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**PERMUTATIONS AS PRODUCTS OF CONJUGATE INFINITE
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Let $S = \{a_i\}_{i=-\infty}^{\infty}$ be a countable set and P any permutation of S with infinite support. Since the subgroup generated by the conjugacy class \mathcal{K} of P must be normal in $\text{Sym}(S)$, we know that every permutation of S is a product of permutations from \mathcal{K} . Since it has recently been discovered that every even permutation in the finite symmetric group $\text{Sym}(n)$ may be expressed as a product of exactly two n -cycles, we are naturally led to a similar question for $\text{Sym}(S)$ and the infinite cycle $C = (\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots)$, with support all of S . In this paper it is proved that for each $k \geq 3$ every permutation of S is a product of exactly k cycles conjugate to C , but that no odd finite permutation is a product of two.

1. Introduction and notation. In 1951 Oystein Ore [4] proved that if S is any infinite set, then every permutation P of S is a commutator, $P = Q(RQ^{-1}R^{-1})$, of permutations Q, R . Throughout his constructions, the conjugacy class to which Q belongs depends upon P . Allan Gray, in his dissertation [2], showed that this dependency of the class of Q upon that of P may be dropped. Our interest here is in the group $\text{Sym}(S)$ of all permutations of a countable set S . Gray showed that if \mathcal{K} is the conjugacy class of permutations whose disjoint cycle decomposition consists of an *infinite* number of infinite (and no other) cycles, then $\text{Sym}(S) = \mathcal{K}\mathcal{K}$, that is *every* permutation is a product of exactly *two* such conjugate permutations.

Recently, in [1], we studied the finite symmetric group $\text{Sym}(n)$, and solved the problem of characterizing those integers $l \leq n$ for which every even permutation of n symbols may be represented as a product of two (not necessarily disjoint) l -cycles. In particular, we showed that *every* even permutation can be expressed as a product of *two* n -cycles. (See also A. M. Gleason in [3, p. 172].)

In this paper we make use of the last result in considering a similar question for the group $\text{Sym}(S)$, S countable: Let \mathcal{C}_∞ denote the conjugacy class of infinite cycles, each of which moves each symbol of S . Is there an integer m such that every permutation in $\text{Sym}(S)$ can be represented as a product of m permutations, each from \mathcal{C}_∞ ? If so, what is the smallest such m ? We are able to show that $\text{Sym}(S) = \mathcal{C}_\infty^k$ for $k \geq 3$, but that $\text{Sym}(S) \neq \mathcal{C}_\infty\mathcal{C}_\infty$. To accomplish this, we prove that every infinite cycle C may be represented as a product of two infinite cycles, where each cycle of the

product moves those and only those symbols moved by C . Our construction is thus simpler than Ore's, who first showed that C is a commutator by constructing C as a product of two permutations, each the (disjoint) product of an infinite number of infinite cycles.

Let $\text{Alt}(S)$ denote the subgroup of finite even permutations; \mathcal{B} denotes the subset of permutations with one infinite cycle, and no restrictions on finite cycles; \mathcal{E} denotes the subset of permutations with infinitely many 2-cycles and no infinite cycles or cycles of length ≥ 4 . Then $\text{Alt}(S) \cup \mathcal{B} \cup \mathcal{E}$ is shown to be a subset of \mathcal{C}_∞^2 . The remainder of the proof consists in showing that if $P \notin \text{Alt}(S) \cup \mathcal{B} \cup \mathcal{E}$ then either $PQ \in \text{Alt}(S) \cup \mathcal{B} \cup \mathcal{E}$ for some $Q \in \mathcal{C}_\infty$ or $PQ \in \mathcal{C}_\infty$ for some $Q \in \text{Alt}(S) \cup \mathcal{B} \cup \mathcal{E}$.

The subset of symbols of S which are moved by $P \in \text{Sym}(S)$ is denoted by $M(P)$. $|M(P)|$ denotes the cardinality of $M(P)$, $(P)_n$ denotes the cardinality of the set of cycles of length n ($n \geq 1$) in the disjoint cycle decomposition of P , and $(P)_\infty$ denotes the cardinality of the set of infinite cycles in this decomposition. *All products of permutations are executed from right to left.* P^m refers to the m th iterate of P , for $m > 0$, and the $-m$ th iterate of P^{-1} (the inverse of P), for $m < 0$.

2. Permutations with $M(P)$ finite. In this section we prove that $\text{Alt}(S) \subseteq \mathcal{C}_\infty^2$, but that no finite odd permutation of S belongs to \mathcal{C}_∞^2 . \denotes set theoretic difference.

THEOREM 2.1. $\text{Alt}(S) \subseteq \mathcal{C}_\infty^2$.

Proof. Given $P \in \text{Alt}(S)$, put $|M(P)| = l (< \infty)$. By Theorem 2 of [1], we know that the restriction of P to $M(P)$ may be represented as a product RS of two l -cycles R and S , each moving precisely the symbols moved by P . If we fix $a \in M(P)$ and put $S \setminus M(P) = \{b_i\}_1^\infty$, we may list the symbols of S in the manner displayed below:

$$\dots, b_5, b_3, b_1, a, S(a), S^2(a), \dots, S^{l-2}(a), S^{-1}(a), b_2, b_4, b_6, \dots$$

Define two new permutations, S^* and R^* , as follows:

$$\begin{array}{lll} S^*(b_{2m+1}) = b_{2m-1} & R^*(b_{2m-1}) = b_{2m+1} & m \geq 1 \\ S^*(b_1) = a & R^*(a) = b_1 & \\ S^*(b_{2m}) = b_{2m+2} & R^*(b_{2m+2}) = b_{2m} & m \geq 1 \\ S^*(S^{-1}(a)) = b_2 & R^*(b_2) = R(a) & \\ S^*(S^k(a)) = S^{k+1}(a) & R^*(S^{k+1}(a)) = R(S^{k+1}(a)) & 0 \leq k \leq l-2. \end{array}$$

It is now a straightforward verification that $R^*, S^* \in \mathcal{C}_\infty$ and $R^*S^* = P$ on all of S .

If we let \mathcal{D} denote the set of all $P \in \text{Sym}(S)$ with $|M(P)| < \infty$ and P an odd permutation of $M(P)$, we have the following result [5].

THEOREM 2.2. $\mathcal{D} \subseteq \text{Sym}(S) \setminus \mathcal{C}_\infty^2$.

Proof. By contradiction. Assume that there exists $P \in \mathcal{D}$ and $Q, R \in \mathcal{C}_\infty$ such that $P = QR$. Since R is an infinite cycle which moves each symbol of S , we may rename the symbols of S with $\{x_i\}_{i=-\infty}^\infty$, where $R(x_i) = x_{i+1}$. Then there exists symbols, say a and c , such that a is the $x_i \in M(P)$ of smallest subscript, and c is that of largest subscript. Since each symbol outside the "segment" $[a, c]$ is fixed by P , we know that $Q^k(a) = R^{-k}(a)$ and $Q^{-k}(R(c)) = R^k(R(c))$, for $k \geq 1$. Thus the list of symbols of S can in fact be given by

$$\dots, Q^3(a), Q^2(a), Q(a), a, R(a), R^2(a), \dots, R^{-1}(c), c, R(c), R^2(c), \dots.$$

Note that $Q(R(c))$ must be among the symbols of the segment $[R(a), R^{-1}(c)]$, and that some symbols of the segment $[R(a), R^{-1}(c)]$ may be fixed by P . Permutations Q^* and R^* are next defined as follows:

$$\begin{aligned} Q^*(x) &= Q(x) \text{ for } x \in [R(a), R(c)]; & R^*(x) &= R(x) \text{ for } x \in [a, c] \\ Q^*(a) &= R(c); & R^*(R(c)) &= a \\ Q^*(x) &= x \text{ for } x \notin [a, R(c)]; & R^*(x) &= x \text{ for } x \notin [a, R(c)]. \end{aligned}$$

Then $Q^*R^* = QR = P$ everywhere on S . But, restricted to the set $[a, R(c)]$, Q^* and R^* are each cycles of length $|M(P)| + 1$, and on this set Q^*R^* represents the restriction of P to the set $[a, R(c)]$. This is a contradiction to the assumption that P is an odd permutation on the set $[a, R(c)]$.

3. Permutations with $(P)_\infty = 1$. We first prove that $\mathcal{C}_\infty \subseteq \mathcal{C}_\infty^2 \subseteq \mathcal{C}_\infty^3 \subseteq \dots$. This, together with Theorem 2.1, yields $\text{Alt}(S) \subseteq \mathcal{C}_\infty^2 \subseteq \mathcal{C}_\infty^3 \subseteq \dots$. Our construction also shows that every infinite cycle C is a product of two permutations, each permutation in the conjugacy class of C . The result is simpler than that of Ore [4, p. 309, Theorem 2], who showed that C may be written as a product of two permutations, each an infinite (disjoint) product of infinite cycles.

Here we regard S as the set \mathbb{Z} of integers $\{\dots, -3, -2, -1, 0, 1, 2, \dots\}$. Let C denote the permutation of \mathbb{Z} given by $C(i) = i + 1$ for each $i \in \mathbb{Z}$. Then \mathcal{C}_∞ is the conjugacy class of C .

THEOREM 3.1. $\mathcal{C}_\infty \subseteq \mathcal{C}_\infty^2 \subseteq \mathcal{C}_\infty^3 \subseteq \dots$.

Proof. We need only show that $C \in \mathcal{C}_\infty^2$. For this, let $A = (\dots, -9, 9, -6, 6, -3, 3, 0, -1, 2, 1, -2, -4, 4, -5, 5, -7, 7, -8, 8, \dots)$

and $B = (\dots, 8, -9, 7, -8, 5, -6, 4, -5, -2, 0, 2, -3, 1, -, 1, 3, -4, 6, -7, 9, -10, \dots)$ be given by

$$\begin{aligned} A(n) &= -n, n \leq -3 & B(n) &= -(n+1), n \geq 3 \\ &\text{and for } n \geq 3: & &\text{and for } n \leq -3 (n \neq -5): \\ A(n) &= -(n-3), n \equiv 0 \pmod{3} & B(n) &= -(n+2), n \equiv 0 \pmod{3} \\ A(n) &= -(n+1), n \equiv 1 \pmod{3} & B(n) &= -(n+3), n \equiv 1 \pmod{3} \\ A(n) &= -(n+2), n \equiv 2 \pmod{3} & B(n) &= -(n+2), n \equiv 2 \pmod{3}. \end{aligned}$$

Obviously, $A \in \mathcal{C}_\infty$. For $n \notin \{-5, -2, -1, 0, 1, 2\}$, by dividing the proof according to the residue class of $n \pmod{3}$, it is quickly verified that $AB(n) = C(n) = n+1$; and this is immediate for $n \in \{-5, -2, -1, 0, 1, 2\}$. B is a permutation, and it follows that $B \in \mathcal{C}_\infty$, since for each $n \in \mathbb{Z}$ there exists an integer $k(n)$ such that $B^k(n) = 0$.

THEOREM 3.2. $\mathcal{B} \subseteq \mathcal{C}_\infty^2$.

Proof. By Theorem 3.1, we may assume that $P \in \mathcal{B}$ satisfies $\sum_{n \geq 1} (P)_n > 0$. Initially, we suppose that $(P)_1 = \mu > 0$ and $\sum_{n \geq 2} (P)_n = \nu > 0$, and if P does not satisfy one (or both) of these it will be apparent what modifications must be made. Denote the set of 1-cycles of P by $\{(f_j) \mid 1 \leq j\}$, and the set of cycles of length ≥ 2 by $\{(a_{i,1}a_{i,2}a_{i,3} \dots a_{i,l(i)}) \mid 1 \leq i\}$. Either set may be infinite, of course. For each set $\{a_{i,j}\}_{j=1}^{l(i)}$ we introduce a new set $\{b_{i,j}\}_{j=1}^{l(i)-1}$ and let S be the set of symbols

$$\begin{aligned} & \mathbb{Z} \cup \bigcup_{i \geq 1} \{a_{i,1}, a_{i,2}, a_{i,3}, \dots, a_{i,l(i)}\} \cup \bigcup_{j \geq 1} \{f_j\} \\ & \cup \bigcup_{i \geq 1} \{b_{i,1}, b_{i,2}, \dots, b_{i,l(i)-1}\} \end{aligned}$$

where the infinite cycle of P is now to be given by

$$(\dots, -2, -1, 0, 1, 2, 3, b_{1,1}, b_{1,2}, \dots, b_{1,l(1)-1}, 4, 5, 6, b_{2,1}, b_{2,2}, \dots, b_{2,l(2)-1}, \dots).$$

In general, the sequence $b_{i,1}, b_{i,2}, \dots, b_{i,l(i)-1}$ is placed between the integers $3i$ and $3i+1$, $1 \leq i$.

Modify the infinite cycle A of the preceding theorem by placing f_i between $-3i$ and $3i$, and by placing the alternating sequence $a_{i,1}, b_{i,1}, a_{i,2}, b_{i,2}, \dots, b_{i,l(i)-1}, a_{i,l(i)}$ between $-(3i+1)$ and $3i+1$.

$$\begin{aligned} A = & (\dots, 9, -6, f_1, 6, -3, f_2, 3, 0, \dots, -4, a_{1,1}, b_{1,1}, a_{1,2}, \dots, b_{1,l(1)-1}, \\ & a_{1,l(1)}, 4, -5, 5, -7, a_{2,1}, b_{2,1}, \dots, a_{2,l(2)}, 7, -8, 8, -11, \dots). \end{aligned}$$

Furthermore, let

$$B = (\dots, -8, 5, f_2, -6, \dots, 2, f_1, -3, \dots, 3, a_{1,1}, b_{1,1}, a_{1,2}, \dots, \\ b_{1,l(1)-1}, a_{1,l(1)}, -4, 6, a_{2,1}, b_{2,1}, \dots, a_{2,l(2)}, -7, 9, \dots),$$

which is now an obvious modification of the B in the preceding theorem. We see that $A, B \in \mathcal{C}_\infty$; and $P = AB$ needs only a straightforward verification.

COROLLARY 3.2.1. $\text{Alt}(S) \cup \mathcal{B} \cup \mathcal{D} \subseteq \mathcal{C}_\infty^3 \subseteq \mathcal{C}_\infty^4 \subseteq \dots$.

Proof. By Theorems 2.1, 3.1 and 3.2, $\text{Alt}(S) \cup \mathcal{B} \subseteq \mathcal{C}_\infty^2 \subseteq \mathcal{C}_\infty^3 \subseteq \dots$. To prove that $\mathcal{D} \subseteq \mathcal{C}_\infty^3$, again let $S = \mathbb{Z}$, and suppose $P \in \mathcal{D}$. Then there exists a $Q \in \text{Alt}(S)$ such that $P = Q \circ (0 \ 1)$. By Theorem 2.1, $Q = AB$, $A, B \in \mathcal{C}_\infty$. Hence $P = AB \circ (0 \ 1)$. If either $B^k(0) = 1$ or $B^k(1) = 0, k \geq 1$, then $B \circ (0 \ 1)$ consists of exactly one infinite cycle and one (disjoint) k -cycle. By Theorem 3.2, since $B \circ (0 \ 1)$ belongs to \mathcal{B} , $B \circ (0 \ 1) \in \mathcal{C}_\infty^2$. But then $P \in \mathcal{C}_\infty^3$.

4. Permutations with infinitely many finite, and no infinite cycles. Let $\mathcal{E} = \{P \mid (P)_2 = \infty, (P)_{n \geq 4} = 0\}$ and $\mathcal{F} = \{P \mid (P)_\infty = 0, \sum_{n \geq 1} (P)_n = \infty\}$. In Theorem 4.1 we prove that $\mathcal{F} \subseteq \mathcal{C}_\infty^3$. Of course $\mathcal{E} \subseteq \mathcal{F}$, so $\mathcal{E} \subseteq \mathcal{C}_\infty^3$, but in order to conclude that $\text{Sym}(S) \subseteq \mathcal{C}_\infty^3$, we will need the fact that $\mathcal{E} \subseteq \mathcal{C}_\infty^2$, and this is shown in Theorem 4.2.

THEOREM 4.1. $\mathcal{F} \subseteq \mathcal{C}_\infty^3$.

Proof. Let $P \in \mathcal{F}$ be expressed as $P = \prod_{i=-\infty}^{\infty} (p_{i,1} p_{i,2} \dots p_{i,l(i)})$, where S is now the collection $\{p_{i,j}\}$. Define the permutation Q by $Q(p_{i,1}) = p_{i+1,1}, -\infty < i < \infty$, and $Q(x) = x$ for $x \notin \{p_{i,1}\}_{i=-\infty}^{\infty}$. Then $Q = (\dots, p_{-2,1}, p_{-1,1}, p_{0,1}, p_{1,1}, p_{2,1}, \dots)$ is in \mathcal{B} , and thus $Q \in \mathcal{C}_\infty^2$. But $QP = (\dots, p_{-1,1}, p_{-1,2}, \dots, p_{-1,l(1)}, p_{0,1}, p_{0,2}, \dots, p_{0,l(0)}, \dots)$ is in \mathcal{C}_∞ , and hence $P \in \mathcal{C}_\infty^3$.

THEOREM 4.2. $\mathcal{E} \subseteq \mathcal{C}_\infty^2$.

Proof. Let $P \in \mathcal{E}$. S is here considered as a sequence of blocks of symbols. On each block, mappings R, S are defined, and when these restrictions are pieced together it will be seen that $P = RS$ and R, S are permutations in \mathcal{C}_∞ . Let $(b_1 b_2)(b_3 b_4)$ and $(b_{4i+1} b_{4i+2})(b_{4i+3} b_{4(i+1)})$, $i \geq 1$, denote the infinitely many pairs of transpositions in the disjoint cycle decomposition of P . We begin by defining R, S as follows: $S(b_1) = b_3, S(b_3) = b_2, S(b_2) = b_4, S(b_4) = b_1, S(b_5) = b_7, R(b_5) = b_2, R(b_2) = b_4, R(b_4) = b_1, R(b_7) = b_3, R(b_1) = b_6$. The most complicated case is when there are 1-cycles and 3-cycles in P ; for each $i \geq 1$ for which there

is a 1-cycle (f_i) and a 3-cycle ($c_{3i-2}c_{3i-1}c_{3i}$), the i th block of S is given by $b_{4i+1}, b_{4i+2}, f_i, b_{4i+3}, b_{4(i+1)}, c_{3i-2}, c_{3i-1}, c_{3i}$. With the exception of the first few b_j noted above, we define R and S on such a block as in Figure 1, where parentheses have been inserted to single out the cycles of P .

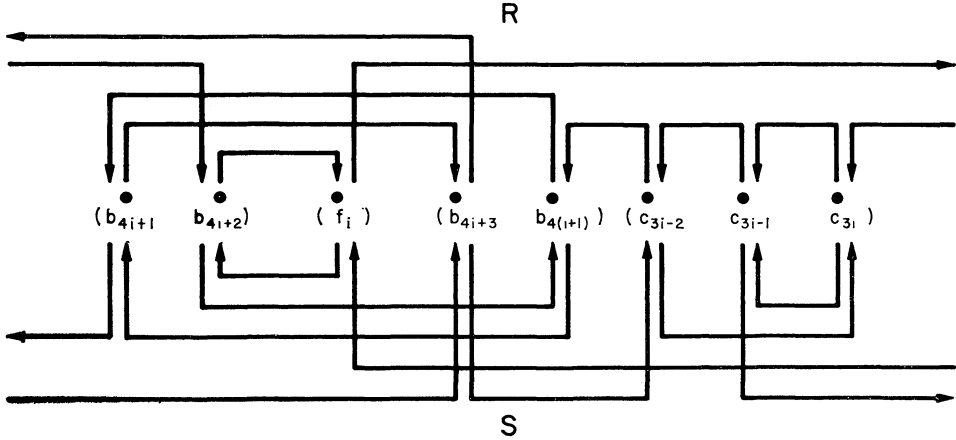


FIGURE 1.

All directed edges above the symbols refer to the mapping R (e.g., $R(b_{4i+1}) = b_{4i+3}$), whereas those below the symbols refer to S (e.g., $S(b_{4(i+1)}) = b_{4i+1}$). If either or both of $(P)_1, (P)_3$ is finite, there will eventually be blocks of symbols either without a 1-cycle, or 3-cycle, or both. In such cases, we simply delete the appropriate incoming and outgoing directed edges, and redefine R, S in the obvious way.

We may now patch the restrictions together in the various cases. The second choice is given for S and R in case there is either no i th 1-cycle, or no i th 3-cycle, or both.

$$\begin{aligned} S(b_{4(i+1)+1}) &= f_i, b_{4i+2} & R(b_{4(i+2)}) &= f_i, b_{4(i+1)+2} \\ S(b_{4i+3}) &= c_{3i-2}, b_{4(i+1)+3} & R(b_{4(i+1)+3}) &= c_{3i}, b_{4(i+1)} \\ S(c_{3i-1}) &= b_{4(i+1)+3} & R(f_i) &= b_{4(i+1)+2} . \end{aligned}$$

In each case $RS = P$ on all of S , and $R, S \in \mathcal{C}_\infty$. Thus $P \in \mathcal{C}_\infty^2$.

5. Permutations with more than one infinite cycle. Let \mathcal{A} denote the set of permutations P of S which satisfy $1 < (P)_\infty \leq \infty$. Then $\text{Sym}(S) = \text{Alt}(S) \cup \mathcal{D} \cup \mathcal{B} \cup \mathcal{F} \cup \mathcal{A}$, and we have shown that the first four sets in this union are subsets of \mathcal{C}_∞^3 . We prove that $\mathcal{A} \subseteq \mathcal{C}_\infty^3$, in two parts, and will thus have proved that $\text{Sym}(S) = \mathcal{C}_\infty^k$, for $k \geq 3$.

THEOREM 5.1. $\{P \in \text{Sym}(S) \mid 1 < (P)_\infty < \infty\} \subseteq \mathcal{C}_\infty^3$.

Proof. We divide the argument according to whether $(P)_\infty$ is odd or even, including more detail for the even case, since for $(P)_\infty$ odd the construction is very similar. Note that P may have cycles of finite length. Let $S = \{a_{i,j}\} \cup \{b_{r,s}\}$, where the infinite cycles of P are denoted by $(\dots, a_{-1,j}, a_{0,j}, a_{1,j}, \dots)$, $1 \leq j \leq m = (P)_\infty$ and the finite cycles by $(b_{r,1}, b_{r,2}, \dots, b_{r,l(r)})_{r \geq 1}$. For purposes of the construction, the $\{a_{i,j}\}$ are realized as the (integral) lattice points $\{(i, j)\}$ in the horizontal strip of the plane given by $-\infty < i < \infty$, $1 \leq j \leq m$; each $\{b_{r,s}\}_{s=1}^{l(r)}$ may be realized as any collection of $l(r)$ points on some r th horizontal line below the horizontal axis.

Let $m (= (P)_\infty)$ be even, and define the mapping Q on S as follows (see Figure 2a, where $m = 6$):

$$\begin{aligned} Q(a_{i,j}) &= \begin{cases} a_{i-1,j+1} & \text{if } i \equiv j \pmod{2} \text{ and } j < m \\ a_{i-1,j-1} & \text{if } i \not\equiv j \pmod{2} \text{ and } 1 < j \end{cases} \\ Q(a_{i,m}) &= a_{-i+1,m} \text{ if } i \equiv 0 \pmod{2} \\ Q(a_{2r,1}) &= b_{r,1} \text{ and } Q(b_{r,1}) = a_{-2r-1,1}, \text{ if } l(r) = 1. \end{aligned}$$

If $l(r) > 1$, put $Q(a_{2r,1}) = b_{r,l(r)}$, $Q(b_{r,j}) = b_{r,j-1}$ ($2 \leq j \leq l(r)$), $Q(b_{r,2}) = a_{-2r-1,1}$, $Q(a_{-2r,1}) = b_{r,1}$, and $Q(b_{r,1}) = a_{2r-1,1}$. If there is no r th [(finite)

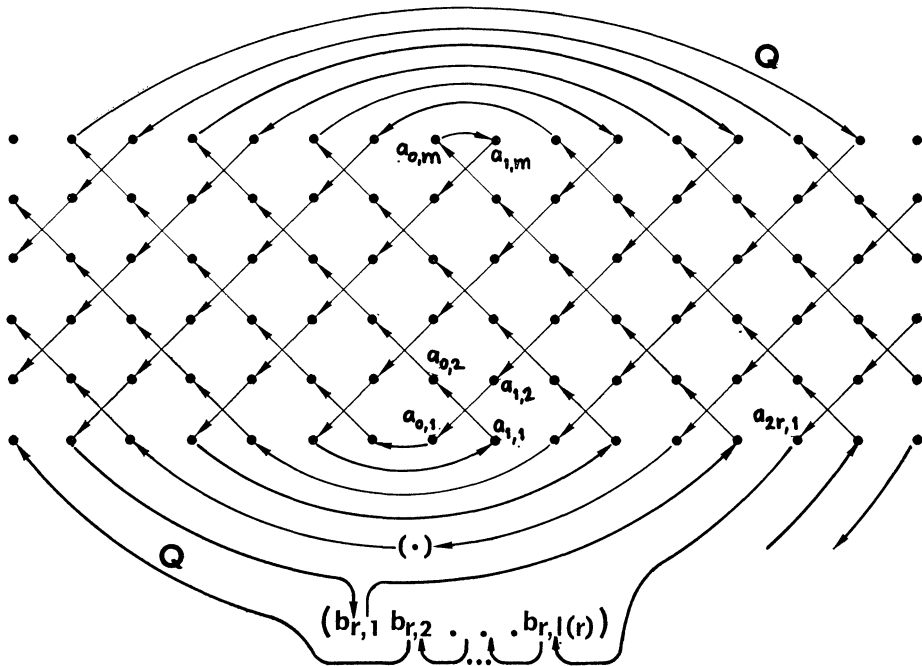


FIGURE 2a. $m \equiv 0 \pmod{2}$

cycle of P , put $Q(a_{2r,1}) = a_{-2r-1,1}$.

In all cases, Q is a permutation in \mathcal{C}_∞ . Furthermore, whenever the r th finite cycle of P is a 1-cycle, PQ contains the 3-cycle $(a_{2r,1}b_{r,1}a_{-2r,1})$. For each finite cycle of length $l(r) \geq 2$, PQ contains $l(r) - 2$ 1-cycles (e.g., PQ fixes $b_{r,j}$, $2 \leq j \leq l(r)$) and two 2-cycles (e.g., $(a_{2r,1}b_{r,1})$ and $(a_{-2r,1}b_{r,2})$). In any case, PQ contains the 2-cycles $(a_{i,j}a_{i,j+1})$ for $i \equiv j \pmod{2}$ and $1 \leq j < m$, and the 2-cycles $(a_{i,m}a_{-i+2,m})$ for $i \equiv 0 \pmod{2}$. Thus, no matter how many or how large the finite cycles of P , $PQ \in \mathcal{E}$. Since $\mathcal{E} \subseteq \mathcal{C}_\infty^2$ and $Q \in \mathcal{C}_\infty$, $P \in \mathcal{C}_\infty^3$.

If $(P)_\infty = m \equiv 1 \pmod{2}$, we may assume (by Theorems 3.1 and 3.2) that $m \geq 3$, and realize the infinite cycles of P as moving the appropriate lattice points, as in the even case. Define Q on these lattice points as follows (see Figure 2b, where $m = 5$):

$$Q(a_{i,j}) = \begin{cases} a_{i-1,j+1} & \text{if } i \equiv j \pmod{2} \text{ and } j < m \\ a_{i-1,j-1} & \text{if } i \not\equiv j \pmod{2} \text{ and } 1 < j \end{cases}$$

$$Q(a_{i,m}) = a_{3-i,m} \text{ if } i \equiv 1 \pmod{2}.$$

If there is no r th finite cycle of P , define $Q(a_{2r,1}) = a_{-2r+1,1}$. Q is defined on the elements $\{b_{r,1}, b_{r,2}, \dots, b_{r,l(r)}\}$ of the r th finite cycle, on $a_{2r,1}$ and $a_{-2(r-1),1}$ just as in the even case, where $a_{-2(r-1),1}$ now plays the role that $a_{-2r,1}$ did there. Again $Q \in \mathcal{C}_\infty$, PQ contains at most 1-cycles, 2-cycles, and 3-cycles, $PQ \in \mathcal{C}_\infty^2$, and $P \in \mathcal{C}_\infty^3$.

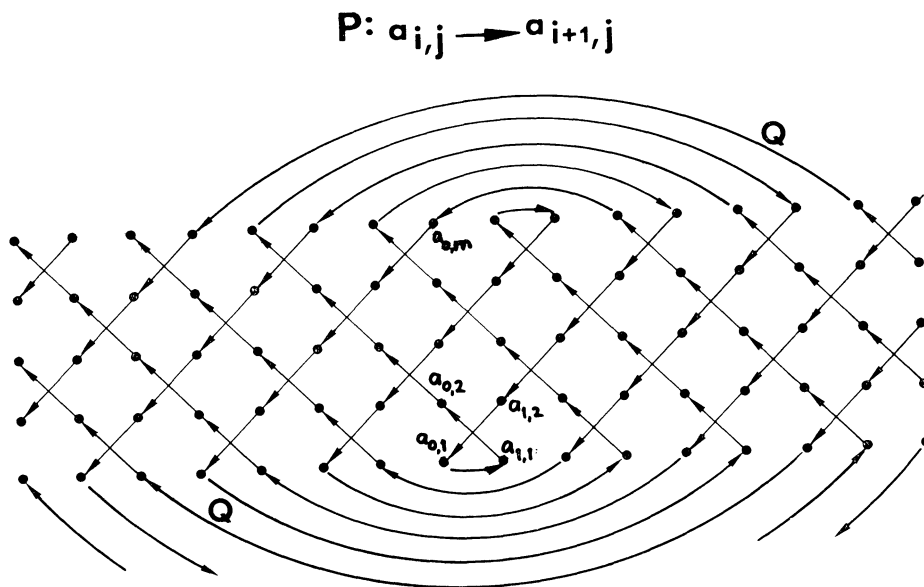


FIGURE 2b. $m \equiv 1 \pmod{2}$

THEOREM 5.2. $\{P \mid (P)_\infty = \infty\} \subseteq \mathcal{C}_\infty^3$.

Proof. Again let $S = \{a_{i,j}\} \cup \{b_{r,s}\}$, with the $a_{i,j}$ realized as the lattice points (i, j) in the upper half plane, $1 \leq j < \infty$, $-\infty < i < \infty$. The infinite cycles of P are $(\dots, a_{-1,j}, a_{0,j}, a_{1,j}, \dots)$ for even j and $(\dots, a_{1,j}, a_{0,j}, a_{-1,j}, \dots)$ if j is odd (see Figure 3). In case $1 \leq \sum_{n \geq 1} (P)_n$, the $b_{r,s}$ represent the symbols of the finite cycles of $P: \{(b_{r,1} b_{r,2} \dots b_{r,l(r)})\}_{r \geq 1}$, and are realized as in the previous theorem.

If $(P)_{n \geq 1} = 0$, define the mapping Q as in Figure 3; it is a straightforward verification that $Q \in \mathcal{C}_\infty$ and PQ contains the 3-cycle $(a_{-1,1} a_{0,1} a_{1,2})$, and otherwise only 2-cycles. Thus $PQ \in \mathcal{C} \subseteq \mathcal{C}_\infty^2$, and $P \in \mathcal{C}_\infty^3$. Otherwise, $1 \leq \sum_{n \geq 1} (P)_n$, and we redefine Q in nearly the same way as in the previous theorem, noting again that Q remains in \mathcal{C}_∞ : For $l(r) = 1$, put $Q(a_{2r,1}) = b_{r,1}$ and $Q(b_{r,1}) = a_{-2r,1}$. For $l(r) > 1$, put $Q(a_{2r,1}) = b_{r,l(r)}$, $Q(b_{r,j}) = b_{r,j-1}$ for $2 < j \leq l(r)$, $Q(b_{r,2}) = a_{-2r,1}$, $Q(a_{-2r-1,1}) = b_{r,1}$ and $Q(b_{r,1}) = a_{2r+1,1}$. For each occurrence of a 1-cycle of P , say $(b_{r,1})$, PQ contains the 3-cycle $(a_{2r,1} b_{r,1} a_{-2r-1,1})$. For each finite cycle of length $l \geq 2$, PQ contains $l - 2$ 1-cycles and one 2-cycle. Again PQ contains at most 1-cycles, 2-cycles, and 3-cycles, $PQ \in \mathcal{C}_\infty^2$ and $P \in \mathcal{C}_\infty^3$.

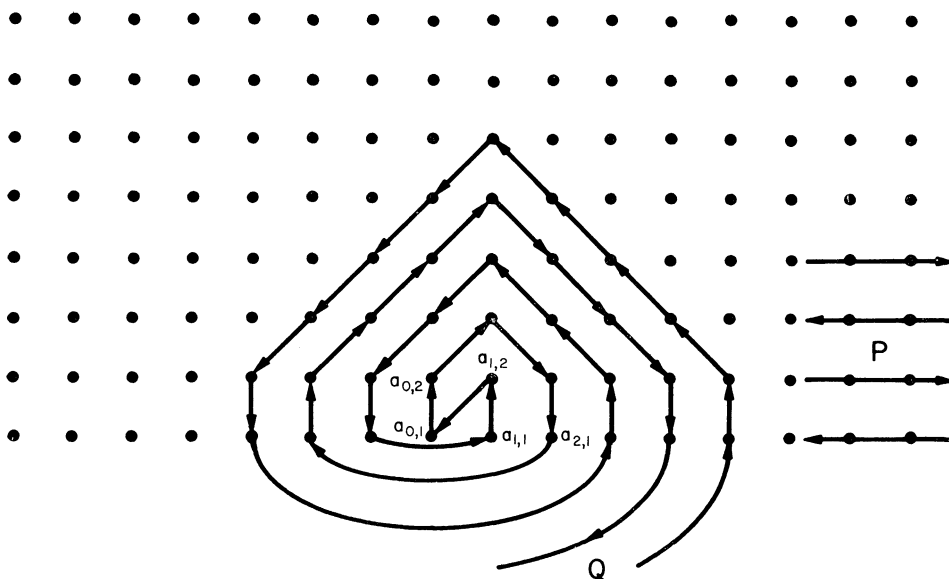


FIGURE 3

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