The object of this paper is to classify (up to inner automorphism) the primitive, maximal rank, reductive subalgebras of the (complex) exceptional Lie algebras. By primitive we mean that the subalgebras correspond to (possibly disconnected) maximal Lie subgroups.

In [3], the corresponding classification for the (complex) classical Lie algebras was completed, as was the classification of the non-reductive maximal rank subalgebras of all the simple Lie algebras.

Using case by case techniques and some more general results proved in §1, we prove the following theorem:

**Theorem 0.** The primitive maximal rank, reductive subalgebras of the exceptional (complex, simple) Lie algebras are listed (up to conjugacy by an inner automorphism) in the table below. Further, all subalgebras isomorphic to one of these are conjugate by an inner automorphism.

We note that Theorem 5.5 (p. 148) in the reductive case of Dynkin [2] is incorrect. In particular $A_3 \oplus D_5$, $A_6 \oplus A_2 \oplus A_1$, $A_7 \oplus A_1$ in $E_6$, $A_5 \oplus A_1$ in $E_7$ and $A_3 \oplus A_1$ in $F_4$ are not maximal subalgebras.

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Subalgebra</th>
<th>Reference in text</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_6$</td>
<td>$A_1 \oplus E_7$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$A_i$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$A_5 \oplus E_6$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$A_4^2$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$A_4^3$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$D_4^2$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$D_8$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$A_8$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$T^8$</td>
<td>Cartan subalgebra</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$A_1 \oplus D_6$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$A_1^2 \oplus D_4$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$A_1^2$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$A_5 \oplus D_6$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$A_5$</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>$A_5^2 \oplus T^1$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$E_6 \oplus T^1$</td>
<td>maximal</td>
</tr>
<tr>
<td></td>
<td>$T^7$</td>
<td>Cartan subalgebra</td>
</tr>
</tbody>
</table>
T^k denotes the center of the subalgebra, where k is the dimension of the center. The other superscripts refer to the number of summands of the corresponding algebra.

(See Table 12, p. 150 of [12]).

The authors wish to thank Robert Steinberg for several helpful remarks.

1. Preliminaries. We now present some basic notation and a characterization of primitivity from [3].

Let \( p \) be a maximal rank subalgebra of a simple Lie algebra \( g \). By maximal rank, we mean that there exists a Cartan subalgebra \( h \) of \( g \) which is contained in \( p \). We fix \( h \).

Let \( W \) be the Weyl group relative to \( h \). \( p \) is then decomposed by \( h \) into

\[
p = h \oplus \sum e_\varphi
\]

where the \( \varphi \)'s are roots in \( g \) determined by \( h \), and the \( e_\varphi \)'s are the corresponding one dimensional root spaces. \( p \) is then uniquely determined by the roots \( \varphi \) for which \( e_\varphi \subset p \). Let \( K_p = \{ \varphi \text{ a root of } g \text{ relative to } h \mid e_\varphi \subset p \} \). Define \( W_p = \{ \alpha \in W \mid \alpha(K_p) = K_p \} \).

**Proposition 1.1.** Let \( p \) be a maximal rank subalgebra of \( g \). Then \( p \) is primitive if and only if the following holds: If \( l \) is a subalgebra of \( g \) such that \( p \subset l \), and \( W_p \subset W_l \), then \( p = l \) or \( l = g \).

**Proof.** This is just Proposition 3.2 of [3].

We will use this as our definition of primitivity.

In our notation \( K_\varphi \) is the set of all roots in \( g \). We introduce an operation \([,]\) on \( K_\varphi \times K_\varphi \) which is induced from the Lie algebra structure of \( g \). Let \( \varphi, \psi \in K_\varphi \), then

<table>
<thead>
<tr>
<th>Algebra</th>
<th>Subalgebra</th>
<th>*</th>
<th>Reference in text</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_6 )</td>
<td>( A_1 \oplus A_4 )</td>
<td>maximal</td>
<td>§ 2 case 1, (a) (iii)</td>
</tr>
<tr>
<td>( A_2 \oplus T_2 )</td>
<td>maximal</td>
<td>§ 2 case 2, (b), (iii)</td>
<td></td>
</tr>
<tr>
<td>( A_1 \oplus C_5 )</td>
<td>maximal</td>
<td>§ 2 case 5, (iii)</td>
<td></td>
</tr>
<tr>
<td>( B_4 )</td>
<td>maximal</td>
<td>§ 2 case 6,</td>
<td></td>
</tr>
<tr>
<td>( G_2 )</td>
<td>( A_2 )</td>
<td>maximal</td>
<td>§ 3 case 1, (a)</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>maximal</td>
<td>§ 3 case 2</td>
<td></td>
</tr>
<tr>
<td>( B_4 )</td>
<td>maximal</td>
<td>§ 3 case 7</td>
<td></td>
</tr>
<tr>
<td>( D_4 )</td>
<td>algebra of longer roots</td>
<td>§ 3 case 8</td>
<td></td>
</tr>
<tr>
<td>( T^6 )</td>
<td>Cartan subalgebra</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We introduce an operation \([,]\) on \( K_\varphi \times K_\varphi \) which is induced from the Lie algebra structure of \( g \). Let \( \varphi, \psi \in K_\varphi \), then
\[
\{\varphi, \psi\} = \begin{cases} 
0 & \text{if } [e_\varphi, e_\psi] = 0 \\
\varphi + \psi & \text{if } [e_\varphi, e_\psi] = e_{\varphi + \psi}.
\end{cases}
\]

Note that \(\alpha[\varphi, \psi] = [\alpha(\varphi), \alpha(\psi)]\) for \(\alpha \in W\). Denote by \((\varphi, \psi)\) the standard inner product on \(h^*\), the dual space to \(h\), given by the Cartan-Killing form.

Now let \(p\) be reductive, i.e., \(p\) is given uniquely as the direct sum of simple algebras and its center. Thus there exists non-isomorphic simple Lie subalgebras of \(g\): \(X_1, X_2, \ldots, X_r\) such that \(p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \oplus T\) where \(T\) is the center of \(p\) and where \(X_i^{k_i}\) denotes the direct sum of all ideals of \(p\) isomorphic to \(X_i\), and \(k_i\) is the number of such ideals. Note that since \(X_i = (h \cap X_i) \oplus \sum_{\psi} e_\psi\), where the \(\varphi\) are unique, \(K_{X_i}\) and \(W_{X_i}\) make sense.

**Lemma 1.2.** Let \(q_i = X_i^{k_i}\). Then \(W_p \subset W_{q_i}\).

**Proof.** Let \(\alpha \in W_p\). Let \(f\) be an inner automorphism of \(g\) representing \(\alpha\). Then \(f(p) = p\). Also \(f(X_i)\) is an ideal of \(p\) isomorphic to \(X_i\). Thus \(f(X_i) \subset q_i\) since all ideals of \(p\) isomorphic to \(X_i\) are in \(q_i\). Hence \(f(q_i) = q_i\) and \(\alpha \in W_{q_i}\).

Let \(z\) be a subalgebra of \(g\) with Cartan subalgebra \(h_z \subset h\). Assume that \(z\) is regular in the sense of Dynkin [2]), i.e., let \(K_z = \{\varphi \in K_g \mid e_\varphi \subset z\}\), then \(z = h_z \oplus \sum_{\varphi \in K_z} e_\varphi\). Denote by \(K_z^\perp\) all of the roots in \(K_g\) orthogonal to the set \(K_z\). Let \(h_z^\perp\) be the subspace of \(h\) orthogonal to \(h_z\).

Let \(z^\perp = h_z^\perp \oplus \sum_{\varphi \in K_z^\perp} e_\varphi\). Note that if \(\varphi, \psi\) are roots of \(g\) such that \([\varphi, \psi] = 0\), then \((\varphi, \psi) = 0\). We then leave it to the reader to show that \(z^\perp\) is a subalgebra of \(g\), and \(K_{z^\perp} = K_z^\perp\).

**Theorem 1.4.** Let \(p\) be a maximal rank, reductive subalgebra. Let \(p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \oplus T\) (as described above). Let \(q_i = X_i^{k_i}\), and let \(Y = q_i^\perp\). If \(p\) is primitive, then either \(Y = q_1 \oplus \cdots \oplus q_r\) or the subalgebra, \(l\), generated by the vector subspace \(q_1 + Y + h\) is \(g\).

**Proof.** Since elements of the Weyl group act as isometries, \(W_{q_1} = W_Y\). Now \([q_1 \oplus \cdots \oplus q_r, q_1] = 0\), thus \((q_1 \oplus \cdots \oplus q_r, q_1) = 0\). Hence \(X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \leq Y\), and \(l \leq W_i\).

Now if \(\alpha \in W_p\) then (a) \(\alpha(K_{q_i}) \subset K_i\) and (b) \(\alpha(K_Y) \subset K_i\). (a) follows from Lemma 1.2 and (b) from the above remark that \(X_i \subset W_Y\). Now \(K_{q_1} \cup K_Y\) forms a set of generators for \(K_i\) (under \([,]\)). Thus \(\alpha(K_i) \subset K_i\) (since \(\alpha\) acts as a “homomorphism” relative to \([,]\)). So we have shown that \(W_p \subset W_i\). By Proposition 1.2 and the primi-
tivity of \( p \), either \( l = p \) or \( l = g \). If \( l = p \), then clearly \( Y = X^{k_2}_1 \oplus \cdots \oplus X^{k_r}_r \). Otherwise \( l = g \).

Let \( \varphi, \psi \) be roots in a simple Lie algebra where all of the roots have the same length. If \( \langle \varphi, \psi \rangle = 0 \), then \( [\varphi, \psi] = 0 \). Otherwise \( [\varphi, \psi] = \varphi + \psi \) and the length of \( \varphi + \psi \) is greater than the length of \( \varphi \) or \( \psi \) since \( \langle \varphi, \psi \rangle = 0 \). Hence the following.

**Corollary 1.5.** Let \( g \) be a simple Lie algebra all of whose roots have the same length (in particular \( E_6 \), \( E_7 \), and \( E_8 \)). Then if \( p \) is a primitive subalgebra, \( Y = X^{k_2}_1 \oplus \cdots \oplus X^{k_r}_r \).

**Proof.** In such an algebra \( X^{k_1}_1 + X + h = l \).

If \( l = g \), then \( g \) can be decomposed into a direct sum of two ideals, which contradicts the fact that \( g \) is simple.

As an immediate application of this corollary we get:

**Corollary 1.6.** Let \( g \) be a simple Lie algebra whose roots all have the same length. Let \( p \) be a subalgebra of \( g \). If \( K_p \neq \emptyset \), then \( p \) is not primitive.

**Note:** This corollary is true even if the roots of \( g \) do not have the same length.

2. \( E_6 \), \( E_7 \) and \( E_8 \). We first describe the roots of the algebras \( E_6 \), \( E_7 \) and \( E_8 \) (see [1]). Let \( z_1, z_2, \ldots, z_8 \) be the standard orthonormal basis for the dual space to a fixed Cartan subalgebra of \( E_8 \). With respect to this basis the roots of \( E_8 \) are given by

\[
(I)_8 = \{ \pm z_i \pm z_j \mid 1 \leq i < j \leq 8 \}
\]

and

\[
(II)_8 = \{ \pm 1/2(z_1 \pm z_2 \pm \cdots \pm z_8) \mid \text{the number of minus signs is even.} \}
\]

We shall refer to these as type I roots and type II roots respectively. The roots of type I will be denoted by \( \pm i \pm j \) when no confusion arises; e.g., \( -z_2 + z_3 \) is denoted by \( -2 + 3 \), etc. The roots of Type II will be denoted by the corresponding sequences of signs, e.g.,

\[
1/2(z_1 + z_2 - z_3 + z_4 - z_5 - z_6 - z_7 + z_8) = (+ + - + - - - +).
\]

We take \( E_6 \) and \( E_7 \) to be regular subalgebras of \( E_8 \). The roots of \( E_7 \) are all of those roots of \( E_8 \) orthogonal to \( 7 + 8 \). Thus

\[
(I)_7 = (I)_8 \cap K_{E_7} = \{ \pm i \pm j \mid 1 \leq i < j \leq 6 \} \cup \{7 - 8\}
\]

\[
(II)_7 = (II)_8 \cap K_{E_7} = \{ \pm (**** - +) \mid \text{an odd number of the * are - and the others are +} \}.
\]
The roots of $E_6 \subseteq E_7$ are as follows:

$$(I)_6 = (I)_8 \cap K_{E_6} = \{\pm i \pm j \mid 1 \leq j < j \leq 5\}$$

$$(II)_6 = (II)_8 \cap K_{E_6} = \{\pm(* * * * * + -) \mid \text{and odd number of the * are } - \text{ and the others are +}\}.$$

When no confusion arises, we shall write merely $(I)$ or $(II)$. Also, we note that $(I)_8$ generates a maximal rank subalgebra, which we also denote by $(I)_8$ (or sometimes by $(I)$). The algebra $(I)_8$ is maximal in $E_8$.

We note that all roots of $E_8$ have the same length, $\sqrt{2}$. Thus the Weyl group acts transitively in each of these algebras. As observed in Corollary 3.3. of [3], the Cartan subalgebras in $E_8$ are primitive.

In order to classify the primitive, reductive, maximal rank subalgebras we need some information about their Weyl groups. We note first that the reflections about roots of type $(I)$ are just determined by signed permutations of $z_1, \cdots, z_8$. For if $\alpha$ and $\beta$ are roots, then

$$S_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha = \beta - (\alpha, \beta)\alpha, \text{ as } (\alpha, \alpha) = 2.$$

If $\alpha = i - j$, then $S_\alpha(z_k) = \begin{cases} z_i & \text{if } k = j \\
 \beta & \text{if } k = i \\
 z_k \text{ otherwise} \end{cases}$. 

If $\alpha = i + j$, then $S_\alpha(z_k) = \begin{cases} -z_i & \text{if } k = j \\
 \beta & \text{if } k = i \\
 z_k \text{ otherwise} \end{cases}$.

We can thus, for example, identify $S_\alpha$ with the transposition $(1 2 3 \cdots 8)$ if $\alpha = 1 - 2$, and with $(-2 -1 3 \cdots 8)$ if $\alpha = 1 + 2$. These reflections, then, generate signed permutations with an even number of sign changes.

**Lemma 2.1.** (a) Any two sets of three mutually orthogonal roots in $E_8$ are conjugate by an inner automorphism (i.e., by a Weyl group element).

(b) Any set of mutually orthogonal roots is conjugate by a Weyl group element to a set of roots (mutually orthogonal) of type $(I)$.

**Proof.** Let $\alpha_1, \alpha_2, \alpha_3$ be mutually orthogonal roots in $E_8$. Since the Weyl group acts transitively on the roots, we may assume $\alpha_1 = 1 - 2$.

Suppose $\alpha_2$ is a root of type $II$. Since $(\alpha_2, \alpha_1) = 0$, $\alpha_2 = \pm (+ + * * * * * *)$, say $ (+ + * * * * * *)$. Let $\beta = (- - * * * * * *)$,
where \( \beta \) is taken to agree with \( \alpha \) in the last six signs. Then \( S_\beta(1 - 2) = 1 - 2, S_\beta(\alpha) = 1 + 2 \). Since no root of type II is orthogonal to both of these, (b) holds. In particular, \( S_\beta(\alpha_3) \) is of type I, say \( i + j, i, j \neq 1 \) or 2. Then there is some signed permutation \( f \) fixing 1 and 2, and taking \( i + j \) into 3 - 4. The permutation can be taken from \( W_{E_6} \) or \( W_{E_7} \) if \( i - j \) is in \( K_{E_6} \) or \( K_{E_7} \), respectively. Thus \( fS_\beta \) takes \( \alpha_1, \alpha_2, \alpha_3 \) into \( 1 \pm 2, 3 - 4 \), and (a) holds.

On the other hand, if \( \alpha_2 \) is a root of type I, and if \( \alpha_3 \neq 1 + 2 \), then we can find a signed permutation \( \gamma \), as above, such that \( f(\alpha_3) = 3 - 4 \). Then let \( \beta = (+ + + - + + + -) \). \( S_\beta(1 - 2) = 1 - 2, S_\beta(3 - 4) = (- - + - - - +) \). Thus we have reduced the problem to the case where \( \alpha_3 \) is of type II, and the lemma is proved.

Let \( N_s \) be the subgroup of the Weyl group of \( E_s \) consisting of the signed permutations. We have the following normal forms for elements of the Weyl group:

**Theorem 2.2.** Let \( g \) be a Weyl group element. Then one of the following three cases holds:

(a) \( g \in N_s \).
(b) \( \exists \gamma \in N_s, \alpha \in (II), \exists g = fS_\gamma \).
(c) \( \exists \gamma \in N_s, \alpha, \beta \in (II), \exists (\alpha, \beta) = 0 \) and \( g = fS_\beta S_\alpha \).

**Proof.** Any Weyl group element \( g \) has the form \( g = S_{\alpha_n} \cdots S_{\alpha_1} \), where the \( \alpha_i \) are roots of the algebra. Recalling that \( S_{\alpha}S_{\beta} = S_{\beta}S_{S_\beta(\alpha)} \), we may move any \( S_{\alpha_i} \), where \( \alpha_i \) is of type I, to the left past all roots of type II. Thus we may assume \( g = S_{\gamma_1} \cdots S_{\gamma_k} S_{\gamma_k} \cdots S_{\gamma_1} (k + l = n) \), where the \( \beta_i \) are of type I, and the \( \gamma_j \) of type II. Let this be a representation of \( g \) so that \( l \) is minimal. Then all the \( \gamma_j \) are mutually orthogonal. For if \( (\alpha, \beta) = 0 \), then \( S_{\alpha}S_{\beta} = S_{\beta}S_{\alpha} \), and thus if there are any \( \gamma_i \) and \( \gamma_j \) not orthogonal, we may assume that we have \( (\gamma_i, \gamma_j) = \pm 1 \) or \( \pm 2 \). If \( (\gamma_i, \gamma_j) = \pm 2 \), then \( \gamma_i = \pm \alpha \), and \( S_{\gamma_i}S_{\gamma_j} = 1 \), the identity. If \( (\gamma_i, \gamma_j) = \pm 1 \), then \( S_{\gamma_i}S_{\gamma_j} = S_{\gamma_j}S_{\gamma_i} \), where \( S_{\gamma_j} \) is a root of type I. In either case, we can decrease \( l \), a contradiction. Thus we may write \( g = fS_{\gamma_1} \cdots S_{\gamma_l} \) where all the \( \gamma_j \) are mutually orthogonal type II roots, and \( f \) is a signed permutation.

Two roots of type II are orthogonal if and only if they agree in four signs, and disagree in four signs, e.g., \( (+ + + + + + + +) \) and \( (+ + + - - - - -) \). Thus in \( E_6 \) it is easy to see that there can be at most two mutually orthogonal roots of type II. Thus, for \( E_6 \) the theorem is proved.

Suppose that for \( E_7 \) or \( E_8 \) we have \( \gamma_1, \gamma_2, \gamma_3 \) mutually orthogonal. Then for an appropriate signed permutation \( h \) we have
Thus $S_{r_1}S_{r_2}S_{r_3} = h^{-1}S_{r_1}h^{-1}S_{r_2}h^{-1}S_{r_3}h = h^{-1}S_{h(r_1)}S_{h(r_2)}S_{h(r_3)}h$. Let $\beta = -++---$. Then $S_{h(r_1)}S_{h(r_2)}S_{h(r_3)}S_{3} = (-1)\cdot(12)(34)(56)(78) = a \in N_i$. Thus $S_{r_1}S_{r_2}S_{r_3} = h^{-1}S_{h(r_1)}S_{h(r_2)}S_{h(r_3)}S_{3}S_{h(r_3)} = h^{-1}S_{3}S_{h(r_3)}h$. Thus $g = fh^{-1}aS_{3}S_{h(r_3)}S_{r_1} \cdots S_{r_3}$. By the arguments above, this reduces $l$ by two, a contradiction. Hence $l \leq 2$. This proves the theorem.

We now proceed with the classification of the primitive, maximal rank, reductive subalgebras, $p$, of $E_n$.

The general plan of attack in classifying the possible algebras $p$ will be to assume that $X_1$ is some particular algebra, say $A_3$ or $D_4$, etc., and then to conjugate $X_1$ (by Weyl group elements) into a form suitable for deciding whether or not the various maximal rank, reductive subalgebras with $X_1$ as an ideal are primitive. Theorem 2.1 is used for the latter. The number of cases to consider is kept small this way since there are only a few choices for $X_1$, and, of course, the rank of $X_1$ is limited by 8. In particular, $X_1$ cannot be $B_n, C_n, G_2$ or $F_4$ since the roots of $E_n$ have only one length. Because the arguments are essentially the same, we treat $E_6$, $E_7$ and $E_8$ together.

We consider the following cases for $X_1$:

$$A_1, A_2, A_3, A_4D_4(l > 4, A_4(l > 4))E_6, E_7, E_8.$$ 

Case 1. $X_1 = A_i$.

(a) $k_i = 1$. (See §1 for the definition of $k_i$).

By Corollary 1.5 and the primitivity of $p$, $p = A_i \oplus Y$, where $Y = A_i^\perp$.

(i) In $E_6$ let $K_{A_i} = \{\pm(7+8)\}$. This can be done without loss of generality since the Weyl group acts transitively. Then $A_i^\perp = E_7$. As is easily checked, $A_i \oplus E_7$ is maximal, and hence primitive.

(ii) In $E_7$, let $K_{A_i} = \{\pm(7-8)\}$. Since the roots of $Y$ must be orthogonal both to $7-8$ ($A_i$) and to $7+8$ (since they are roots of $E_7$), $Y$ must have only roots of type $I$. Then

$K_Y = \{\pm i \pm j | 1 \leq i < j \leq 6\}$,

or $Y = D_6$. $A_i \oplus D_6$ is the algebra $(I)_7$, which is maximal, and thus primitive.

(iii) In $E_8$, let $K_{A_i} = \{\pm(1-2)\}$. Then $Y = A_i^\perp$, and $K_Y = \{\pm(1+2), \pm i \pm j, 3 \leq i < j \leq 5\}$, and all type $II$ roots of the form $\pm(+\cdots+-)$ and $\pm(+\cdots-\cdots+)$. We see, then, that $Y = A_3$. In this case $p = A_i \oplus A_3$ is maximal,
and thus primitive.

(b) \( k_i > 1 \)

Here we let \( p = A^{k_i} \oplus Y \).

(i) In \( E_8 \), since each \( A_i \) summand has a root orthogonal to the others, we can assume that for the first two summands \( A_i \oplus A_i = A^i, K_{A^i} = \{ \pm 1 \pm 2 \} \), by Lemma 2.1. Let \( \bar{Y} = A^{k_i-2} \oplus Y \). Then \( \bar{Y} \subseteq (A^i) \), and thus \( K_{\bar{Y}} \subseteq (I) \).

If \( k_i = 2 \), then \( K_Y = K_{(A^i)^{-1}} = D_6 \).

Here we digress to prove a lemma about \( D_i \) factors of reductive subalgebras.

**Lemma 2.3.** \( D_i \subseteq E_8 \) is always conjugate to the algebra with roots \( \{ \pm i \pm j \mid 1 \leq i < j \leq l \} \). If \( D_i \) has only type I roots, and if \( l > 4 \), then \( W_{D_i} \subseteq W_{(I)} \).

**Proof.** Let \( D_i \subseteq E_8 \). Let \( h \) be the Cartan subalgebra of \( E_8 \). Let \( \tilde{h} \) be the Cartan subalgebra of \( D_i \). Note that \( \dim \tilde{h} = l \). We can choose an orthonormal basis \( w_1, w_2, \ldots, w_l \) of \( \tilde{h} \) so that the roots of \( D_i \) are just \( \pm w_i \pm w_j \), \( 1 \leq i < j \leq l \).

Now \( w_i \pm w_j \) are orthogonal. Thus by Lemma 2.1 we can assume \( \{ \pm w_i \pm w_j \} = \{ \pm 1 \pm 2 \} \) in \( E_8 \). \( w_2 - w_3 \) is not orthogonal to \( w_1 + w_2 \) nor to \( w_1 - w_2 \). But any type II root is orthogonal either to \( 1 - 2 \) or to \( 1 + 2 \). Thus \( w_2 - w_3 \) is of type I. By use of a signed permutation fixing the set \( \{ \pm 1 \pm 2 \} \), we can take \( w_2 - w_3 \) to be \( 2 - 3 \). Clearly \( w_2 + w_3 \) then becomes \( 2 + 3 \) under this same signed permutation. Thus we may assume \( \pm 1 \pm 2, \pm 2 \pm 3 \) and therefore \( \pm 1 \pm 3 \) are roots of \( D_i \). Continuing in this way, we can assume \( \pm i \pm j, 1 \leq i < j \leq l \), are roots of \( D_i \). These are all the roots of \( D_i \), and hence the first part of the lemma is proved.

Now in fact what we saw was that if \( D_i \) has only type I roots, then up to conjugacy by a signed permutation the roots of \( D_i \) are \( \pm i \pm j, 1 \leq i < j \leq l \).

Thus it is sufficient for establishing the last part of the lemma to assume that the roots of \( D_i \) are \( \pm i \pm j, 1 \leq i < j \leq l \).

Let \( g \in W_{D_i} \). Suppose, contrary to the assertion, that \( g \) is not a signed permutation. Then by Theorem 2.1, either \( g = fS_{\alpha} \) or \( g = fS_{\beta}S_{\alpha} \), \( f \) a signed permutation, and \( \alpha, \beta \) roots of type II with \( (\alpha, \beta) = 0 \). In the first case, either \( S_{\alpha}(1 + 2) \) or \( S_{\alpha}(1 - 2) \) is of type II, and thus \( fS_{\alpha}(1 + 2) \) or \( fS_{\alpha}(1 - 2) \) is of type II, and hence not in \( K_{D_i} \), contradicting the assumption that \( g \in W_{D_i} \).

For the second case, let \( g = fS_{\alpha}S_{\beta} \), \( (\alpha, \beta) = 0 \), and let \( i \pm j \in K_{D_i} \).

As above, \( g(i \pm j) \in K_{D_i} \subseteq (I) \). Thus \( S_{\alpha}S_{\beta}(i \pm j) \subseteq (I) \). Now if \( \alpha \) and \( \beta \) disagree in sign in one of the \( i \) or \( j \) positions, and agree in
the other, then $S_\alpha S_\beta (i \pm j)$ are both in (II). (We see this by observing $S_\alpha S_\beta = S_\beta S_\alpha$, and if $(\gamma, \alpha) = 0$, then $S_\alpha (\gamma) = \gamma$. $i - j$ is orthogonal to one of $\alpha$ and $\beta$, and $i + j$ to the other). Thus $\alpha$ and $\beta$ must agree in both the $i$ and $j$ positions, or both disagree. Changing $i$ and $j$, we see that $\alpha$ and $\beta$ must totally agree or totally disagree on positions $1, 2, \ldots, l$. If $l > 4$, this contradicts $(\alpha, \beta) = 0$. This completes the proof of the lemma.

We now return to the case at hand, namely $A^\circ \oplus D_4$ in $E_8$. By Lemma 1.2 we have $W_\gamma \subseteq W_{D_8}$ and by Lemma 2.3, $W_{D_6} \subseteq W_{(1)}$. But $A^\circ \oplus D_6 \subseteq (I)$. Hence $A^\circ \oplus D_6$ is not primitive, by Prop. 1.1.

This means that we may assume $k_1 > 2$. By Lemma 2.1 we may assume that $K_{A^\circ} = \{\pm (1 \pm 2), \pm (3 - 4)\}$, the roots for the first three factors $A_1$. Now $3 + 4$ is in $K_{A^\circ}$. If $3 + 4 \in K_\gamma$, then $\gamma$ is not primitive, by Corollary 1.6. If $3 + 4 \in K_\gamma$, then since $Y$ has no summands of type $A_1$ in it, $K_\gamma \subseteq (I)$ contains another root $\alpha$ such that $[3 + 4, \alpha] \neq 0$. But then either $[-3 + 4, \alpha] = 0$ or $[3 - 4, \alpha] = 0$, contradicting the fact that $\pm (3 - 4)$ are the roots for a direct summand $A_1$ of $\gamma$.

This all implies that $k_1$ can't just be 3. In fact, the same kind of reasoning implies that $k_1$ must be even if $k_1 > 1$. $k_1 = 6$ is not possible, since, by Lemma 2.1, we can assume $K_{A^\circ}$ consists of six of the following eight roots, together with their negatives: $1 \pm 2, 3 \pm 4, 5 \pm 6, 7 \pm 8$. But then $K_{A^\circ}$ consists of the other two, and we have two more summands of $A_1$. Thus $k_1 = 4$ and $k_1 = 8$ are the only possibilities. By Corollary 1.5, then, the only remaining possibilities are $A^\circ \oplus D_1$ and $A^\circ$, where $K_{A^\circ} = \{\pm (1 \pm 2), \pm (3 \pm 4), \pm i \pm j, 5 \leq i < j \leq 8\}$ and $K_{A^\circ} = \{\pm (1 \pm 2), \pm (3 \pm 4), \pm (5 \pm 6) \pm (7 \pm 8)\}$.

We first show that $A^\circ \oplus D_1$ is not primitive by showing $W_{A^\circ \oplus D_1} = W_{D_8 \oplus D_1}$, where $K_{D_8 \oplus D_1} = \{\pm i \pm j | i, j \leq 4$ or $i, j \geq 5\}$. Let $g \in W_{A^\circ \oplus D_1}$. If $g$ is a signed permutation, then $g \in W_{D_8 \oplus D_1}$. If $g = f S_\alpha$, $f$ a signed permutation, $\alpha \in (II)$, then $f S_\alpha$ must take one of $1 + 2$ and $1 - 2$ into (II), contradicting $g \in W_{A^\circ \oplus D_1}$.

If $g = f S_\beta S_\alpha$, $(\alpha, \beta) = 0$ then, just as in the proof of Lemma 2.3, $\alpha$ and $\beta$ must totally agree or totally disagree in sign in positions 5, 6, 7, 8, say they agree. Then since $(\alpha, \beta) = 0$, they must totally disagree in positions 1, 2, 3, 4. Thus $S_\beta S_\alpha$ are in $W_{D_8 \oplus D_1}$ (by computation). Since $g \in W_{A^\circ \oplus D_1}$, $f = g S_\beta S_\alpha$ is in $W_{A_1 \oplus D_1}$, and thus in $W_{D_8 \oplus D_1}$ by the observations above. Hence $g \in W_{D_8 \oplus D_1}$. This completes the proof that $W_{A_1 \oplus D_1} \subseteq W_{D_8 \oplus D_1}$. Hence $A^\circ \oplus D_1$ is not primitive.

$A^\circ$, the last case, is primitive. We show this by noting that the following elements are in $W_{A^\circ}$: (c) (13)(24), (d) (35)(46), (e) (57)(68), (f) arbitrary sign changes (even number), and (g) $S_\beta S_\alpha$, where $\alpha =
Let $q$ be an algebra strictly containing $A_1^8$, and invariant under $W_{A_1^8}$. If $K_q$ contains a root of (I) not in $K_\alpha$, then using (c), (d) and (e) above we see that $K_q$ must contain all of (I). But $S_\beta S_\alpha(4 + 5)$ is in (II), and thus $K_q$ also has a root of type II. But then $K_q$ has all roots of type II since (I) is a maximal subalgebra, and $q = E_8$. If $K_q$ contains a root of type I, say $\gamma = \alpha$, then we can assume $\alpha = \beta$, using (f). Similarly, using (f) we see that $(- + + + + + + +) \in K_q$. But then $S_\beta S_\alpha(- + + + + + +) = -2 - 3 \in K_q$, and $q = E_8$ by the case above, as this is a new root of type I. Thus the only algebra larger than $A_1^8$ invariant under $W_{A_1^8} \otimes D_4$ is $E_8$, and $A_1^8$ is primitive. This completes the case $X_3 = A_1$ for $E_8$. (ii) In $E_7$ we may assume that the first copy of $A_1$ is given by $K_{A_1} = \{ \pm(7 - 8) \}$, and the second by $\{ \pm(1 - 2) \}$, using Lemma 2.1. Just as $k_1$ had to be even in the $E_8$ case above, the same argument shows that $k_1$ must be odd for $E_7$, and thus $k_1 = 3, 5$ or 7, as $k_3 > 1$. But if $k_1 = 5$, then $K_{A_1^4} = \{ \pm 1 \pm 2, \pm 3 \pm 4, \pm(7 - 8) \}$ up to signed permutations, and thus $K_{A_1^4} = \{ \pm 5 \pm 6 \}$, and $(A_1^4) = A_2^4$, contradicting $k_1 = 5$. Hence this case is impossible. The only remaining possibilities are $A_1^6$ and $A_1^7$, by rank considerations and Corollary 1.5.

$$K_{A_1^6 \otimes D_4} = \{ \pm 1 \pm 2, \pm i \pm j, 3 \leq i < j \leq 6, \pm(7 - 8) \}.$$ We show that $A_1^6 \otimes D_4$ is primitive. Let $\alpha = (+ - + + + + +)$, $\beta = (- + + + + + +)$. Then $S_\beta S_\alpha$ is in $W_{A_1^6 \otimes D_4}$. Let $q$ be a subalgebra of $E_7$ properly containing $A_1^6 \otimes D_4$ and invariant under $W_{A_1^6 \otimes D_4}$. Suppose $K_q$ contains a root of type I not in $K_{A_1^6 \otimes D_4}$. Then $K_q$ contains all of (I), using the algebra multiplication $[,]$ of $q$. Now $S_\beta S_\alpha(2 - 3) \in K_q$ a root of type II. Thus, since (I) is a maximal subalgebra of $E_7$, $q = E_7$. On the other hand, if $K_q$ contains a root, $\gamma$, of type (II), then, since $f = (1 - 2 - 3 4 \cdots 8)$ is in $W_{A_1^6 \otimes D_4}$, $f(\gamma)$ must be in $K_q$. Thus $[\gamma, f(\gamma)] \in K_q$. But this is one of the four roots $\pm 2 \pm 3$ which is not in $K_{A_1^6 \otimes D_4}$. Thus $q = E_7$ by the previous argument, and $A_1^6 \otimes D_4$ is primitive. 

Now consider $A_1^7$. We claim that this is primitive also,

$$K_{A_1^7} = \{ \pm 1 \pm 2, \pm 3 \pm 4, \pm 5 \pm 6, \pm(7 - 8) \}.$$ Let $q$ be a subalgebra properly containing $A_1^7$ and invariant under $W_{A_1^7}$, and in particular under (c) (13)(24), (d) (35)(46), (e) $(1 - 2 - 3 4 \cdots 8) = f$ and $S_\beta S_\alpha$, where $\alpha = (+ - + + + + + +)$ and $\beta = (- + + + + + + +)$. If $K_q$ contains a root of type I not in $K_{A_1^7}$, then using (c), (d) and the algebra multiplication we see $K_q$ contains all of (I). Further, $K_q$ contains $S_\beta S_\alpha(2 - 3)$, which is a root of type II. Thus, since (I),
is maximal, \( q = E_7 \). On the other hand, if \( K_q \) has a root \( \gamma \), of type \( II \), then \( f(\gamma) \in K_q \) and \( [\gamma, f(\gamma)] \in K_q \) is a root of type \( I \) not in \( K_{A_i^4}^+ \). Then the previous argument yields \( q = E_7 \). Thus \( A_i^4 \) is primitive.

**Note.** As can be seen in the last few arguments, the crucial thing in demonstrating that a given subalgebra \( p \) is primitive is knowing which elements of \( W_p \) to use in showing that larger algebra invariant under \( W_p \) is \( E_7 \). The arguments thereafter are just like the ones above. Thus, from now on, rather than show in detail why a given algebra is primitive, we shall simply indicate which elements of \( W_p \) one uses.

(iii) In \( E_6 \), we can assume \( K_{A_i^4}^+ = \{ \pm 1 \pm 2 \} \), by Corollary 1.5. Then \( K_{A_i^4}^+ \subseteq (I)_6 \). In this case, as in \( E_6 \), we must have \( k_i \) even, by the same arguments. \( k_i = 6 \) is impossible since \( (I)_6 \) doesn't contain \( K_{A_i^4}^+ \). Thus only \( A_i^4 \oplus A_3 \) and \( A_i^4 \) remain. We claim that neither of these is primitive.

\[ K_{A_i^4 \oplus A_3} = \{ \pm 1 \pm 2, \pm 3 \pm 4, \pm 4 \pm 5, \pm 3 \pm 5 \} \]

We show that \( W_{A_i^4 \oplus A_3} \subseteq W_{(I)_6} \). Let \( g \in W_{A_i^4 \oplus A_3} \). \( g \) cannot be of the form \( fS_\alpha, f \) a signed permutation and \( \alpha \in (II)_6 \), for either \( fS_\alpha(1 - 2) \) or \( fS_\alpha(1 + 2) \) would be in \( (II)_6 \) and thus not in \( K_{A_i^4 \oplus A_3} \). Suppose \( g = fS_\beta S_\alpha, (\alpha, \beta) = 0, f \) a signed permutation, \( \alpha, \beta \in (II)_6 \). As we saw in the proof of Lemma 2.3 (since \( A_3 = D_4 \), \( \beta \) and \( \alpha \) must agree in each of positions 3, 4 and 5, or disagree in these three position. Similarly they must agree or disagree in both positions 1 and 2. Since they are both in \( (II)_6 \), they must totally agree or totally disagree in positions 6, 7, 8. It is impossible, then, for \( \alpha \) and \( \beta \) to agree in exactly four positions, contradicting \( (\alpha, \beta) = 0 \). Thus \( g \) must be in \( W_{(I)_6} \) and \( W_{A_i^4 \oplus A_3} \subseteq W_{(I)_6} \). So \( A_i^4 \oplus A_3 \) is not primitive.

We now consider the last case, \( A_i^4 \). \( K_{A_i^4} = \{ \pm 1 \pm 2, \pm 3 \pm 4 \} \). Here we show that \( W_{A_i^4} \subseteq W_{D_4} \), where \( K_{D_4} = \{ \pm i \pm j \mid 1 \leq i < j \leq 4 \} \), and thus that \( A_i^4 \) is not primitive. Let \( g \in W_{A_i^4} \). As above, \( g = fS_\alpha \) is not possible, as \( fS_\alpha(K_{A_i^4}) \subseteq (I)_6 \). If \( g = fS_\beta S_\alpha \), as above, \( \alpha \) and \( \beta \) must agree in both positions 1 and 2 or disagree in both, and similarly for positions 3 and 4. Also, as \( \alpha, \beta \in (II)_6 \), they totally agree or totally disagree in positions 6, 7 and 8. Since \( (\alpha, \beta) = 0 \), \( \alpha \) and \( \beta \) must agree in exactly four positions, and these must be 1, 2, 3 and 4 or 5, 6, 7 and 8. In either case, \( S_\alpha S_\beta \in W_{A_i^4} \) and \( S_\alpha S_\beta \in W_{D_4} \) (by computation). Then \( f = gS_\beta S_\alpha \in W_{A_i^4} \). But a signed permutation in \( W_{A_i^4} \) is clearly also in \( W_{D_4} \). Thus \( f \in W_{D_4} \) and \( g = fS_\beta S_\alpha \in W_{D_4} \), and \( A_i^4 \) is not primitive.

Before beginning Case 2, we prove a lemma about subalgebras

\[ A_i \subseteq E_i \]
Lemma 2.4. Let $A_i \subset E_s$.

(a) If $l \leq 6$ for $s = 8$, $l \leq 4$ for $s = 7$, or $l \leq 3$ for $s = 6$, then all $A_i$ are conjugate, and in particular conjugate to $A_1$ with $K_{A_1} = \{ \pm (i-j) | 1 \leq i < j \leq l + 1 \}$.

(b) If $l = 7$ for $E_8$, $l = 5$ for $E_7$, or $l = 4$ for $E_6$, then there are two conjugacy classes of $A_i$ whose simple roots are respectively $\pm (i-(i+1)), 1 \leq i \leq l$, and $\pm (i-(i+1)), 1 \leq i \leq l-1$, together with $\pm (l+(l+1))$. (Note: where we write $(i+1)$ and $(l+1)$ above we mean numerical addition, of course).

Proof. Let $h$ be the fixed Cartan subalgebra of $E_s$. Let $h_i \subseteq h$ be the Cartan subalgebra of $A_i$. We can choose vectors $w_1, \ldots, w_{l+1}$ in $h$ such that $h_i$ is the hyperplane in the span of $\{w_1, \ldots, w_{l+1}\}$ determined by the $w_i - w_{i+1}, 1 \leq i \leq l$. We can assume that the $w_i - w_{i+1}$ are the simple roots of $A_i$, since $l < s$ in all cases. We can assume $w_1 - w_2$ is 1 - 2, by conjugating with a Weyl group element. If $w_2 - w_3$ is of type $I$, then a signed permutation can conjugate $\{w_1 - w_2, w_2 - w_3\}$ to $\{1 - 2, 2 - 3\}$. If $w_2 - w_3$ is of type $II$, it must have the form $\pm 1$ 2. 3. 4. Using a signed permutation, we can assume that $w_2 - w_3$ is $(- + + + + * * *)$. Let $\beta = (+ - + + + * * *), \text{ where } \beta$ and $w_2 - w_3$ agree in the last four positions. Then $S_\beta(1 - 2) = 1 - 2, S_\beta( - + + + + * * *) = -2 + 3$. Hence we can assume $w_2 - w_3$ is a root of type $I$, and $(w_1 - w_2, w_2 - w_3) = \{1 - 2, 2 - 3\}$. Next consider $w_3 - w_4$. If it in $(I)_s$, then using a signed permutation we can assume it is $(- + - + + * * *).$ Letting $\beta = (+ - + + + * * *), \text{ and using } S_\beta, \text{ we can assume } w_3 - w_4 \in (I)_s$. Continuing in this way yields (a).

For (b) we do the $E_8$ case, $E_6$ and $E_7$ being entirely similar. By (a) we can assume $w_i - w_{i+1} = i - (i+1)$ for $1 \leq i \leq 6$. Now if $w_7 - w_8$ is a root of type $I$, then since $\langle w_7 - w_8, i - (i+1) \rangle = 0$ for $1 \leq i \leq 5$, and $\langle w_7 - w_8, 6 - 7 \rangle \neq 0$, we can assume $w_7 - w_8$ is $7 - 8$ or $7 + 8$, and if $w_7 - w_8$ is of type $II$, then it must be $\pm ( + + + + + + + + )$ for the same reasons. Let $\alpha = (+ + + + + + + + )$. Then $S_\alpha$ fixes $i - (i+1), i \leq 6$, and takes $w_7 - w_8$ into a root of type $I$. Thus it is reduced to the previous case. The two forms (the one with $7 - 8$ and the one with $7 + 8$) obtained here are not conjugate, since in the first case all the roots are orthogonal to $(+ + + + + + + + )$, whereas in the second case no root is orthogonal to all of them. This completes the proof of (b).

Case 2. $X_1 = A_2$

(a) $k_1 = 1, Y = A_2^\perp$
(i) In $E_8$, we can take $K_{A_2} = \{\pm(6 - 7), \pm(7 + 8), \pm(6 + 8)\}$, since by Lemma 2.4 all $A_2$ in $E_8$ are conjugate. Then $A^+_2 = E_8$. Now $A_2 \oplus E_8$ is maximal and thus primitive.

(ii) In $E_7$, take $K_{A_2} = \{\pm(7 - 8), \pm(+ + + + + - + -), \pm(+ + + + + - + -)\}$ (by Lemma 2.4). Then $A^+_2 = A_6$, where $K_{A_2} = \{\pm(i - j), \pm(i + 6) \mid i \neq j, i, j = 1, \ldots, 5\}$. This is maximal, and thus primitive.

(iii) In $E_6$, by Lemma 2.4, we can take

$$K_{A_2} = \{\pm(1 - 2), \pm(2 - 3), \pm(1 - 3)\}.$$ 

Then $K_{A_2} = \{\pm 4 \pm 5, \pm(+ + + \varepsilon_2 \varepsilon_3 + + -), \pm(+ + + \varepsilon_3 \varepsilon_4 - +) \mid \varepsilon_1 \neq \varepsilon_2, \varepsilon_3 = \varepsilon_4\}$. Thus $A^+_2 = A_2 \oplus A_2$. Either we have a summand of $A_1$, and Case 1 applies, another summand of $A_2$, violating $k_1 = 1$, or neither and hence a root orthogonal to $p$, contradicting primitivity by Corollary 1.6.

(b) $k_i > 1$

(i) In $E_8$, let one copy of $A_2$ have $K_{A_2} = \{\pm \alpha, \pm \beta, \pm \delta\}$ where $\delta = \alpha + \beta$, and let $\gamma$ be a root in another copy of $A_2$. By Lemma 2.1 we can assume $\alpha = 1 + 2, \gamma = 1 - 2$. Now $(\beta, 1 + 2) \neq 0$, and $(\beta, 1 - 2) = 0$. Thus $\beta$ must be of type $II$, and using a signed permutation we may assume $\beta = (+ + + + + + + +)$. Since $[\alpha, \beta] = \delta$, we have $\delta = (- - + + + + + +)$.

The roots in the second copy of $A_2$ are orthogonal to $\delta$ and $\beta$, but not to $1 - 2$. Thus they are $\pm(1 - 2), \pm \eta, \pm \tau$, where $\eta$ and $\tau$ are of type $II$. Using a signed permutation, then, which fixes $z_1$ and $z_2$, we can assume $\eta = (+ + + + + - - -)$ and $\tau = (- + + + + + +)$.

Then $K_{A_2} = \{\pm(3 - 4), \pm(4 - 5), \pm(3 - 5), \pm(6 - 7), \pm(7 - 8), \pm(6 - 8)\}$, and $A^+_2 = A_2^2$ (two more copies). Thus $Y \subseteq A^2_2$. If $Y \neq A^2_2$, then either $Y = A_1 \oplus Z$, which is already covered by Case 1, or $Y = A_4$ or $A_8$. If $Y = A_2$, then there is a root orthogonal to $p$ in $E_8$, which can't happen by Corollary 1.6. Thus $Y = A^2_2$ and $p = A^2_8$. This is primitive, as can be seen using the following elements of $W_{A_2}^2$: $(46)(57)(38)$ and $(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8)$.

(ii) In $E_7$, we can assume that for one copy of $A_2$, $K_{A_2} = \{\pm(7 - 8), \pm(+ + + + + + +), \pm(+ + + + + + + +)\}$. $A^+_2 \subseteq (I)_7$, since any root of $A^+_2 \subseteq E_7$ must be orthogonal to $7 + 8$ and, in this case, to $7 - 8$. Thus, by using signed permutations, we can assume that for the second copy of $A_2$ we have $K_{A_2} = \{\pm(1 - 2), \pm(2 - 3), \pm(1 - 3)\}$. Then $K_{A_2} = \{\pm(4 - 5), \pm(5 + 6), \pm(4 + 6)\}$. Hence $A^+_2$ is the only possibility. $A^+_2$ is primitive, which can be seen using the elements of $W_{A_2}^2$: $(\begin{array}{cccc} 12 & 3 & 4 & 5 \\ 45 & 6 & 12 & 378 \end{array})$ and $fS_pS_\alpha$, where $f = (\begin{array}{cccc} 12 & 3 & 4 & 5 \\ 12 & -3 & -4 & 5 \end{array})$ $\alpha = (+ - - + + + +)$, $\beta = (+ - - + + + +)$. 


(iii) In $E_6$ we can assume that the first copy of $A_2$ has $K_{A_2} = \{\pm(1 - 2), \pm(2 - 3), \pm(1 - 3)\}$, by Lemma 2.4. Then $A_2^1 = A_2 \oplus A_2$. As in subcase (a) (iii) of case 2, $p = A_2^0$ is the only possibility. This is maximal in $E_6$ and hence primitive.

Case 3. $X_1 = A_3$

(a) $k_1 = 1$.

In $E_8$ we can assume by Lemma 2.4, that

$$K_{A_3} = \{\pm 1 \pm 2, \pm 2 \pm 3, \pm 1 \pm 3\}.$$  

(i) In $E_8$, then $A_3^1 = D_5$, with $K_{D_5} = \{\pm i \pm j | 4 \leq i < j \leq 8\}$. By Lemma 2.3, we have $W_{A_3 \oplus D_5} \subseteq W_{D_5} \subseteq W_{(I)_8}$. Thus $A_3 \oplus D_5$ is not primitive.

(ii) In $E_7$, $A_3^1 = A_3 \oplus A_1$, where $K_{A_3} = \{\pm 4 \pm 5, \pm 5 \pm 6, \pm 7 \pm 8\}$. Either this violates $k_1 = 1$ or it has been included in a former case.

(iii) In $E_6$, $A_3^1 = A_3^1$ with $K_{A_3^1} = \{\pm 4 \pm 5\}$. This was treated in Case 1.

(b) $k_1 > 1$. Then $k_1 = 2$ since the rank of $A_3^1$ is greater than 8.

( i ) In $E_8$ there are two non-conjugate ways to imbed $A_3^1$ in $E_8$. We see this as follows. By Lemma 2.4 we can assume that the first copy of $A_3$ has $K_{A_3} = \{\pm(i - j) | 1 \leq i < j \leq 4\}$. Let $\bar{h}$ be the Cartan subalgebra of the second copy of $A_3$. We can assume the coordinates are chosen so that the roots are given by

$$\pm(w_i - w_j)(1 \leq i < j \leq 4).$$

Now $w_i - w_j$ is orthogonal to the roots of the first copy of $A_3$. So if $w_1 - w_2$ is a root of type $I$, by use of a signed permutation we can assume that $w_1 - w_2$ is $5 - 6$. If $w_1 - w_2$ is of type $II$, it must be of the form $\pm(+ + + + + * * *)$ in order to be orthogonal to the first copy of $A_3$. But then conjugating by an element $\beta$ which agrees with $w_1 - w_2$ in all positions except 5 and 6 gives us a root of type $I$ while keeping fixed the first copy of $A_3$. Thus we may assume $w_1 - w_2 = 5 - 6$.

Now if $w_2 - w_3$ is of type $II$, then it must be $\pm(+ + + + + \varepsilon_5 \varepsilon_6 * * *)$, where $\varepsilon_5$ and $\varepsilon_6$ disagree, in order that it not be orthogonal to $5 - 6$. Now conjugating by a root $\beta$ which agrees with $w_2 - w_3$ everywhere except in positions 6 and 7 we fix the first copy of $A_3$ and $5 - 6$ as well. $w_2 - w_3$ becomes a root of type $I$, which can be assumed to be $6 - 7$, using a signed permutation. Thus, we can assume $w_2 - w_3 = 6 - 7$.

Now $w_3 - w_4$ must be orthogonal to $5 - 6$ but not to $6 - 7$, and, of course, orthogonal to the first copy of $A_3$. If $w_3 - w_4$ is of type $II$, then it must be of the form $\pm(+ + + + + + - - -)$ or
±(+ + + + + + + +) Conjugating by (+ + + + + + + + + + + + +) or (+ + + + + +), respectively, we fix 5 - 6, 6 - 7 and the first copy of $A_3$, while conjugating $w_3 - w_4$ into a root of type $I$, in particular $\pm(7 + 8)$. Hence we may assume $w_3 - w_4$ is of type $I$. There are two cases: $w_3 - w_4 = 7 + 8$ or $7 - 8$.

Thus we have the two non-conjugate imbeddings of $A_3$ in $E_8$, namely

$$K_{A_3} = \{ \pm(i - j) | 1 \leq i < j \leq 4, 5 \leq i < j \leq 8 \}$$

and

$$K_{A_6} = \{ \pm(i - j) | 1 \leq i < j \leq 4, 5 \leq i < j \leq 7 \} \cup \{ \pm(i + 8) | 5 \leq i \leq 7 \}.$$  

These are not conjugate since in the first case there is a root, (+ + + + + + + +), orthogonal to $K_{A_3}$, whereas in the second case there is no such root.

But then in the first case, $(A_3)^- \neq \emptyset$, and thus $p$ has a factor of lower rank, which has already been covered in Cases 1 and 2. Thus only the second case need be considered. Here we claim that $A_3$ is not primitive, and in fact that $W_{A_3} \subseteq W_{D_4}$, where

$$K_{D_4} = \{ \pm i \pm j | 1 \leq i < j \leq 4, 5 \leq i < j \leq 8 \}.$$  

Let $g \in W_{A_3}$. If $g$ is a signed permutation, then clearly $g \in W_{D_4}$. Suppose $g = fS_\alpha$, where $f$ is a signed permutation and $\alpha \in (II)_3$. Then $S_\alpha$ must take all roots of $A_3$ into roots of type $I$, and $\alpha = \pm(+ + + + + + + +)$ or $\pm(+ + + + + + + +)$ and neither of these is a root of $E_8$. Thus $g \neq fS_\alpha$. Finally, then, let $g = fS_\beta S_\alpha$, $(\beta, \alpha) = 0$, $f$ a signed permutation. But $\beta$ and $\alpha$ must agree completely or disagree completely on the first four positions, by the same reasons as were used in the proof of Lemma 2.3. Thus, as $(\alpha, \beta) = 0$, $\alpha$ and $\beta$ must disagree completely or agree completely, respectively, on positions 5, 6, 7 and 8. But then $S_\beta S_\alpha \in W_{A_6}^2$, and therefore $f = gS_\beta S_\alpha$ is in $W_{A_3}^2$. We already saw that this implies $f \in W_{D_4}$ since $\alpha$ and $\beta$ agree or disagree completely on 1, 2, 3 and 4. Thus $g \in W_{D_4}$, and $A_3$ is not primitive.

(ii) In $E_7$ we can assume for the first copy of $A_3$ that $K_{A_3} = \{ \pm 1 \pm 2, \pm 2 \pm 3, \pm 1 \pm 3 \}$. Then as in Case 2, (a), (ii) above, $A_3 = A_2 \oplus A_1$. Then either $p = A_3 \oplus A_3 \oplus A_1$, which was covered in Case 1, or $p = A_3 \oplus A_3$ and there is a root, namely the root of $A_4$, orthogonal to $p$, and hence $p$ isn't primitive (Corollary 1.6).

(iii) In $E_6$ $A_3 = A_3$, so $A_3$ isn't a subalgebra of $E_6$.

Case 4. $X = A_4$. 

(i) In $E_6$, by Lemma 2.4, we can assume
Then $K_{A_4} = \{ \pm (i - j) \mid 1 \leq i < j \leq 5 \}$.

Then $K_{A_4} = \{ \pm 6 \pm 7, \pm 7 \pm 8, \pm 6 \pm 8; \text{all roots of type II having the same sign in positions 1 through 5} \}$. Thus $A_t^i = A_i$, and $A_t^i$ is the only possibility in this case. This subalgebra is maximal, and thus primitive.

(ii) In $E_7$, by Lemma 2.4, we can assume

$$K_{A_4} = \{ \pm (i - j) \mid 1 \leq i < j \leq 5 \}.$$  

Then $K_{A_4} \neq \emptyset$, and there must be another summand. This has rank less than 4, and hence is covered by previous cases.

(iii) In $E_6$ Lemma 2.4 implies that there are two possible embeddings of $A_t$, up to conjugacy. In both cases $K_{A_4} \neq \emptyset$, and thus we obtain only cases previously considered.

**Case 5. $X_1 = D_t$.**

We can assume by Lemma 2.3 that $K_{D_t} = \{ \pm i \pm j \mid 1 \leq i < j \leq 4 \}$.

(i) In $E_8$, $K_{D_t} = \{ \pm i \pm j \mid 5 \leq i < j \leq 8 \}$. Thus $D_t^i = D_t$, and the only case not previously considered is $D_t \oplus D_t$. This algebra is primitive in $E_8$. To see this we use $S_\alpha S_\beta \in W_{D_t \oplus D_t}$, where $\alpha = (+ + + + + + + +)$, $\beta = (+ + + + + + + +)$, and sign changes, all of which are in $W_{D_t^i \oplus D_t^i}$.

(ii) In $E_7$, $K_{D_t} \neq \emptyset$. The only possibilities here were covered by earlier cases.

(iii) In $E_6$, $K_{D_t} \neq \emptyset$. In fact, $D_t$ is primitive in $E_6$, which can be seen by using $S_\alpha S_\beta$ from (i) above, together with sign changes on the first five coordinates.

**Case 6. $D_t, l > 4$.**

By Lemma 2.3 we can assume that $K_{D_t} = \{ \pm i \pm j \mid 1 \leq i < j \leq l \}$. Let $p$ be a primitive subalgebra of $E_8$ with a $D_t$ summand. Since $K_{D_t} \subseteq (I)_s$, we can apply Lemma 2.3 to get

$$W_p \subset W_{D_t} \subset W_{(I)_s}.$$  

Since $p$ is primitive and $p \subseteq (I)_s$, we have that $p = (I)_s$. Thus the only possibilities are $D_8$ in $E_8$, $D_5$ in $E_8$, (both are maximal and hence primitive) and $D_8 \oplus A_1$ in $E_7$ (which is covered by a previous case).

**Case 7. $X_1 = A_t$, $l > 4$.**

(i) In $E_8$, suppose $l \leq 7$ and suppose

$$K_{A_t} = \{ \pm (i - j) \mid 1 \leq i < j \leq l + 1 \}.$$  

Then $K_{A_t} \neq \emptyset$, as it contains $(+ + + + + + + +)$, and since $l > 4$,
the rank of $A_l$ is less than 4, and these possibilities have been covered by previous cases. Thus, by Lemma 2.4, we have one other possibility, namely $l = 7$ and

$$K_{A_l} = \{ \pm (i - j) \mid 1 \leq i < j \leq 7 \} \cup \{ \pm (i + 8) \mid 1 \leq i \leq 7 \},$$

$$K_{A_l} = \emptyset.$$ We claim that $A_l$ is not primitive. In particular, we show $W_{A_l} \subseteq W_{D_8}.$

To see this, let $g \in W_{A_l}.$ Then if $g$ is a signed permutation, $g \in W_{D_8}.$ Let $g = fS\alpha$, $f$ a signed permutation, $\alpha \in (II)_s.$ Then $S\alpha$ must not take any root of $K_{A_l}$ into a root of type $II$. The only possibilities are $\alpha = \pm(+-+-+-+),$ which is not a root. Finally, suppose $g = fS\beta S\alpha$, $(\beta, \alpha) = 0$, $f$ a signed permutation. But $\beta$ and $\alpha$ must either agree completely or disagree completely in all positions, or else some root of $K_{A_l}$ would be taken by $S\beta S\alpha$ into a root of type $II$, and $fS\beta S\alpha \in W_{A_l}$. This violates $(\alpha, \beta) = 0$. Hence if $g \in W_{A_l}$, then $g$ is a signed permutation, and $W_{A_l} \subseteq W_{D_8}$.

The only other case of $A_l \subseteq E_8$ is $l = 8$. Since $A_8 \in A_7 \subseteq A_8$, using Lemma 2.4, we can assume that $K_{A_8}$ contains

$$\{ \pm (i - j) \mid 1 \leq i < j \leq 7 \}.$$ Then it is easy to see that this can only happen when

$$K_{A_8} = \{ \pm (i - j) \mid 1 \leq i < j \leq 7 \} \cup \{ i + 8 \mid 1 \leq i \leq 7 \}$$

$$\cup \{ \pm (+ + + + + + +), \pm (+ + + + + - - -) \}.$$ Then $A_8$ is maximal, and hence primitive.

(ii) In $E_7$, $A_5 \subseteq A_5$, and we can assume by Lemma 2.4 that $K_{A_5} \subseteq \{ \pm i \pm j \mid 1 \leq i < j \leq 6 \}.$ Thus $K_{A_5} \neq \emptyset$, as it contains $7 - 8$. Since $K_{A_5}$ has rank less than 4, we have already covered these possibilities by previous cases.

Let $\alpha_1, \alpha_2, \ldots, \alpha_6$ be the simple roots of $A_6$. By Lemma 2.4 we can assume that for $i \leq 4$, $\alpha_i = i - (i + 1)$, and $\alpha_6$ is either $5 - 6$ or $5 + 6$. We have $(\alpha_6, \alpha_i) = 0$ for $i \leq 4$, and $(\alpha_6, \alpha_5) \neq 0$. Thus $\alpha_6$ must be a root of type $II$ of the form $\pm(+-+-+-+)$ or $\pm(+-+-+-+)$. Thus we are in $E_7$, and thus $\alpha_6$ must be fact be $5 - 6$.

Now $A_7 \supset A_6$, and hence we can assume $K_{A_7} = K_{A_6} \cup \{ \pm (7 - 8) \}.$ $A_7$ is maximal, and thus primitive.

$A_6$ itself is not primitive, and in fact $W_{A_6} \subseteq W_{A_l}$. For let $g \in W_{A_6}$. If $g$ is a signed permutation, then $g \in W_{A_l}$ also. Suppose $g = fS\alpha$, $f$ a signed permutation, $\alpha \in (II)_s$. $S\alpha$ must take one of $1 - 2, 2 - 3, \ldots$, or $5 - 6$ into a root of type $II$, for not all of the first six positions of $\alpha$ can have the same sign (in $(II)_s$). Thus we
can assume $S_a(1 - 2)$ is in $(II)_7$. Now if $S_a(3 - 4)$, $S_a(4 - 5)$ or $S_a(5 - 6)$ is a root of type II, say $fS_a(3 - 4)$, then $fS_a(3 - 4)$ and $fS_a(1 - 2)$ would be two orthogonal roots of type II in $A_6$. Such a pair of roots does not exist. Thus $S_a$ must not take $3 - 4, 4 - 5, 5 - 6$ into roots of type II. $\alpha$ has the same sign on positions 3, 4, 5, 6, and must have in positions 1 and 2 different signs. There are four possibilities:

(a) $\pm(+ - + + + + -)$
(b) $\pm(+ - - - - - +)$
(c) $\pm(- + + + + + -)$
(d) $\pm(- - - - - - +)$.

We note that in all four cases, $\alpha \in K_{A_7}$. In cases (a) and (c), $\alpha \in K_{A_6}$, hence $S_a \in W_{A_6}$, and $f \in W_{A_6}$. By the remark above, $f \in W_{A_7}$. Since $K_{A_6} \subset K_{A_7}$, $S_a \in W_{A_7}$. Thus $g \in W_{A_7}$.

In case (b), for $i = 2, 3, 4, 5$, $fS_a(i - (i + 1)) = f(i - (i + 1))$, and these must not be $\pm(7 - 8)$, as $\pm(7 - 8)$ is orthogonal to all other roots of type I. Thus $f$ fixes $\pm(7 - 8)$. Now $f = f''f'$, where $f'$ is a permutation, and $f''$ changes some signs. Clearly, if $f''$ changes any signs on $f''(2), \cdots, f''(6)$, then it changes them all, or else some $f(i - (i + 1)) \in K_{A_6}$. Since $f''$ must change an even number of signs, and $f'$ must fix $\{7, 8\}$, then if $f''$ changes any signs on $f''(2), \cdots, f''(6)$, it must also change sign on either $f''(1)$ or $7, 8$. In either case, $f''$ changes sign on $\{f''(1), \cdots, f''(6)\} = \{1, \cdots, 6\}$. Thus $f''$ either changes all signs or no signs on $1, 2, \cdots, 6$. Then $f \in W_{A_5}$, and therefore $f \in W_{A_7}$. But $\alpha \in K_{A_7}$, and thus $S_a$ and $fS_a$ are in $W_{A_7}$.

In case (d), the argument is the same with 1 and 2 interchanged.

Finally, let $g = fS_aS_\beta$, $(\alpha, \beta) = 0$. Then since $\alpha$ and $\beta$ are orthogonal, they can’t totally agree or totally disagree in sign on 1, 2, 3, 4, 5 and 6. Hence there is some root $i - j$ which $S_\beta S_a$ takes into a type II root. We can assume $S_\beta S_a(1 - 2) \in (II)_7$. Now since all of $i - j$, $3 \leq i < j \leq 6$ are orthogonal to 1 - 2, none of them are taken into roots of type II, as there aren’t two orthogonal type II roots in $K_{A_6}$, as noted above. Thus $\alpha$ and $\beta$ must totally agree or totally disagree in sign on 3, 4, 5 and 6. Since $S_\beta S_a(1 - 2) \in (II)_7$, $\beta$ and $\alpha$ must agree on exactly one of positions 1 and 2. Thus $\alpha$ and $\beta$ totally agree or totally disagree on five positions, contradicting $(\alpha, \beta) = 0$. Thus $g$ cannot be of the form $fS_\beta S_a$.

(iii) In $E_6$ let $A_5 \subset E_6$. We have $A_4 \subset A_5$. Now by Lemma 2.4, $K_{A_4}$ is either $\{\pm(i - j) \mid 1 \leq i < j \leq 5\}$ or $\{\pm(i - j) \mid 1 \leq i < j \leq 4\} \cup \{(i + 5) \mid 1 \leq i \leq 4\}$.

In the first case, there is a simple root in $A_5$ orthogonal to $1 - 2, 2 - 3, 3 - 4$ and not to $4 - 5$. This root must be of type II,
so it is $\pm(+-+-++-+-)$, and $A_5$ is the subalgebra generated by $1 - 2, 2 - 3, 3 - 4, 4 - 5$ and $(+-+-++-+-)$. But then $K_{A_5}^+$ contains $(+-+-++-+-)$, and thus we have considered this in previous cases.

In the second case, the argument is the same.

Finally, $A_6$ cannot be imbedded in $E_6$. For $A_6 \subset A_6$, and we can assume that $K_{A_6}$ is one of the two possibilities described above. Then there would be a simple root $\alpha$ in $K_{A_6}$ orthogonal to $1 - 2, 2 - 3, 3 - 4, 4 - 5$ (resp. $4 - 5$), and not orthogonal to $(+-+-++-+-)$ (resp. $(+-+-++-+-)$). This is impossible in $E_6$.

Case 8. $X_1 = E_7$.

We need only consider $E_7 \subset E_6$. There are seven mutually orthogonal roots of $E_7$. By Lemma 2.1 we can assume that these roots are all of type I. There is a root of type I orthogonal to these seven. This root is in $K_{E_7}$, and thus we are done by previous cases.

Case 9. $X_1 = E_6$.

Let $E_6 \subset E_6$, $s = 7, 8$. $D_5 \subset E_6$. By Lemma 2.3 we can assume that $K_{D_5} = \{\pm i \pm j \mid 1 < i < j \leq 5\}$. There is no further root of type I in $E_6$ or else either we obtain $D_6 \subset E_6$ or we obtain five mutually orthogonal roots in $E_6$, both impossible. Thus all other roots of $E_6$ are of type II. We can assume that $(+-+-++-+-)$ is in $K_{E_6}$. But then $K_{E_6} \neq \emptyset$ in $E_8$, and hence we are done by previous cases. In $E_7$, $E_6$ is maximal and hence primitive.

3. $F_4$. The roots of $F_4$ are described as follows. Let $z_1, z_2, z_3, z_4$ be the standard orthonormal basis for the dual space to a fixed Cartan subalgebra $h$ of $F_4$. With respect to this basis the roots of $F_4$ are given by

$$(I) = \{\pm z_i \mid i = 1, 2, 3, 4\}$$

$$(II) = \{\pm z_i \pm z_j \mid 1 \leq i, j \leq 4\}$$

$$(III) = \left\{\frac{1}{2}(\pm z_1 \pm z_2 \pm z_3 \pm z_4)\right\}.$$

As in the case of the $E_s$, we denote the roots of type III by the corresponding sequence of $+$ and $-$ signs, the roots of type II by the corresponding $i-j$, and the roots of type I merely by the corresponding $\pm j$. Thus $1/2(z_1 + z_2 + z_3 - z_4)$ is denoted $(+-+-)$, $z_3 - z_2$ is denoted $3 - 2$, $z_4$ is denoted 4, and so on. Also, we use $(I) \cup (II), (II), \text{ etc.}$ to denote the subalgebras determined by these
roots, when no confusion results.

We note that the roots have two lengths. Roots in (I) and (III) have length 1, and roots in (II) have length \( \sqrt{2} \). The Weyl group \( W \) of \( F_4 \) acts transitively on the roots of each length.

As in the case of \( E_6 \) above, using Theorem 1.4 we see that if \( p \) is a maximal rank, reductive, primitive subalgebra, and if \( p = X_i^{k_1} \oplus \cdots \oplus X_i^{k_r} \), then either

\[
(X_i^{k_1})^\perp = X_i^{k_2} \oplus \cdots \oplus X_i^{k_r}, \quad \text{or} \quad (X_i^{k_1} \oplus (X_i^{k_1})^\perp) + h
\]

generates (as a subalgebra) \( F_4 \).

**Case 1.** \( X_i = A_i \).

(a) \( k_i = 1 \).

By transitivity of the Weyl group on roots of each length, we may assume that \( K_{A_i} = \{ \pm 1 \} \) or \( K_{A_i} = \{ \pm (1 - 2) \} \). In the first case \( K_{A_i} = \{ \pm 1, (i \pm j) \mid 2 \leq i, j \leq 4 \} \). Thus \( A_i = B_2 \). But \( K_{A_1} \cup K_{B_3} \) generates \((I) \cup (II)\), which is not \( K_{A_1} \cup K_{B_3} \) nor \( K_{F_4} \). Thus \( A_i \oplus Z \) can't be primitive, by Theorem 1.4. In the second case,

\[
K_{A_i} = \{ \pm (1 + 2), \pm 3, \pm 4, \pm 3 \pm 4, \pm (+ + + +) \},
\]
\[
\pm (+ + + -), \pm (+ + - +), \pm (+ + - -) \}.
\]

Then \( A_i^\perp = C_3 \), and \( A_i \oplus C_3 \) is maximal, and hence primitive.

(b) \( k_i = 2 \).

First we observe that if \( K_{A_i} = \{ \pm \alpha \} \) and the other \( K_{A_i} = \{ \pm \beta \} \), then not both \( \alpha \) and \( \beta \) can be shorter roots, or else \([A_i, A_i] \neq 0\) contradicting fact that the two copies of \( A_i \) were ideals in \( p \). Thus it is sufficient to consider only three cases for \( \alpha, \beta \) (up to conjugacy):

- (1) \( 1, 2 - 3 \)
- (2) \( 1 + 2, 1 - 2 \)
- (3) \( 1 - 2, 3 - 4 \).

In the first case, \( K_{A_i} = \{ \pm 4 \} \), and the Lie algebra generated by \( A_i^\perp \oplus (A_i^\perp)^\perp = A_i \oplus A_i \) is more than \( A_i^\perp \oplus A_i \) (e.g., \( 1 + 4 \)), but not all of \( F_4 \) (e.g., \( (+ + + +) \)). Thus \( A_i^\perp \oplus Z \) can't be primitive.

In the second case \( K_{A_i} = \{ \pm 3, \pm 4, \pm 3 \pm 4 \} \), and \( (A_i^\perp)^\perp = B_2 \). In the third case, \( K_{A_i} = \{ \pm (1 + 2), \pm (3 + 4), \pm (+ + + +), \pm (+ + - -) \} \), and again \( (A_i^\perp)^\perp = B_2 \). Since the Weyl group is transitive on the shorter roots, there is some element \( w \) such that \( w(+ + + +) = 1 \). Then \( w(+ + - -) \) is a shorter root orthogonal to 1, and hence, using a signed permutation if necessary, we may assume \( w(+ + - -) = 2 \). Then since \( w(1 + 2) \) and \( w(3 + 4) \) are orthogonal to neither 1 nor 2, and they are orthogonal to each other, we must have \( w(\pm (3 + 4), \pm (1 + 2)) = \{ \pm 1 \pm 2 \} \). Hence \( w \) takes the \( B_2 \) from case (3) into the \( B_2 \) of case (2). Thus \( w \) takes \( B_2^\perp \) into \( B_2^\perp \), which is just \( A_i \) in each...
case. Hence cases (2) and (3) are conjugate, and we only need to consider case (2).

In this case we claim \( A_i \oplus B_2 \) is not primitive. In particular \( A_i \oplus B_2 \) is a subalgebra of the algebra \((I) \cup (II)\), and \( W_{A_i \oplus B_2} \subseteq W_{(I) \cup (II)} \). For let \( w \in W_{A_i \oplus B_2} \). Then \( w \) must preserve each summand, and within each summand it must preserve the roots of each length. Hence \( w \) takes \([\pm 3, \pm 4]\) onto itself. Also \( w \) preserves \([\pm 1 \pm 2]\).

If \( w(1) \in (III) \) (and thus \( w(2) \in (III) \), as \( w(1) \) and \( w(2) \) are orthogonal), then we may assume \( w(1) = (++++) \), \( w(2) = (++--) \), or \( w(2) = (+--+) \). If \( w(2) = (++++) \), then \( w(1-2) = 3+4 \), contradicting the fact that \( w \) fixes \([\pm 3, \pm 4]\). If \( w(2) = (+-+-) \), then \( w(1-2) = 2+3 \), which is not orthogonal to \( w(3-4) \in [\pm 3 \pm 4] \). Thus neither case can occur, and \( w(1) \) and \( w(2) \) are of type I. Hence \( w \) is a signed permutation, therefore fixing \((I) \cup (II)\) as well. Thus \( W_{A_i \oplus B_2} \subseteq W_{(I) \cup (II)} \), and \( A_i \oplus B_2 \) is not primitive.

(c) \( k = 3 \).

Let the three copies of \( K_{A_1} \) be \([\pm \alpha], [\pm \beta], [\pm \gamma] \). Then at most one of \( \alpha, \beta, \gamma \) can be a shorter root, or else two copies of \( A_1 \) would have \([A_1, A_1] \neq 0 \), as above in the \( k = 2 \) case. Thus there are two cases to consider (up to conjugacy) for \( \alpha, \beta, \gamma \):

1. \( 1, 2-3, 2+3 \)
2. \( 1-2, 1+2, 3-4 \),

In the first case, \( K_{A_1}^3 = [\pm 4] \). Then \( (A_1)^3 = A_1 \), and \( A_1 \oplus A_1 \) generates as a Lie algebra \( B_2 \oplus A_2 \). Thus there is no primitive subalgebra in this case.

In the second case, \( K_{A_1}^3 = [\pm (3+4)] \), and \( (A_1)^3 = A_1 \). Here we get \( A_1 \) which is treated below.

(d) \( k = 4 \).

\( A_1 \) contains \( A_i \) as a summand, and by the \( k = 3 \) case we can assume \( K_{A_1}^4 = [\pm 1 \pm 2, \pm 3 \pm 4] \). But this is not primitive, since \( A_1 \subseteq (II) \), and \( W_{A_1} = W_{A_1} \cup W_{A_1} \).

Case 2. \( X = A_2 \).

If \( K_{A_2} \subseteq (I) \cup (III) \), then we may assume \( K_{A_2} = [\pm 1, \pm (++++), \pm (-++++)]. \) Then \( K_{A_2} = [\pm (2-3), \pm (3-4), \pm (2-4)] \). Thus we get \( A_2 \oplus A_2 \). Similarly, if \( K_{A_2} \subseteq (II) \), then we may assume \( K_{A_2} = [\pm (2-3), \pm (3-4), \pm (2-4)] \), and then \( K_{A_2} = [\pm 1, \pm (++++), \pm (-++++)]. \)

Hence we get in either case \( A_2 \oplus A_2 \) with one \( K_{A_2} \subseteq (I) \cup (III) \) and the other \( K_{A_2} \subseteq (II) \). This algebra is maximal and thus primitive.

Case 3. \( X = B_2 \).

As we observed in Case 1 above, all \( B_2 \subseteq F_4 \) are conjugate. Thus
we can assume $K_{B_2} = \{\pm 1, \pm 2, \pm 1 \pm 2\}$. Then $K_{B_2} = \{\pm 3, \pm 4, \pm 3 \pm 4\}$, and we have $B_2 \oplus B_2$. But this generates $(I) + (II) = B_t$. Thus there are no primitive subalgebras in this case.

**Case 4.** $X_1 = A_3$.
If $K_{A_3} \subset (II)$, then we may assume $K_{A_3} = \{\pm 1 \pm 2, \pm 2 \pm 3, \pm 1 \pm 3\}$. Then $K_{A_3} = \{\pm 4\}$, and we have $A_1 \oplus A_3$. This is covered by Case 1. On the other hand, $K_{A_3}$ can have at most one root (and its negative) in $(I)$, or else it would also have one in $(II)$, contradicting the fact that all roots of $A_3$ have the same length. Thus, if $K_{A_3} \subset (I) \cup (III)$, then all but at most one root (and its negative) are in $(III)$. This is impossible.

**Case 5.** $X_1 = A_4$.
This is not contained in $F_4$ as a maximal rank subalgebra.

**Case 6.** $X_1 = B_3$.
Just as in Case 3, there is only one way, up to conjugacy, to have $B_3 \subset F_4$, namely $K_{B_3} = \{\pm i, \pm i \pm j | 2 \leq i, j \leq 4\}$. Then $K_{B_3} = \{\pm 1\}$. Thus $B_3 = A_1$. $A_1 + B_3$ generates $(I) + (II) = B_t$. Thus there is no primitive subalgebra in this case.

**Case 7.** $X_1 = B_4$.
As above, all $B_i \subset F_4$ are conjugate, and we may assume $K_{B_4} = (I) \cup (II)$. This is maximal, and thus primitive.

**Case 8.** $X = D_4$.
$K_{D_4} = (II)$ is the only possibility. This is primitive, since $W_{D_4} = W_{F_4}$, and $W_{F_4}$ is transitive on shorter roots. However $D_4$ is not maximal, since $D_4 \subset B_4 = (I) \cup (II)$.

**Case 9.** $X_1 = C_3$.
All $C_3$ are conjugate $K_{C_3} = K_{A_1}$ as in Case 1 (a).

**Case 10.** $X_1 = C_4$.
This is not a maximal rank subalgebra of $F_4$.

4. $G_2$. Let $\alpha_1$, $\alpha_2$ be simple roots of $G_2$. Then $K_{G_2} = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2), \pm (3\alpha_1 + \alpha_2), \pm (3\alpha_1 + 2\alpha_2) \}$

**Case 1.** $p = A_2$.
The roots of $A_2$ are all of the same length.
The only way to imbed $A_2$, then, is as roots of longer length,
i.e., $K_{A_2} = \{ \pm \alpha_2, \pm (3\alpha_1 + \alpha_2) \pm (3\alpha_1 + 2\alpha_2) \}$. This is maximal and hence primitive.

**Case 2.** $p = A_1$.

There are two possibilities up to conjugacy for $K_{A_1} = \{ \pm \beta \}$, namely, $\beta = \alpha_1$ and $\beta = \alpha_2$, since the Weyl group acts transitively on roots of each length, and there are two lengths of roots in $K_{A_2}$, that of $\alpha_1$ and that of $\alpha_2$.

If $\beta = \alpha_1$, this is not primitive, since $W_{A_1} \subseteq W_{A_2} = W_{A_2}$. If $\beta = \alpha_2$, then the only root $\gamma$ with $[\gamma, \alpha_1] = [\gamma, -\alpha_1] = 0$ is $\gamma = \pm (3\alpha_1 + 2\alpha_2)$. Thus we have $A_1$ with $K_{A_2} = \{ \pm \alpha_1, \pm (3\alpha_1 + 2\alpha_2) \}$.

Further, if $f \in W_{A_1}$, i.e., $f(\pm \alpha_i) = \pm \alpha_i$, then

$$0 = f(0) = f[\pm \alpha_1, 3\alpha_1 + 2\alpha_2] = [f(\pm \alpha_1), f(3\alpha_1 + 2\alpha_2)] = [\pm \alpha_1, f(3\alpha_1 + 2\alpha_2)] .$$

Thus $f(\pm (3\alpha_1 + 2\alpha_2)) = \pm (3\alpha_1 + 2\alpha_2)$. Hence $f \in W_{A_1}$. Thus $A_1$ is not primitive.

**Case 3.** $p = A_2$.

Since $K_{A_2}$ is not contained in the roots of one length, we have, up to conjugacy, $K_{A_2} = \{ \pm \alpha_1, \pm (3\alpha_1 + 2\alpha_2) \}$. This is maximal, and thus primitive.

**References**


Received April 12, 1971. This paper was partially supported by NSF Grant GP-23482.

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