

Pacific Journal of Mathematics

GLOBALIZATION THEOREMS FOR LOCALLY FINITELY GENERATED MODULES

ROGER ALLEN WIEGAND

GLOBALIZATION THEOREMS FOR LOCALLY FINITELY GENERATED MODULES

ROGER WIEGAND

Each commutative ring has a coreflection \hat{R} in the category of commutative regular rings. We use the basic properties of \hat{R} to obtain globalization theorems for finite generation and for projectivity of R -modules.

1. Preliminaries. A detailed description of the ring \hat{R} may be found in [8]. Here we list without proofs the facts that will be needed. We assume that everything is unitary, but not necessarily commutative. However, R will always denote an arbitrary commutative ring. All unspecified tensor products are taken over R . For each $a \in R$ and each $P \in \text{Spec}(R)$, let $a(P)$ be the image of a under the obvious map $R \rightarrow R_P/PR_P$. Then \hat{R} is the subring $\coprod_P R_P/PR_P$ consisting of finite sums of elements $[a, b]$, where $[a, b]$ is the element whose P^{th} coordinate is 0 if $b \in P$ and $a(P)/b(P)$ if $b \notin P$. There is a natural homomorphism $\varphi: R \rightarrow \hat{R}$ taking a to $[a, 1]$. The ring \hat{R} is regular (in the sense of von Neumann). The statement that \hat{R} is a coreflection means simply that each homomorphism from R into a commutative regular ring factors uniquely through φ .

The map $\text{Spec}(\varphi): \text{Spec}(\hat{R}) \rightarrow \text{Spec}(R)$ is one-to-one and onto; for each $P \in \text{Spec}(R)$ we let \hat{P} be the corresponding prime (= maximal) ideal of \hat{R} .

If A is an R -module and $P \in \text{Spec}(R)$, then A_P/PA_P and $(A \otimes \hat{R})_{\hat{P}}$ are vector spaces over R_P/PR_P and $\hat{R}_{\hat{P}}$ respectively. The map $\varphi: R \rightarrow \hat{R}$ induces an isomorphism $R_P/PR_P \cong \hat{R}_{\hat{P}}$, and, under the identification, A_P/PA_P and $(A \otimes \hat{R})_{\hat{P}}$ are isomorphic vector spaces.

2. Globalization theorems.

LEMMA. *If $A \otimes \hat{R} = 0$ and A_R is locally finitely generated then $A = 0$.*

Proof. For each prime P , $A_P/PA_P = 0$, by the last paragraph of § 1. Since A_P is finitely generated over R_P , Nakayama's lemma implies that $A_P = 0$ for each $P \in \text{Spec}(R)$. Therefore $A = 0$.

THEOREM 1. *Assume $(A \otimes \hat{R})$ is finitely generated over \hat{R} , and that A_R is either locally free or locally finitely generated. Then A_R is finitely generated.*

Proof. Assume A_R is locally free. Then, for each prime P , A_P is a direct sum of, say, κ copies of R_P . Then A_P/PA_P is a direct sum of κ copies of R_P/PR_P . But since $(A \otimes \hat{R})$ is finitely generated over \hat{R} , A_P/PA_P is finite dimensional over R_P/PR_P . Thus κ is finite, and we conclude that A_R is locally finitely generated.

Now, if A_R is not finitely generated, we can express A as a well-ordered union of submodules A_α , each of which requires fewer generators than A . We will get a contradiction by showing that some $A_\alpha = A$. Let $B_\alpha = \text{Im}(A_\alpha \otimes \hat{R} \rightarrow A \otimes \hat{R})$. Since

$$A \otimes \hat{R} = \lim_{\rightarrow \alpha} (A_\alpha \otimes \hat{R}), \quad A \otimes \hat{R} = \bigcup_\alpha B_\alpha.$$

Since the B_α are nested and $(A \otimes \hat{R})$ is finitely generated over \hat{R} , some $B_{\alpha_0} = A \otimes \hat{R}$, that is, $A_{\alpha_0} \otimes \hat{R} \rightarrow A \otimes \hat{R}$. Let $C = A/A_{\alpha_0}$. Then $C \otimes \hat{R} = \text{Coker}(A_{\alpha_0} \otimes \hat{R} \rightarrow A \otimes \hat{R}) = 0$, and C_R is certainly locally finitely generated. By the lemma, $C = 0$, and $A_{\alpha_0} = A$.

THEOREM 2. *Let A_R be finitely generated and flat, and assume $(A \otimes \hat{R})$ is \hat{R} -projective. Then A_R is projective.*

Proof. By Chase's theorem [3, Theorem 4.1] it is sufficient to show that A_R is finitely related. Let $0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ be an exact sequence, with F_R free of finite rank. This sequence splits locally, so K is locally finitely generated. Since A_R is flat, the long exact sequence of Tor shows that $0 \rightarrow K \otimes \hat{R} \rightarrow F \otimes \hat{R} \rightarrow A \otimes \hat{R} \rightarrow 0$ is exact. This sequence splits, so $(K \otimes \hat{R})$ is finitely generated over \hat{R} . By Theorem 1, K_R is finitely generated.

3. Applications. The following result generalizes the well-known fact that over a noetherian ring every finitely generated flat module is projective.

PROPOSITION 1. *If R has a.c.c. on intersections of prime ideals then every finitely generated flat R -module is projective.*

Proof. In [8] these rings are characterized as those for which $(A \otimes \hat{R})$ is \hat{R} -projective for every finitely generated A_R . The conclusion follows from Theorem 2.

Suppose A_R is locally finitely generated. For each prime ideal P let $r_A(P)$ denote the number of generators required for A_P over R_P . By Nakayama's lemma, $r_A(P) = d_A(\hat{P})$, the dimension of $(A \otimes \hat{R})_{\hat{P}}$ as a vector space over $\hat{R}_{\hat{P}}$. Since the map $\hat{P} \rightarrow P$ is continuous, it follows that if r_A is continuous on $\text{Spec}(R)$ then d_A is continuous on $\text{Spec}(\hat{R})$. Using these observations we can give easy proofs of the

following two theorems:

THEOREM 3 (Bourbaki [1, Th. 1]): *Assume A_R is finitely generated and flat, and that r_A is continuous. Then A_R is projective.*

THEOREM 4 (Vasconcelos [7, Prop. 1.4]): *Assume A_R is projective and locally finitely generated, and that r_A is continuous. Then A_R is finitely generated.*

Proof of Theorem 3. By Theorem 3 we may assume R is regular. A proof of Theorem 3 in this case may be found in [5], but we include one here for completeness. For each $k \geq 0$ let

$$U_k = \{P \in \text{Spec}(R) \mid r_A(P) = k\} .$$

By hypothesis the sets U_k are clopen, and we let e_k be the idempotent with support U_k . Then $A = A e_0 \oplus \cdots \oplus A e_n$, and $r_{A e_k}$ is constant on $\text{Spec}(R e_k)$. Therefore we may assume r_A is constant on $\text{Spec}(R)$, say $r_A(P) = n$ for all P . Given a prime P , choose $a_1, \dots, a_n \in R$ such that $a_1(P), \dots, a_n(P)$ span A_P . Then $a_1(Q), \dots, a_n(Q)$ span R_Q for all Q in some neighborhood of P . (Here we need A_R finitely generated.) In this way we get a partition of $\text{Spec}(R)$ into disjoint clopen sets V_1, \dots, V_m together with elements $a_{ij} \in R$ such that $a_{ij}(P), \dots, a_{nj}(P)$ span A_P for each $P \in V_j$. Let e_j be the idempotent with support V_j , and set $b_i = \sum_j e_j a_{ij}$. Then, if P_R is free on u_1, \dots, u_n , the map $P \rightarrow A$ taking u_i to b_i is an isomorphism locally, and therefore globally.

Proof of Theorem 4. By Theorem 1 and the proof of Theorem 3 we can assume R is regular and $r_A(P) = n$ for all P . Write $A = \bigoplus_{i \in I} R e_i$, $e_i^2 = e_i \neq 0$, by [4]. Given $P \in \text{Spec}(R)$, since $(R e_i)_P$ is 0 if $e_i \in P$ and R_P if $e_i \notin P$, we see that there are precisely n indices i for which $e_i \notin P$. For each n -element subset $J \subseteq I$ let

$$U(J) = \{P \in \text{Spec}(R) \mid e_j \notin P \text{ for each } j \in J\} .$$

These open sets cover $\text{Spec}(R)$, so $\text{Spec}(R) = U(J_1) \cup \cdots \cup U(J_m)$. If $j \in J_1 \cup \cdots \cup J_m$ then e_j is in every prime ideal, contradicting $e_j \neq 0$. Therefore $|I| \leq mn$, and A_R is finitely generated.

As a final application we give the following:

PROPOSITION 2. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of flat R -modules. Assume A_R is finitely generated and $(B \otimes \hat{R})_{\hat{R}}$ is projective. Then A_R is projective.*

Proof. Since C_R is flat, $0 \rightarrow A \otimes \hat{R} \rightarrow B \otimes \hat{R} \rightarrow C \otimes \hat{R} \rightarrow 0$ is

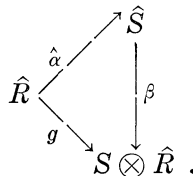
exact. Since \hat{R} is semihereditary $(A \otimes \hat{R})$ is R -projective. By Theorem 2, A_R is projective.

If B_R is projective this proposition contains no new information. (In fact, a trivial extension of Chase's Theorem shows that the sequence splits.) On the other hand, if we let M_R be projective, take $f \in R$, and let $B = M_f = \{[m/f^n]\}$, then B_R is not in general projective; but by the second corollary to Theorem 5 (next section), $B \otimes \hat{R}$ is \hat{R} -projective.

4. Epimorphisms. Suppose M is a multiplicative subset of R , and let $S = M^{-1}R$. Since $S \otimes \hat{R}_{\hat{P}} = S_P/PS_P$ for each prime P , we see that $S \otimes \hat{R}_{\hat{P}}$ is $\hat{R}_{\hat{P}}$ if $P \cap M = \emptyset$, and 0 if $P \cap M \neq \emptyset$. If we could show that $(S \otimes \hat{R})_{\hat{R}}$ is finitely generated, it would follow easily that $S \otimes \hat{R} = \hat{R}/K$, where K is the intersection of those primes \hat{P} for which $P \cap M = \emptyset$. We give an indirect proof of this fact in a more general setting.

Suppose R and S are commutative rings and that $\alpha: R \rightarrow S$ is an epimorphism in the category of rings. By a theorem of Silver [6] this is equivalent to the natural map $S \otimes S \rightarrow S$ being an isomorphism. It is known [8] that $R \rightarrow \hat{R}$ is an epimorphism, and it follows readily that the natural maps $f: S \rightarrow S \otimes \hat{R}$ and $g: R \rightarrow S \otimes \hat{R}$ are epimorphisms.

THEOREM 5. *Let R and S be commutative rings and let $\alpha: R \rightarrow S$ be an epimorphism in the category of rings. Then there is a unique ring homomorphism $\beta: \hat{S} \rightarrow S \otimes \hat{R}$ making the following diagram commute:*



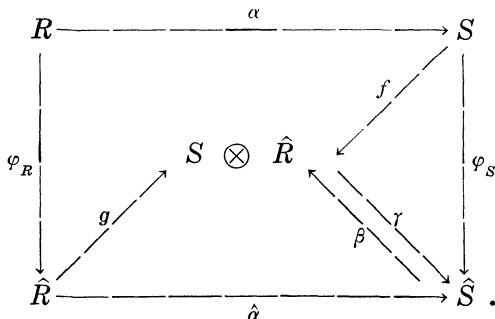
Moreover, β is an isomorphism, and $\hat{\alpha}$ and g are surjections with kernel $K = \cap \{\hat{P} \mid S_P \neq PS_P\}$.

Proof. We first show that $S \otimes \hat{R}$ is regular. Suppose A and B are $(S \otimes \hat{R})$ -modules. Then by Silver's Theorem $B = S \otimes_R B$, and by [2, p.165] we have

$$A \otimes_{S \otimes \hat{R}} B = A \otimes_{S \otimes \hat{R}} (S \otimes_R B) = (A \otimes_S S) \otimes_{\hat{R} \otimes R} B = A \otimes_{\hat{R}} B.$$

It follows that tensor products over $S \otimes \hat{R}$ are exact, and therefore

$S \otimes \hat{R}$ is regular. Hence there is a unique map $\beta: \hat{S} \rightarrow S \otimes \hat{R}$ such that $\beta\varphi_s = f$, where $\varphi_s: S \rightarrow \hat{S}$ is the natural map. Consider the diagram:



Here γ is defined by the equations $\gamma f = \varphi_s$, $\gamma g = \hat{\alpha}$. Now $\gamma\beta\varphi_s = \gamma f = \varphi_s$ and $\beta\gamma f = \beta\varphi_s = f$. Since φ_s and f are both epimorphisms, we see that $\gamma = \beta^{-1}$. Also, $B\hat{\alpha} = B\gamma g = g$, as required. Uniqueness of β follows from the fact that $\hat{\alpha}$ is an epimorphism (since both α and φ_s are).

Next, we show $\hat{\alpha}$ is onto. To simplify notation, we assume R is regular and $\alpha: R \rightarrow S$ is an epimorphism. Then $S \otimes S \xrightarrow{\mu} S$ is an isomorphism. But then $S_P \otimes_{R_P} S_P \rightarrow S_P$ is an isomorphism for each $P \in \text{Spec}(R)$. If $s \in S_P$ then $1 \otimes s - s \otimes 1 \in \ker \mu_P = 0$. It follows that the dimension of S_P as a vector space over R_P is either 0 or 1. Therefore α_P is surjective for each P , ($\alpha(1) = 1$), and we conclude that α is surjective.

Finally, we compute $\ker g = K$. If $P \in \text{Spec}(\hat{R})$, then

$$K \subseteq \hat{P} \iff K_{\hat{P}} = 0 \iff \hat{S}_{\hat{P}} \neq 0 \iff S \otimes \hat{R}_{\hat{P}} \neq 0 \iff S_P/PS_P \neq 0.$$

COROLLARY 1. *Let M be a multiplicative subset of R and let $S = M^{-1}R$. Then $S \otimes \hat{R}$ is a cyclic \hat{R} -module, and $S \otimes \hat{R}$ is \hat{R} -projective if and only if $\{\hat{P} \mid M \cap P \neq \emptyset\}$ is closed in $\text{Spec}(\hat{R})$.*

Proof. Let K be as in Theorem 5. Then $S \otimes \hat{R} = \hat{R}/K$ is \hat{R} -projective if and only if K is a principal ideal, that is, if and only if the set of primes containing K is open in $\text{Spec}(\hat{R})$. But

$$\hat{P} \supseteq K \iff PS_P \neq S_P \iff M \cap P = \emptyset.$$

The next corollary shows that Theorem 2 is false if A_R is not assumed to be finitely generated.

COROLLARY 2. *For each $f \in R$, $R_f \otimes \hat{R}$ is \hat{R} -projective.*

Proof. Set $M = \{f^n: n \geq 0\}$. Then $P \cap M \neq \emptyset$ if and only if $\varphi(f) \in \hat{P}$. Thus K is the principal ideal of \hat{R} generated by $\varphi(f)$, and \hat{R}/K is \hat{R} -projective.

REFERENCES

1. N. Bourbaki, *Eléments de mathématique. Algèbre commutative*, Chapter II, Hermann, Paris, 1961.
2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, 1956.
3. S. U. Chase, "Direct products of modules", *Trans. Amer. Math. Soc.*, **97** (1960), 457-473.
4. I. Kaplansky, Projective modules, *Ann. of Math.*, **68** (1958), 372-377.
5. R. S. Pierce, *Modules over commutative regular rings*, *Memoirs Amer. Math. Soc.*, **70** (1967).
6. L. Silver, *Non-commutative localizations and applications*, *J. Algebra*, **7** (1967), 44-76.
7. W. Vasconcelos, *On projective modules of finite rank*, *Proc. Amer. Math. Soc.*, **22**, No. 2 (1969), 430-433.
8. R. Wiegand, *Modules over universal regular rings*, (to appear in the *Pacific J. Math.*).

Received March 19, 1971. This research was partially supported by NSF Grant GP-19102.

UNIVERSITY OF WISCONSIN

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Charles A. Akemann, <i>A Gelfand representation theory for C^*-algebras</i>	1
Sorrell Berman, <i>Spectral theory for a first-order symmetric system of ordinary differential operators</i>	13
Robert L. Bernhardt, III, <i>On splitting in hereditary torsion theories</i>	31
J. L. Brenner, <i>Geršgorin theorems, regularity theorems, and bounds for determinants of partitioned matrices. II. Some determinantal identities</i>	39
Robert Morgan Brooks, <i>On representing F^*-algebras</i>	51
Lawrence Gerald Brown, <i>Extensions of topological groups</i>	71
Arnold Barry Calica, <i>Reversible homeomorphisms of the real line</i>	79
J. T. Chambers and Shinnosuke Oharu, <i>Semi-groups of local Lipschitzians in a Banach space</i>	89
Thomas J. Cheatham, <i>Finite dimensional torsion free rings</i>	113
Byron C. Drachman and David Paul Kraines, <i>A duality between transpotence elements and Massey products</i>	119
Richard D. Duncan, <i>Integral representation of excessive functions of a Markov process</i>	125
George A. Elliott, <i>An extension of some results of Takesaki in the reduction theory of von Neumann algebras</i>	145
Peter C. Fishburn and Joel Spencer, <i>Directed graphs as unions of partial orders</i>	149
Howard Edwin Gorman, <i>Zero divisors in differential rings</i>	163
Maurice Heins, <i>A note on the Löwner differential equations</i>	173
Louis Melvin Herman, <i>Semi-orthogonality in Rickart rings</i>	179
David Jacobson and Kenneth S. Williams, <i>On the solution of linear G.C.D. equations</i>	187
Michael Joseph Kallaher, <i>On rank 3 projective planes</i>	207
Donald Paul Minassian, <i>On solvable O^*-groups</i>	215
Nils Øvrelid, <i>Generators of the maximal ideals of $A(\bar{D})$</i>	219
Mohan S. Putcha and Julian Weissglass, <i>A semilattice decomposition into semigroups having at most one idempotent</i>	225
Robert Raphael, <i>Rings of quotients and π-regularity</i>	229
J. A. Siddiqi, <i>Infinite matrices summing every almost periodic sequence</i>	235
Raymond Earl Smithson, <i>Uniform convergence for multifunctions</i>	253
Thomas Paul Whaley, <i>Multiplicity type and congruence relations in universal algebras</i>	261
Roger Allen Wiegand, <i>Globalization theorems for locally finitely generated modules</i>	269