GLOBALIZATION THEOREMS FOR LOCALLY FINITELY GENERATED MODULES

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Each commutative ring has a coreflection \( \hat{R} \) in the category of commutative regular rings. We use the basic properties of \( \hat{R} \) to obtain globalization theorems for finite generation and for projectivity of \( R \)-modules.

1. Preliminaries. A detailed description of the ring \( \hat{R} \) may be found in [8]. Here we list without proofs the facts that will be needed. We assume that everything is unitary, but not necessarily commutative. However, \( R \) will always denote an arbitrary commutative ring. All unspecified tensor products are taken over \( R \).

For each \( a \in R \) and each \( P \in \text{Spec}(R) \), let \( a(P) \) be the image of \( a \) under the obvious map \( R \to R_P/PR_P \). Then \( \hat{R} \) is the subring \( \coprod_P R_P/PR_P \) consisting of finite sums of elements \([a, b]\), where \([a, b]\) is the element whose \( P \)-th coordinate is 0 if \( b \in P \) and \( a(P)/b(P) \) if \( b \notin P \). There is a natural homomorphism \( \varphi: R \to \hat{R} \) taking \( a \) to \([a, 1]\). The ring \( \hat{R} \) is regular (in the sense of von Neumann). The statement that \( \hat{R} \) is a coreflection means simply that each homomorphism from \( R \) into a commutative regular ring factors uniquely through \( \varphi \).

The map \( \text{Spec}(\varphi): \text{Spec}(\hat{R}) \to \text{Spec}(R) \) is one-to-one and onto; for each \( P \in \text{Spec}(R) \) we let \( \hat{P} \) be the corresponding prime (= maximal) ideal of \( \hat{R} \).

If \( A \) is an \( R \)-module and \( P \in \text{Spec}(R) \), then \( A_P/PA_P \) and \((A \otimes \hat{R})_{\hat{P}}\) are vector spaces over \( R_P/PR_P \) and \( \hat{R}_{\hat{P}} \) respectively. The map \( \varphi: R \to \hat{R} \) induces an isomorphism \( R_P/PR_P \cong \hat{R}_{\hat{P}} \), and, under the identification, \( A_P/PA_P \) and \((A \otimes \hat{R})_{\hat{P}}\) are isomorphic vector spaces.

2. Globalization theorems.

**Lemma.** If \( A \otimes \hat{R} = 0 \) and \( A_R \) is locally finitely generated then \( A = 0 \).

**Proof.** For each prime \( P \), \( A_P/PA_P = 0 \), by the last paragraph of §1. Since \( A_P \) is finitely generated over \( R_P \), Nakayama's lemma implies that \( A_P = 0 \) for each \( P \in \text{Spec}(R) \). Therefore \( A = 0 \).

**Theorem 1.** Assume \((A \otimes \hat{R})\) is finitely generated over \( \hat{R} \), and that \( A_R \) is either locally free or locally finitely generated. Then \( A_R \) is finitely generated.
Proof. Assume $A_R$ is locally free. Then, for each prime $P$, $A_P$ is a direct sum of, say, $\kappa$ copies of $R_P$. Then $A_P/PA_P$ is a direct sum of $\kappa$ copies of $R_P/PR_P$. But since $(A \otimes \hat{R})$ is finitely generated over $\hat{R}$, $A_P/PA_P$ is finite dimensional over $R_P/PR_P$. Thus $\kappa$ is finite, and we conclude that $A_P$ is locally finitely generated.

Now, if $A_R$ is not finitely generated, we can express $A$ as a well-ordered union of submodules $A_\alpha$, each of which requires fewer generators than $A$. We will get a contradiction by showing that some $A_\alpha = A$. Let $B_\alpha = \text{Im}(A_\alpha \otimes \hat{R} \to A \otimes \hat{R})$. Since

$$A \otimes \hat{R} = \lim_{\alpha} (A_\alpha \otimes \hat{R}) , \quad A \otimes \hat{R} = \bigcup_\alpha B_\alpha .$$

Since the $B_\alpha$ are nested and $(A \otimes \hat{R})$ is finitely generated over $\hat{R}$, some $B_{\alpha_0} = A \otimes \hat{R}$, that is, $A_{\alpha_0} \otimes \hat{R} \to A \otimes \hat{R}$. Let $C = A/A_{\alpha_0}$. Then $C \otimes \hat{R} = \text{Coker}(A_{\alpha_0} \otimes \hat{R} \to A \otimes \hat{R}) = 0$, and $C_R$ is certainly locally finitely generated. By the lemma, $C = 0$, and $A_{\alpha_0} = A$.

**Theorem 2.** Let $A_R$ be finitely generated and flat, and assume $(A \otimes \hat{R})$ is $\hat{R}$-projective. Then $A_R$ is projective.

**Proof.** By Chase’s theorem [3, Theorem 4.1] it is sufficient to show that $A_R$ is finitely related. Let $0 \to K \to F \to A \to 0$ be an exact sequence, with $F_R$ free of finite rank. This sequence splits locally, so $K$ is locally finitely generated. Since $A_R$ is flat, the long exact sequence of Tor shows that $0 \to K \otimes \hat{R} \to F \otimes \hat{R} \to A \otimes \hat{R} \to 0$ is exact. This sequence splits, so $(K \otimes \hat{R})$ is finitely generated over $\hat{R}$. By Theorem 1, $K_R$ is finitely generated.

3. Applications. The following result generalizes the well-known fact that over a noetherian ring every finitely generated flat module is projective.

**Proposition 1.** If $R$ has a.c.c. on intersections of prime ideals then every finitely generated flat $R$-module is projective.

**Proof.** In [8] these rings are characterized as those for which $(A \otimes \hat{R})$ is $\hat{R}$-projective for every finitely generated $A_R$. The conclusion follows from Theorem 2.

Suppose $A_R$ is locally finitely generated. For each prime ideal $P$ let $r_A(P)$ denote the number of generators required for $A_P$ over $R_P$. By Nakayama’s lemma, $r_A(P) = d_A(\hat{P})$, the dimension of $(A \otimes \hat{R})_{\hat{P}}$ as a vector space over $\hat{R}_{\hat{P}}$. Since the map $\hat{P} \to P$ is continuous, it follows that if $r_A$ is continuous on Spec($R$) then $d_A$ is continuous on Spec($\hat{R}$). Using these observations we can give easy proofs of the
following two theorems:

**Theorem 3 (Bourbaki [1, Th. 1]):** Assume $A_R$ is finitely generated and flat, and that $r_A$ is continuous. Then $A_R$ is projective.

**Theorem 4 (Vasconcelos [7, Prop. 1.4]):** Assume $A_R$ is projective and locally finitely generated, and that $r_A$ is continuous. Then $A_R$ is finitely generated.

*Proof of Theorem 3.* By Theorem 3 we may assume $R$ is regular. A proof of Theorem 3 in this case may be found in [5], but we include one here for completeness. For each $k \geq 0$ let

$$U_k = \{P \in \text{Spec}(R) \mid r_A(P) = k\}.$$  

By hypothesis the sets $U_k$ are clopen, and we let $e_k$ be the idempotent with support $U_k$. Then $A = A e_0 \oplus \cdots \oplus A e_n$, and $r_{A e_k}$ is constant on Spec $(Re_k)$. Therefore we may assume $r_A$ is constant on Spec $(R)$, say $r_A(P) = n$ for all $P$. Given a prime $P$, choose $a_1, \ldots, a_n \in R$ such that $a_i(P), \ldots, a_n(P)$ span $A_P$. Then $a_1(Q), \ldots, a_n(Q)$ span $R_Q$ for all $Q$ in some neighborhood of $P$. (Here we need $A_R$ finitely generated.) In this way we get a partition of Spec $(R)$ into disjoint clopen sets $V_1, \ldots, V_m$ together with elements $a_{ij} \in R$ such that $a_{ij}(P), \ldots, a_{nj}(P)$ span $A_P$ for each $P \in V_j$. Let $e_j$ be the idempotent with support $V_j$, and set $b_i = \sum_j e_j a_{ij}$. Then, if $P_R$ is free on $u_1, \ldots, u_n$, the map $P \rightarrow A$ taking $u_i$ to $b_i$ is an isomorphism locally, and therefore globally.

*Proof of Theorem 4.* By Theorem 1 and the proof of Theorem 3 we can assume $R$ is regular and $r_A(P) = n$ for all $P$. Write $A = \bigoplus_{i \in I} R e_i$, $e_i^2 = e_i \neq 0$, by [4]. Given $P \in \text{Spec}(R)$, since $(Re_i)_P$ is 0 if $e_i \in P$ and $R_P$ if $e_i \notin P$, we see that there are precisely $n$ indices $i$ for which $e_i \notin P$. For each $n$-element subset $J \subseteq I$ let

$$U(J) = \{P \in \text{Spec}(R) \mid e_j \notin P \text{ for each } j \in J\}.$$  

These open sets cover Spec $(R)$, so Spec $(R) = U(J_1) \cup \cdots \cup U(J_m)$. If $j \in J_1 \cup \cdots \cup J_m$ then $e_j$ is in every prime ideal, contradicting $e_j \neq 0$. Therefore $|I| \leq mn$, and $A_R$ is finitely generated.

As a final application we give the following:

**Proposition 2.** Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of flat $R$-modules. Assume $A_R$ is finitely generated and $(B \otimes \hat{R})_R$ is projective. Then $A_R$ is projective.

*Proof.* Since $C_R$ is flat, $0 \rightarrow A \otimes \hat{R} \rightarrow B \otimes \hat{R} \rightarrow C \otimes \hat{R} \rightarrow 0$ is
exact. Since \( \hat{R} \) is semihereditary \((A \otimes \hat{R}) \) is \( R \)-projective. By Theorem 2, \( A_R \) is projective.

If \( B_R \) is projective this proposition contains no new information. (In fact, a trivial extension of Chase's Theorem shows that the sequence splits.) On the other hand, if we let \( M_R \) be projective, take \( f \in R \), and let \( B = M_f = \{[m/f^*] \} \), then \( B_R \) is not in general projective; but by the second corollary to Theorem 5 (next section), \( B \otimes \hat{R} \) is \( \hat{R} \)-projective.

4. Epimorphisms. Suppose \( M \) is a multiplicative subset of \( R \), and let \( S = M^{-1}R \). Since \( S \otimes \hat{R}_P = S_P/PS_P \) for each prime \( P \), we see that \( S \otimes \hat{R}_P \) is \( \hat{R}_P \) if \( P \cap M = \emptyset \), and 0 if \( P \cap M \neq \emptyset \). If we could show that \( (S \otimes \hat{R})_R \) is finitely generated, it would follow easily that \( S \otimes \hat{R} = \hat{R}/K \), where \( K \) is the intersection of those primes \( \hat{P} \) for which \( P \cap M = \emptyset \). We give an indirect proof of this fact in a more general setting.

Suppose \( R \) and \( S \) are commutative rings and that \( \alpha: R \to S \) is an epimorphism in the category of rings. By a theorem of Silver \([6]\) this is equivalent to the natural map \( S \otimes S \to S \) being an isomorphism. It is known \([8]\) that \( R \to \hat{R} \) is an epimorphism, and it follows readily that the natural maps \( f: S \to S \otimes \hat{R} \) and \( g: R \to \hat{S} \) are epimorphisms.

**Theorem 5.** Let \( R \) and \( S \) be commutative rings and let \( \alpha: R \to S \) be an epimorphism in the category of rings. Then there is a unique ring homomorphism \( \beta: \hat{S} \to S \otimes \hat{R} \) making the following diagram commute:

\[
\begin{array}{ccc}
\hat{R} & \xrightarrow{\hat{\alpha}} & \hat{S} \\
\downarrow{\hat{\beta}} & & \uparrow{\beta} \\
S \otimes \hat{R} & \rightarrow & \end{array}
\]

Moreover, \( \beta \) is an isomorphism, and \( \hat{\alpha} \) and \( g \) are surjections with kernel \( K = \cap \{\hat{P} | S_P \neq PS_P \} \).

**Proof.** We first show that \( S \otimes \hat{R} \) is regular. Suppose \( A \) and \( B \) are \((S \otimes \hat{R})\)-modules. Then by Silver's Theorem \( B = S \otimes_R B \), and by \([2, p. 165]\) we have

\[
A \otimes_{S \otimes \hat{R}} B = A \otimes_{S \otimes \hat{R}} (S \otimes_R B) = (A \otimes_S S) \otimes_{\hat{R}} B = A \otimes_{\hat{R}} B .
\]

It follows that tensor products over \( S \otimes \hat{R} \) are exact, and therefore
$S \otimes \hat{R}$ is regular. Hence there is a unique map $\beta: \hat{S} \to S \otimes \hat{R}$ such that $\beta \varphi_s = f$, where $\varphi_s: S \to \hat{S}$ is the natural map. Consider the diagram:

Here $\gamma$ is defined by the equations $\gamma f = \varphi_s$, $\gamma g = \hat{\alpha}$. Now $\gamma \beta \varphi_s = \gamma f = \varphi_s$ and $\beta \gamma f = \beta \varphi_s = f$. Since $\varphi_s$ and $f$ are both epimorphisms, we see that $\gamma = \beta^{-1}$. Also, $B\hat{\alpha} = B\gamma g = g$, as required. Uniqueness of $\beta$ follows from the fact that $\hat{\alpha}$ is an epimorphism (since both $\alpha$ and $\varphi_s$ are).

Next, we show $\hat{\alpha}$ is onto. To simplify notation, we assume $R$ is regular and $\alpha: R \to S$ is an epimorphism. Then $S \otimes S \to S$ is an isomorphism. But then $S \otimes S \to S$ is an isomorphism for each $P \in \text{Spec}(R)$. If $s \in S_P$ then $1 \otimes s - s \otimes 1 \in \ker \mu_P = 0$. It follows that the dimension of $S_P$ as a vector space over $R_P$ is either 0 or 1. Therefore $\alpha_P$ is surjective for each $P$, ($\alpha(1) = 1$), and we conclude that $\alpha$ is surjective.

Finally, we compute $\ker g = K$. If $P \in \text{Spec}\ (\hat{R})$, then

$$K \subseteq \hat{P} \iff K \beta = 0 \iff \hat{S} \beta = 0 \iff S \otimes \hat{R} \beta = 0 \iff S_P/PS_P \neq 0.$$ 

**Corollary 1.** Let $M$ be a multiplicative subset of $R$ and let $S = M^{-1}R$. Then $S \otimes \hat{R}$ is a cyclic $\hat{R}$-module, and $S \otimes \hat{R}$ is $\hat{R}$-projective if and only if $\{P \mid M \cap P \neq \emptyset\}$ is closed in $\text{Spec}(\hat{R})$.

**Proof.** Let $K$ be as in Theorem 5. Then $S \otimes \hat{R} = \hat{R}/K$ is $\hat{R}$-projective if and only if $K$ is a principal ideal, that is, if and only if the set of primes containing $K$ is open in $\text{Spec}(\hat{R})$. But

$$\hat{P} \ni K \iff PS_P \neq S_P \iff M \cap P = \emptyset.$$ 

The next corollary shows that Theorem 2 is false if $A_R$ is not assumed to be finitely generated.

**Corollary 2.** For each $f \in R$, $R_f \otimes \hat{R}$ is $\hat{R}$-projective.
Proof. Set $M = \{f^n : n \geq 0\}$. Then $P \cap M \neq \emptyset$ if and only if $\varphi(f) \in \hat{P}$. Thus $K$ is the principal ideal of $\hat{R}$ generated by $\varphi(f)$, and $\hat{R}/K$ is $\hat{R}$-projective.

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