{n-1}\left|\sin\frac{\pi}{2}\left\{\log\left(1+\frac{1}{k}\right)^k + \log(k+1)\right\}\right| + o(1)\ .$$

Since

$$k\log\left(1+\frac{1}{k}\right) \to 1$$

as $k\to\infty$, we have

$$\sin\frac{\pi}{2}\left\{k\log\left(1+\frac{1}{k}\right) + \log(k+1)\right\} = \cos\frac{\pi}{2}\log(k+1) + o(1)$$

so that

$$\frac{2}{n}\sum_{1}^{n-1}\left|\sin\frac{\pi}{2}\left\{k\log\left(1+\frac{1}{k}\right) + \log(k+1)\right\}\right|$$
$$= \frac{2}{n}\sum_{1}^{n-1}\left|\cos\frac{\pi}{2}\log(k+1)\right| + o(1)\ (n\to\infty)\ .$$

We assert that

$$\frac{1}{n}\sum_{1}^{n-1}\left|\cos\frac{\pi}{2}\log(k+1)\right|$$

does not tend to zero. In fact, if we put

$$u_k = \left|\cos\frac{\pi}{2}\log(k+1)\right| - \left|\cos\frac{\pi}{2}\log k\right|,$$

then we have

$$|u_k| \leqq 2\left|\sin\frac{\pi}{4}\log(k^2+k)\right|\left|\sin\frac{\pi}{4}\log\left(1+\frac{1}{k}\right)\right| = O\left(\frac{1}{k}\right)\ (k\to\infty).$$

It is known (cf. Zygmund [17], p. 78) that if a series Σu_k is summable $(C, 1)$ and $u_k = 0(1/k)$, then Σu_k is convergent. Hence, if in our case the series Σu_n were summable $(C, 1)$ to zero i.e., if

$$\frac{1}{n}\sum_{1}^{n-1}\left|\cos\frac{\pi}{2}\log(k+1)\right|$$

were to tend to zero as $n\to\infty$, the series Σu_k would be convergent which is not the case since

$$\sum_{1}^{n} u_k = \left|\cos\frac{\pi}{2}\log(n+1)\right|$$

does not tend to a limit as $n\to\infty$.

As a corollary of Theorem 5, we get that there exist sequences that are almost convergent without being almost periodic.

4. Let $V[0, 2\pi]$ denote the class of all normalized functions F of bounded variation in $[0, 2\pi]$ such that $F(x+2\pi) - F(x) = F(2\pi) - F(0)$ for all x and let $\{C_n\}$ be the sequence of Fourier-Stieltjes coefficients of F. We now show that almost periodic matrices enter naturally in the solution of the problem of the determination of the jump or the total quadratic jump of a function $F \in V[0, 2\pi]$ by means of the limits of the matrix transforms of $\{C_k e^{ikx}\}$ or $\{|C_k|^2\}$ respectively.

THEOREM 6. *Let* $A = (a_{n,k})$ *be such that* $||A|| < \infty$. *Then for every* $F \in V[0, 2\pi]$ *and for every* $x \in [0, 2\pi]$, *the sequence* $\{C_k e^{ikx}\}$ *is summable* A *or* $\mathscr{F}_A$ *to* $(2\pi)^{-1} D(x)$ *where,* $D(x) = F(x+0) - F(x-0)$, *if and only if* A *is normal almost periodic.*

Proof. We prove the assertion for summability A, the proof for summability $\mathscr{F}_A$ being similar. The condition is necessary, for if we choose F: $F(t) = 2\pi$ for $0 < t \leqq 2\pi$ and $F(0) = 0$, then $C_k = 1$ for all k, $D(0) = 2\pi$ and $D(x) = 0$ for $0 < x < 2\pi$ so that $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikx) = 0$ for all $x \in (0, 1)$ and $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} = 1$.

Suppose that A is a normal almost periodic matrix. Then

$$\sum_{k=0}^{\infty} a_{n,k} C_k e^{ikx} = \sum_{k=0}^{\infty} a_{n,k} \frac{1}{2\pi} \sum_{j=0}^{\infty} D(x_j) e^{ik(x-x_j)} + (2\pi)^{-1} \int_0^{2\pi} K_n\left(\frac{x-t}{2\pi}\right) dF_c(t) ,$$

where $K_n(t) = \sum_{k=0}^{\infty} a_{n,k} \exp(2\pi ikt)$, $\{x_j\}$ are the points of jump of F in $[0, 2\pi)$ and F_c is the continuous part of F. Clearly the first term on the right tends to $D(x)/2\pi$ as $n \to \infty$. The second term on the right tends to 0 as $n \to \infty$, for, given an $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\int_{x-\delta}^{x+\delta} \left| dF_c(t) \right| < \frac{\varepsilon}{2} \, || A ||^{-1}$$

so that

$$\left| \int_{x-\delta}^{x+\delta} K_n\left(\frac{x-t}{2\pi}\right) dF_c(t) \right| < \frac{\varepsilon}{2}$$

and, by bounded convergence theorem,

$$\left| \left(\int_0^{x-\delta} + \int_{x+\delta}^{2\pi} \right) K_n\left(\frac{x-t}{2\pi}\right) dF_c(t) \right| < \frac{\varepsilon}{2}$$

for large n. Thus $\{C_k e^{ikx}\}$ is summable A to $D(x)/2\pi$.

Theorem 6 generalizes a theorem of Fejér [4] (cf. also Zygmund [17] p. 107) and, in particular, it shows that in Fejér's theorem the summability $(C, 1)$ can be replaced by almost convergence.

THEOREM 7. *Let $A = (a_{n,k})$ be such that $|| A || < \infty$. Then for every $F \in V[0, 2\pi]$, the sequence $\{|C_k|^2\}$ is summable A and $\mathscr{F}_A$ to $(4\pi^2)^{-1} \sum_{j=0}^{\infty} | D(x_j)|^2$, where $\{x_j\}$ are the points of jump of F in $[0, 2\pi)$ if and only if A is a normal almost periodic matrix.*

Proof. If we put $F^*(x) = (2\pi)^{-1} \int_0^{2\pi} F(x + t) \, d\bar{F}(t)$, then $F^* \in V[0, 2\pi]$, $F^*(+0) - F^*(-0) = (2\pi)^{-1} \sum_{j=0}^{\infty} | D(x_j) |^2$ and the Fourier-Stielt,jes coefficients of F^* are $\{|C_k|^2\}$. Applying Theorem 6 to F^* at $x = 0$ we get the proof of the sufficiency part of the above theorem.

To prove the necessity part, we observe if $\{C_k\}$ and $\{C'_k\}$ are the Fourier-Stieltjes coefficients of F and F' in $V[0, 2\pi]$, then

$$\{C_k \bar{C}'_k + \bar{C}_k C'_k\} = \left\{ \frac{1}{2} (|C_k + C'_k|^2 - | C_k - C'_k|^2) \right\}$$

is summable A to $(4\pi^2)^{-1}\sum_{l=0}^{\infty}\{D(y_l)\overline{D'(y_l)}+\overline{D(y_l)}D'(y_l)\}$, where $\{y_l\}$ denotes the set of all points of jump of F and F'. On replacing F' by iF', we get that $\{C_k\bar{C}'_k-\bar{C}_kC'_k\}$ is summable A to $(4\pi^{-2})$ $\sum_{l=0}^{\infty}\{D(y_l)\ \overline{D'(y_l)}-\overline{D(y_l)}\ D'(y_l)\}$ so that $\{C_k\bar{C}'_k\}$ is summable A to $(4\pi^2)^{-1}\sum_{l=0}^{\infty}\{D(y_l)\overline{D'(y_l)}\}$. If we choose $F'\in V[0,2\pi]$ such that $F'(t)=0$ for $0\leqq t<x$, $F'(t)=2\pi$ for $x<t\leqq 2\pi$, then $C'_k=e^{-ikx}$ so that $\{C_k e^{ikx}\}$ is summable A to $D(x)/2\pi$ for each $x\in[0,2\pi]$ and Theorem 6 applies. Thus we conclude that A is normal almost periodic.

Theorem 7 generalizes a theorem of Wiener [16] (cf. also Zygmund [17] p. 108) and, in particular, it shows that in Wiener's theorem the summability $(C,1)$ can be replaced by almost convergence.

As an immediate consequence we have the following:

THEOREM 8. *For functions $F\in V[0,2\pi]$, the following conditions are equivalent:*

(1) *F is continuous,*

(2) *$\{|C_k|^2\}$ is summable A or $\mathscr{F}_A$ to 0 by a normal almost periodic matrix A,*

(3) *$\{|C_k|\}$ is summable A or $\mathscr{F}_A$ to 0 by a normal almost periodic matrix $A=(a_{n,k})$ for which $\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n,k}|\exp(2\pi ikt)=0$ for all $t\in(0,1)$.*

Proof. The equivalence of (1) and (2) is a direct consequence of Theorem 7. Suppose that F is continuous. Then the convolution F^* as defined in the proof of Theorem 7 is continuous and belongs to $V[0,2\pi]$. If we go through the steps of the proof of Theorem 6 for F^* with $x=0$ and $D^*(0)=(2\pi)^{-1}\sum_{j=0}^{\infty}|D(x_j)|^2$ and note that the Fourier-Stieltjes coefficients of F^* are $\{|C_k|^2\}$, we conclude that the hypothesis $\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n.k}|\exp(2\pi ikt)=0$ for all $t\in(0,1)$ without the requirement that $\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n,k}|$ exists, assures that $\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n,k}|\ |C_k|^2=0$. Applying Schwarz inequality, we get that $\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n,k}|\ |C_k|=0$ and consequently that $\{|C_k|\}$ is summable A to 0. Similarly we show that $\{|C_k|\}$ is summable $\mathscr{F}_A$ to 0. Thus (1) implies (3). Suppose that $\{|C_k|\}$ is summable A to 0. If we write $C_k=C'_k+C''_k$, where C'_k and C''_k are respectively the Fourier-Stieltjes coefficients of the saltus part and the continuous part of F, we have

$$\left|\sum_{k=0}^{\infty}a_{n\ k}(|C_{k+p}|-|C'_{k+p}|)\right|\leqq\sum_{k=0}^{\infty}|a_{n.k}|\ |C''_{k+p}|\leqq B\sum_{k=0}^{\infty}|a_{n\ k}|\ |C''_{k+p}|^2\,.$$

Since the last term tends to zero in view of the equivalence of (1) and (2) already proved and since the almost periodic sequence $\{|C'_k|\}$ is summable $\mathscr{F}_A$, we have

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k} \, | C_{k+p} | = 0$$

uniformly in p. Similarly we prove that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} | a_{n,k} | \, |C_{k+p} | = L \,(\text{say}) \;,$$

exists uniformly in p. If we set

$$\sigma_{n,p} = \sum_{k=0}^{\infty} a_{n,k} \, | C_{k+p} | \;,$$

we see that

$$\begin{aligned}\sum_{p=0}^{\infty} | a_{n,p} | \, \sigma_{n,p} &= \sum_{p=0}^{\infty} | a_{n,p} | \sum_{k=0}^{\infty} a_{n,k} | C_{k+p} | \\ &= \sum_{k=0}^{\infty} a_{n,k} \sum_{p=0}^{\infty} | a_{n,p} | \, | C_{k+p} | \;.\end{aligned}$$

If for an $\varepsilon > 0$, we choose an $N = N(\varepsilon)$ such that for all $n \geqq N$

$$| \sigma_{n,p} | < \varepsilon, \; \left| \left(\sum_{p=0}^{\infty} | a_{n,p} | \, | C_{k+p} | \right) - L \right| < \varepsilon$$

uniformly in p and k respectively, it follows that for $n \geqq N$

$$\begin{aligned}\left| \sum_{k=0}^{\infty} a_{n,k} \right| L &\leqq \left| \sum_{k=0}^{\infty} a_{n,k} \left[L - \sum_{p=0}^{\infty} | a_{n,p} | \, | C_{k+p} | \right] \right| \\ &\quad + \left| \sum_{k=0}^{\infty} a_{n,k} \sum_{p=0}^{\infty} | a_{n,p} | \, | C_{k+p} | \right| \\ &\leqq || A || \, \varepsilon + \sum_{p=0}^{\infty} | a_{n,p} | \, \sigma_{n,p} \\ &\leqq || A || \, \varepsilon + || A || \, \varepsilon = 2 \, || A || \, \varepsilon \;.\end{aligned}$$

Making $n \to \infty$, we get $L \leqq 2 \, || A || \, \varepsilon$ so that $L = 0$. Thus $\{| C_k |\}$ is summable $\mathscr{F}_{|A|}$ to 0. Hence $\{| C_k |^2\}$ is summable $\mathscr{F}_{|A|}$ to 0 and therefore summable $\mathscr{F}_A$ to 0. Since (1) and (2) have already been shown to be equivalent, we conclude that F is continuous. Thus (3) implies (1).

Theorem 8 generalizes a theorem of Wiener [16] (cf. Zygmund [17] p. 108) and contains as special cases various generalizations of that theorem including those given by Lozinskiĭ [9] and Matveev (cf. Bari [1] p. 256).

Theorem 8 can be reformulated in the following strengthened forms which we state separately.

THEOREM 9. *For $F \in V[0, 2\pi]$ to be continuous, it is necessary* *that* $\{|C_k|^2\}$ *should be summable* $\mathscr{F}_A$ *to* 0 *by each normal almost periodic matrix* A *and sufficient that* $\{|C_k|^2\}$ *should be summable* A *to* 0 *by some normal almost periodic matrix* A.

THEOREM 10. *For* $F \in V[0, 2\pi]$ *to be continuous, it is necessary that* $\{|C_k|^2\}$ *and* $\{|C_k|\}$ *should be summable* $\mathscr{F}_{|A|}$ *to* 0 *by each normal almost periodic matrix* $A = (a_{n,k})$ *for which* (1) $\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}| \exp(2\pi ikt) = 0$ *for all* $t \in (0, 1)$ *and sufficient that either* $\{|C_k|^2\}$ *or* $\{|C_k|\}$ *should be summable* A *by some normal almost periodic matrix satisfying* (1).

It may be pointed out that the assertion regarding summability $\mathscr{F}_{|A|}$ in Theorem 10 has been established in the course of the proof of Theorem 8. Theorem 10 generalizes the following strengthened form of Wiener's theorem given by Keogh and Petersen [7].

THEOREM A. *For* $F \in V[0, 2\pi]$ *to be continuous, it is necessary that* $\{|C_k|^2\}$ *and* $\{|C_k|\}$ *should be almost convergent to zero and sufficient that either* $\{|C_k|^2\}$ *or* $\{|C_k|\}$ *should be summable to zero by some summation method which contains almost convergence.*

Since every strongly regular matrix $A = (a_{n,k})$ is an almost periodic matrix satisfying (1) and the $(C, 1)$ matrix is strongly regular, the direct proposition of Theorem A is a particular case of the corresponding assertion in Theorem 10. We have already remarked earlier (§ 3) that Lorentz [8] has shown that a matrix sums all almost convergent sequences to their almost convergence limits if and only if it is strongly regular. The sufficiency part of Theorem A is therefore also a special case of the corresponding assertion in Theorem 10.

Lorentz [8] has proved that (a) if A is regular, then summability $\mathscr{F}_A$ implies almost convergence and that (b) if A is strongly regular, then summability $\mathscr{F}_A$ and almost convergence are equivalent. Although not explicitly stated by Lorentz, it follows that summability $\mathscr{F}_A$ and almost convergence are equivalent if and only if A is strongly regular. For, if A is not strongly regular, there exists an almost convergent sequence that is not summable A and hence a fortiori not summable $\mathscr{F}_A$. Hence if A is not strongly regular, summability $\mathscr{F}_A$ is strictly weaker then almost convergence. Since there exist non-strongly regular normal almost periodic matrices satisfying (1) (cf. Theorem 5), Theorem 10 is sharper than Theorem A in both directions.

A particularly interesting corollary of Theorem 10 is the following:

COROLLARY. *For a continuous $F \in V[0, 2\pi]$ with Fourier coefficients $\{c_k\}$, we have $\sum_p^{n+p} |c_k| = o(\log n)$ $(n \to \infty)$ uniformly in p.*

This result is significant since there exist continuous functions of bounded variation for which $c_k \neq o(1)$ $(k \to \infty)$.

5. In Theorems 6 and 7 of the preceding section, we started with matrices A satisfying the condition that $\|A\| < \infty$ and then found the necessary and sufficient condition in order that these be effective in the problem of the determination of jump or total quadratic jump of functions belonging to $V[0, 2\pi]$. However, as we shall see below, this restriction is not necessary. In fact, if we call a matrix $A = (a_{n,k})$ for which

$$K_n:\ K_n(t) = \sum_{n=0}^{\infty} a_{n,k} \exp(2\pi ikt)$$

is continuous in $[0, 1]$ for each n, a matrix with continuous kernel, we have the following:

THEOREM 11. *Let $A = (a_{n,k})$ be a matrix with continuous kernel $\{K_n\}$. Then for every $F \in V[0, 2\pi]$ and for every $x \in [0, 2\pi]$, the sequence $\{C_k e^{ikx}\}$ is summable A or $\mathscr{F}_A$ to $(2\pi)^{-1}D(x)$, where*

$$D(x) = F(x+0) - F(x-0)$$

if and only if

(i) $\sup\limits_{N \geqq 0} \max\limits_{t} |K_n^N(t)| = M_n < \infty$ *for every* n,

(ii) $\sup\limits_{n \geqq 0} \max\limits_{t} |K_n(t)| = M < \infty$,

(iii) $\lim\limits_{n\to\infty} K_n(t) = 0$ for $t \in (0, 1)$ *and* $=1$ *otherwise*,

where $K_n^N(t) = \sum_{k=0}^N a_{n,k} \exp(2\pi ikt)$, $N = 0, 1, \cdots$.

Proof. If A sums every sequence $\{C_k e^{ikx}\}$ for each x in $[0, 2\pi]$, and for each $F \in V[0, 2\pi]$, it follows that for each fixed n the sequence of continuous functions $\{K_n^N\}$ converges weakly in $C[0, 1]$ so that, by the uniform boundedness principle, we get (i). Since

$$\sum_{k=0}^{\infty} a_{n,k} C_k = \lim_{N\to\infty} \frac{1}{2\pi}\int_0^{2\pi} K_n^N\left(\frac{-t}{2\pi}\right) dF(t) = \frac{1}{2\pi}\int_0^{2\pi} K_n\left(\frac{-t}{2\pi}\right) dF(t)$$

and

$$\lim_{n\to\infty}\int_0^{2\pi} K_n\left(\frac{-t}{2\pi}\right) dF(t)$$

exists for all $F \in V[0, 2\pi]$, again, by the uniform boundedness principle, we get (ii). If for each $t \in [0, 1]$, we choose $F: F(x) = 0$ in $[0, t]$, $F(x) = 2\pi$ in $(t, 2\pi]$, we get $C_k = e^{-ikt}$ so that (iii) holds. Thus conditions (i), (ii) and (iii) are necessary. The proof of the sufficiency of these conditions is the same as in case of Theorem 6, if we observe that the continuity of K_n and (i) assure that

$$\sum_{k=0}^{\infty} a_{n,k}\, C_k\, e^{ikx} = \frac{1}{2\pi}\int_0^{2\pi} K_n\Big(\frac{x-t}{2\pi}\Big)\, dF(t)\ .$$

The assertion for summability F_A can be similarly proved.

We call a matrix $A = (a_{n,k})$ normal Fejér effective if it satisfies conditions (i), (ii) and (iii) of Theorem 11.

We can similarly prove the following analogues of Theorems 7 and 8 respectively.

THEOREM 12. *Let $A = (a_{n,k})$ be a matrix with continuous kernel. Then for every $F \in V[0, 2\pi]$, the sequence $\{|C_k|^2\}$ is summable A and $\mathscr{F}_A$ to $(4\pi^2)^{-1}\sum_{j=0}^{\infty}|D(x_j)|^2$, where $\{x_j\}$ are the points of jump of F in $[0, 2\pi)$, if and only if A is a normal Fejér effective matrix.*

THEOREM 13. *For functions $F \in V[0, 2\pi]$, the following conditions are equivalent:*

(1) *F is continuous,*

(2) *$\{|C_k|^2\}$ is summable A or $\mathscr{F}_A$ to 0 by a normal Fejér effective matrix A with continuous kernel,*

(3) *$\{|C_k|\}$ is summable A or $\mathscr{F}_A$ to 0 by a normal Fejér effective matrix $A = (a_{n,k})$ with continuous kernel, for which $\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n,k}|\exp(2\pi ikt) = 0$ for all $t \in (0, 1)$.*

Theorems analogous to Theorems 9 and 10 can also be established.

A normal almost periodic matrix is clearly a normal Fejér effective matrix since the hypothesis $\|A\| < \infty$ implies that the conditions (i) and (ii) of Theorem 11 are satisfied. But the converse is not true. Consider the matrix $A = (a_{n,k})$, where

$$\begin{aligned} a_{n,0} &= 0\ , & \\ a_{n,k} &= \frac{1}{n} & \text{for } 1 \leqq k \leqq n\ , \\ a_{n,k} &= \frac{\exp\{i\pi(k-n)\log(k-n)\}}{n^{3/4}(k-n)^{1/2+\alpha}}\Big(0<\alpha<\frac{1}{2}\Big) & \text{for } k > n\ . \end{aligned}$$

It can be verified that A is a normal Fejér effective matrix with continuous kernel that does not satisfy the condition $\|A\| < \infty$, since even $\sum_{k=0}^{\infty}|a_{n,k}| = \infty$ so that applying Theorem 2 one concludes that the matrix A is not an almost periodic matrix. It follows that for the validity of the theorems of this section we need normal matrices that may not be conservative.

6. Hayman [6] and Petersen [12] have applied Wiener's theorem and its generalization Theorem A respectively to the study of coefficient properties of holomorphic functions with positive real part. We can apply Theorem 10 to obtain the following:

THEOREM 14. *Let* $\psi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = u + iv$ *be holomorphic in* $|z| < 1$ *satisfying the condition* $u > 0$ *there and let*

$$\psi(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, dg(\theta)$$

be the Herglotz representation of ψ *where* g *is non-decreasing on* $[0, 2\pi]$. *Let* g_1 *denote the saltus part of* g.

a. *If* $A = (a_{n,k})$ *is a normal almost periodic matrix for which* $\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}| \exp(2\pi ikt) = 0$ *for all* $t \in (0, 1)$, *then there exists a complex Borel measure* μ *uniquely determined by* g_1 *and* A, *defined on the disk* $\Delta = \{w: |w| \leqq 2\}$ *such that*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\, \phi(b_{n+p}) = \int_\Delta \phi(w)d\mu \qquad \text{for all } \phi \in C(\Delta)$$

uniformly in p, *where* $C(\Delta)$ *denotes the space of all complex continuous functions on* Δ.

b. *If, moreover,* $\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}|$ *exists, then there exists a positive Borel measure* ν *uniquely determined by* g_1 *and* A, *defined on* Δ *such that*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}|\, \phi(b_{n+p}) = \int_\Delta \phi(w)d\nu \qquad \text{for all } \phi \in C(\Delta)$$

uniformly in p.

c. *If we define* χ_E: $\chi_E(b_k) = 1$ *if* $b_k \in E$ *and* $= 0$ *if* $b_k \notin E$ *where* E *is a Borel set, then under the same assumption on* $A = (a_{n,k})$ *as in* **b**, *we have*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}|\, \chi_E(b_{k+p}) = \nu(E)$$

uniformly in p.

Proof. If $\{\theta_\nu\}$ denote the points of discontinuity and $\{\alpha_\nu\}$ the jumps of g in $[0, 2\pi)$, and $g_2 = g - g_1$, we get

$$\begin{aligned}\psi(z) &= \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, dg(\theta) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, dg_1(\theta) + \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, dg_2(\theta) \\ &= \sum_{\nu=0}^{\infty} \alpha_\nu \frac{1+ze^{-i\theta_\nu}}{1-ze^{-i\theta_\nu}} + \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, dg_2(\theta) \\ &= \sum_{n=0}^{\infty} c_n z^n + \sum_{n=0}^{\infty} d_n z^n ,\end{aligned}$$

where $c_n = \sum_{\nu=0}^{\infty} 2\alpha_\nu e^{-in\theta_\nu} (n \geqq 1)$. It is known (cf. Hayman [5] p. 12) that $|b_n| \leqq 2$. Since

$$1 = \psi(0) = \int_0^{2\pi} dg(\theta) = \sum \alpha_\nu + g_2(2\pi) - g_2(0) \geqq \sum \alpha_\nu ,$$
$$|c_n| \leqq 2 \sum \alpha_\nu = 2c_0 \leqq 2 .$$

If we set $\phi(Re^{i\theta}) = (3-R)\,\phi\,(2e^{i\theta})\,(2 < R < 3)$ and $\phi(Re^{i\theta}) = 0(R \geqq 3)$, then ϕ is continuous in the whole plane, $\phi = 0$ for $|w| \geqq 3$ and hence ϕ is uniformly continuous. This extension of ϕ outside the disk $|w| \leqq 2$ does not alter $\phi\,(b_n)$ and $\phi\,(c_n)$ since $|b_n| \leqq 2$ and $|c_n| \leqq 2$.

If we put

$$L(\phi) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\,\phi\,(c_{k+p}) ,$$

then this limit exists uniformly in p, since $\{\phi(c_k)\}$ is almost periodic. We now show that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\,[\phi(c_{k+p}) - \phi(b_{k+p})] = 0$$

uniformly in p. Since ϕ is uniformly continuous, given any $\varepsilon > 0$, we can choose $\delta > 0$ so that *if* $|w - w'| < \delta$, then $|\phi(w) - \phi(w')| < \varepsilon$. Now

$$\begin{aligned}\sum_{k=0}^{\infty} |a_{n,k}|\,|\phi\,(b_{k+p}) - \phi\,(c_{k+p})| &\leqq \Big(\sum_{|d_{k+p}|<\delta} + \sum_{|d_{k+p}|\geqq\delta} \Big) \\ &\quad \times |a_{n,k}|\,|\phi\,(b_{k+p}) - \phi(c_{k+p})| \\ &\leqq \varepsilon\,\|A\| + 2M \sum_{|d_{k+p}|\geqq\delta} |a_{n,k}| .\end{aligned}$$

Since A is normal almost periodic and is such that $\lim_{n\to\infty} \sum_{k=0}^{\infty} |a_{n,k}| \exp(2\pi ikt) = 0$ for all $t \in (0, 1)$ and $\{d_n\}$ are the Fourier-Stieltjes coefficients of continuous part of g, it follows from Theorem 10 and the inequalities

$$\delta \sum_{|d_{k+p}|\geqq\delta} |a_{n,k}| \leqq \sum_{|d_{k+p}|\geqq\delta} |a_{n,k}|\,|d_{k+p}| \leqq \sum_{k=0}^{\infty} |a_{n,k}|\,|d_{k+p}|$$

that $\lim_{n\to\infty} \sum_{|d_{k+p}|\geqq\delta} |a_{n,k}| = 0$ uniformly in p. It follows that

$$L(\phi) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\,\phi(c_{k+p}) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\,\phi(b_{k+p})$$

exists uniformly in p and depends only on g_1. Since L is a bounded linear functional on the space of all continuous functions in the plane with compact support, there exists a complex Borel measure μ in the plane such that

$$L(\phi) = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}\,\phi\,(b_{k+p}) = \int_{\Delta} \phi\,(w)\,d\mu .$$

This establishes (a), and (b) follows from it.

Suppose E is a set, whose frontier has ν-measure zero. Then we can find a compact set $K \subset \operatorname{Int} E$ such that $\nu(K) > \nu(E) - \varepsilon$. Further, we can construct ϕ continuous in the plane such that $0 \leqq \phi \leqq 1$, $\phi(K) = 1$ and $\phi(cE) = 0$. Then

$$\sum_{k=0}^{\infty} | a_{n,k} | \phi(b_{k+p}) \leqq \sum_{k=0}^{\infty} | a_{n,k} | \chi_E(b_{k+p})$$

so that

$$\varliminf_{n\to\infty} \sum_{k=0}^{\infty} | a_{n,k} | \chi_E(b_{k+p}) \geqq \int_{\Delta} \phi(w) d\nu \geqq \nu(K) > \nu(E) - \varepsilon .$$

Similarly there exists a bounded open set $U \supset E$ such that $\nu(U) < \nu(E) + \varepsilon$. Choose a continuous function ψ in the plane such that $0 \leqq \psi \leqq 1$, $\psi(E) = 1$ and $\psi(cU) = 0$. Then

$$\sum_{k=0}^{\infty} | a_{n,k} | \psi(b_{k+p}) \geqq \sum_{k=0}^{\infty} | a_{n,k} | \chi_E(b_{k+p})$$

so that

$$\varlimsup_{n\to\infty} \sum_{k=0}^{\infty} | a_{n,k} | \chi_E(b_{k+p}) \leqq \int \psi(w) d\nu \leqq \nu(U) < \nu(E) + \varepsilon .$$

Since ε is arbitrary, we get

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} | a_{n,k} | \chi_E(b_{k+p}) = \nu(E) ,$$

uniformly in p.

We remark that in the above theorem we can replace the normal almost periodic matrix by a Fejér effective matrix with continuous kernel.

Finally, I would like to express my thanks to Professor B. Kuttner for kindly reading the manuscript of this paper and making valuable suggestions.

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Received December 7, 1970. The research was supported in part by the National Research Council of Canada Grant A-4057 and in part by the grant of a fellowship of the Summer Research Institute of the Canadian Mathematical Congress.

UNIVERSITE DE SHERBROOKE, SHERBROOKE, CANADA
SUMMER RESEARCH INSTITUTE OF CANADIAN MATHEMATICAL CONGRESS
UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, CANADA

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 39, No. 1 May, 1971