TWO REMARKS ON ELEMENTARY EMBEDDINGS OF THE UNIVERSE

THOMAS J. JECH
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OF THE UNIVERSE

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The paper contains the following two observations: 1. The existence of the least submodel which admits a given elementary embedding \( j \) of the universe. 2. A necessary and sufficient condition on a complete Boolean algebra \( B \) that the Cohen extension \( V^B \) admits \( j \).

A function \( j \) defined on the universe \( V \) is an elementary embedding of the universe if there is a submodel \( M \) such that for any formula \( \varphi \),

\[
(*) \quad \forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \iff M \models \varphi(jx_1, \ldots, jx_n)].
\]

Let \( j \) be an elementary embedding of the universe. If \( N \) is a submodel, let \( j_N = j|N \) be the restriction of \( j \) to \( N \). \( N \) admits \( j \) if

\[
(**) \quad N \models j_N \text{ is an elementary embedding of the universe.}
\]

If \( B \) is a complete Boolean algebra, let \( V^B \) be the Cohen extension of \( V \) by \( B \). \( V^B \) admits \( j \) if

\[
(***) \quad V^B \models \text{there exists an elementary embedding } i \text{ of the universe such that } i \equiv j.
\]

**THEOREM 1.** There is a submodel \( L(j) \) which is the least submodel which admits \( j \). \(^1\)

**THEOREM 2.** The Cohen extension \( V^B \) admits \( j \) if and only if the identity mapping on \( j''B \) can be extended to a \( j(V) \) - complete homomorphism of \( j(B) \) onto \( j''B \).

Before giving the proof, we have a few remarks. The underlying set theory is the axiomatic theory \( BG \) of sets and classes of Bernays and Gödel [1]. The formula \( \varphi \) in (*) is supposed to have only set variables. However, if for any class \( C \) we let \( j(C) = \bigcup_{a \in \text{on}} j(C \cap V_a) \), then (*) holds also for formulas having free class variables ("normal formulas" of [1].) Incidentally, "\( j \) is an elementary embedding of the universe" is expressible in the language of \( BG \) (viz.: \( j \) is an \( \varepsilon \)-isomorphism and \( \forall C_1 \forall C_2 [\mathcal{F}_i(jC_1, jC_2) = j(\mathcal{F}_i(C_1, C_2))] \) where \( \mathcal{F}_i \) are the Gödel operations).

\(^1\) This was observed independently by K. Hrbáček, giving a different proof.
A submodel $M$ is a transitive class containing all ordinals which is a model of GB; the classes of $M$ are all those subclasses $C$ of $M$ which satisfy the condition $\forall \alpha (C \cap V_\alpha \in M)$. The submodel $M$ in (*) is unique and $M = j(V)$. It is a known fact that if $j$ is not the identity then there exists a measurable cardinal. And, as proved recently by Kunen [2], $j(V) \neq V$. On the other hand, if there exists a measurable cardinal, then there exists a nontrivial elementary of the universe (cf. Scott [6]).

The notion $L(j)$ differs somewhat from the notion of relative constructibility, introduced by Lévy [4]; in general, $L(j) \supseteq L[j]$. A homomorphism is C-complete, if it preserves all Boolean sums $\sum_{i \in I} u_i$ where $\{u_i : i \in I\} \in C$. As usual, $j''B$ is the algebra \{\{j(u): u \in B\}; $j(B)$ is an algebra, $j(B) \supseteq j''B$, and $j(B)$ is not necessarily complete (although $jV$-complete).

A similar observation as our Theorem 2 was used recently by J. Silver in his result about extendable cardinals.

As a corollary of Theorem 2, we get the following theorem of Lévy and Solovay [5]: If $\kappa$ is measurable and $|B| < \kappa$, then $\kappa$ is measurable in $V^B$.

Let $j$ be a fixed elementary embedding of the universe. First we prove Theorem 1.

Let $M$ be a submodel.

**Lemma 1.** If $j_M$ is a class of $M$ then $M$ admits $j$.

**Proof.** We must show that for any formula $\varphi$,

$$(\forall \bar{x} \in M)M \models (\varphi(\bar{x}) \rightarrow jM \models \varphi(j\bar{x})).$$

If $M \models \varphi(\bar{x})$, then since $M \models \varphi(\bar{x})$ is a normal formula, we have $jV \models (jM \models \varphi(j\bar{x}))$. However, $\models$ is absolute, so that $M \models (jM \models \varphi(j\bar{x}))$.

**Lemma 2.** If $j \cap M$ is a class of $M$ and if $M$ is closed under $j$ (i.e., $j''M \subseteq M$), then $M$ admits $j$.

**Proof.** It suffices to show that $j_M$ is a class of $M$. Obviously, $j_M \cap M = j \cap M$, and because $M$ is closed under $j$, we have $j_M \subseteq M$, and $j_M = j_M \cap M = j \cap M$.

Now we define the model $L(j)$:

- (i) $L_0(j) = 0$,
- (ii) $L_\alpha(j) = \bigcup_{\beta \subseteq \alpha} L_\beta(j)$ if $\alpha$ is a limit ordinal.

\[\textit{Lj}\text{m}	ext{S} m\text{S} c\] An example of models which are not mild extensions but still admit $j$ are the models constructed by Kunen and Paris in [3].
(iii) \( L_{\alpha+1}(j) = \text{Def} \langle L_\alpha(j), \varepsilon, j \cap L_\alpha(j) \rangle \) if \( \alpha \) is even,
(iv) \( L_{\alpha+1}(j) = L_\alpha(j) \cup [j'' L_\alpha(j) \cap \mathcal{P}(L_\alpha(j))] \) if \( \alpha \) is odd,
(v) \( L(j) = \bigcup_{\alpha \in \mathbb{N}} L_\alpha(j) \).

(iii) means that \( L_{\alpha+1}(j) \) consists of all subsets of \( L_\alpha(j) \) which are definable in \( L_\alpha(j) \) from \( j \cap L_\alpha(j) \). \( \mathcal{P}(L_\alpha(j)) \) is the set of all subsets of \( L_\alpha(j) \).

By standard methods it follows that \( L_\alpha(j) \) is a submodel. That \( L_\alpha(j) \) satisfies the axiom of choice is proved in Lemma 4.

**Lemma 3.** \( i = j \cap L(j) \) is a class of \( L(j) \) and
\[
L(j) = L(i) = L^{L(j)}(i).
\]

**Proof.** By induction on \( \alpha \), we prove
\[
L_\alpha(j) = L_\alpha(i) = L^{L(j)}(i).
\]

If \( \alpha \) is a limit ordinal or \( \alpha = \beta + 1 \) with \( \beta \) even, then the proof is standard. Let \( \beta \) be odd:
\[
x \in L_{\beta+1}(j) \iff x \in L_\beta(j) \lor [x \subseteq L_\beta(j) \land x \in L(j) \land (\exists y \in L_\beta(j))[x = j(y)]]
\]
\[
\iff x \in L_\beta(i) \lor [x \subseteq L_\beta(i) \land (\exists y \in L_\beta(i))[x = i(y)]]
\]
\[
\iff x \in L_{\beta+1}(i)
\]
\[
\iff x \in L^{L(j)}(i).
\]

**Corollary.** \( L(j) \models V = L(i) \).

**Lemma 4.** \( L(j) \models \text{Axiom of Choice} \).

**Proof.** If \( V = L(i) \) then there is a well ordering of the universe, definable from \( i \); hence \( L(j) \models \text{Axiom of Choice} \).

**Lemma 5.** \( L(j) \) is closed under \( j \).

**Proof.** (a) If \( X \subseteq \text{On} \) and \( X \in L(j) \) then there exists \( \alpha \) such that \( X \in L_\alpha(j) \) and \( j(X) \subseteq \alpha \subseteq L_\alpha(j) \); hence \( j(X) \in L_{\alpha+1}(j) \) and so \( j(X) \in L(j) \). Similarly, if \( X \subseteq \text{On} \times \text{On} \).

(b) If \( X \in L(j) \) is arbitrary, then since \( L(j) \models AC \), there exists a well founded relation \( R \in L(j) \) on ordinals which is isomorphic to \( TC(\{X\}) \), the transitive closure of \( \{X\} \). Hence \( j(TC(\{X\})) = TC(\{jX\}) \) is isomorphic to \( j(R) \) which is well founded and by (a), \( jR \in L(j) \); thus \( j(X) \in L(j) \).

**Lemma 6.** If \( M \) admits \( j \) then
\[ L(j) = L^x(j \cap M) \subseteq M. \]

*Proof.* Same as of Lemma 3.

Now, Theorem 1 follows.

Let \( B \) be a complete Boolean algebra. The *Cohen extension* \( V^n \) is the Boolean-valued model of Scott [7] or Vopěnka [8]. There is a natural embedding \( x \mapsto \bar{x} \) of \( V \) into \( V^n \) and \( C \mapsto \check{C} \) can be defined also for classes, in a natural way (in (**), we should rather write \( i \supseteq j \)). More generally, if \( M \) is a submodel satisfying the axiom of choice and if \( B \in M \) is an \( M \)-complete Boolean algebra then \( M^n \) is the Cohen extension of \( M \) by \( B \).

**Lemma 7.** The condition in Theorem 2 is necessary.

*Proof.* Let \( i \) be such that

(1) \( V^n \models i \) is an elementary embedding of the universe and \( i \supseteq j \).

Let \( G \) be the canonical generic ultrafilter on \( \check{B} \), i.e.,

\[ G \in V^{(B)}, \quad \text{dom}(G) = \{ \hat{u}: u \in B \}, \]

\[ G(\hat{u}) = u \text{ for all } u \in B. \]

From (1) it follows that

(3) \( V^n \models i(G) \) is an \( i(\check{V}) \)-complete ultrafilter on \( i(\check{B}) \), i.e.,

(4) \( V^n \models j(V)^{\check{V}} \)-complete ultrafilter on \( (jB)^{\check{V}} \).

Let \( f \) be the following function from \( j(B) \) into \( B \):

\[ f(v) = \llbracket \check{v} \in i(G) \rrbracket. \]

By (4), \( f \) is a \( j(V) \)-complete homomorphism of \( j(B) \) into \( B \) and for all \( u \in B, \) \( f(ju) = \llbracket (ju)^{\check{V}} \in i(G) \rrbracket = \llbracket i(\hat{u}) \in i(G) \rrbracket = \llbracket \hat{u} \in G \rrbracket = u. \) If we let \( h = j \circ f \) then \( h \) is a \( j(V) \)-complete homomorphism of \( j(B) \) onto \( j''B \) and \( h \mid j''B \) is the identity.

**Lemma 8.** The condition is sufficient.

*Proof.* Let \( h \) be a \( j(V) \)-complete homomorphism of \( j(B) \) onto \( j''B \) such that \( h(ju) = ju \) for all \( u \in B \). We are supposed to find \( i \) such that (1) holds. To simplify the considerations, assume that \( G \) is some \( V \)-complete ultrafilter on \( B \) and that \( V[G] \) is the universe. (This is possible because \( V^n \models \check{V}[G] \) is the universe,
where $G$ is the canonical generic ultrafilter defined in (2).

Let $i(G) = h_\gamma(j''G)$. We have $i(G) \supseteq j''G$, and

$$i(G)$$

is a $j(V)$-complete ultrafilter on $j(B)$.

Let $\pi_\gamma: V^\beta \to V[G]$ be the $G$-interpretation of $V^\beta$:

$$\pi_\gamma(0) = 0, \quad \pi_\gamma(x) = \{\pi_\gamma(y): x(y) \in G\}.$$  

Since $j(B) \subseteq j(V)$ is an $j(V)$-complete Boolean algebra, $j(V)^{j(B)} = j(V''')$ is the Cohen extension of $j(V)$ by $j(B)$; it follows from the definition of $i(G)$ that $i(G)$ is a $j(V)$-complete ultrafilter on $j(B)$. Let $\pi_{i\gamma}:(jV)^{j(B)} \to (jV)[iG]$ be the $i(G)$-interpretation of $(jV)^{j(B)}$ and let

$$i(\pi_\gamma x) = \pi_{i\gamma}(jx), \text{ for all } x \in V^\beta.$$  

Now we claim that $i$ is a function, $i$ is an elementary embedding of $V[G]$ into $(jV)[iG]$ and that $i \supseteq j$. To prove that, note that for any formula $\varphi$ and for all $\bar{x} \in V^n$,

$$\models_{j} \varphi(j\bar{x}) \equiv j(\models V \varphi(\bar{x}));$$

This can be proved by induction on the rank of $\bar{x}$ and on the complexity of $\varphi$. In particular, if $\pi_\gamma \bar{x} = \pi_\gamma \bar{y}$, then $\models_{V} \varphi(\bar{x}) \in G$, so that $\models_{jV} \varphi(j\bar{x}) \in i''G \supseteq i(G)$ and so $i(\pi_\gamma \bar{x}) = \pi_{i\gamma}(j\bar{x}) = \pi_{i\gamma}(j\bar{y}) = i(\pi_\gamma \bar{y})$. Similarly, if $V[G] \models \varphi(\pi_\gamma \bar{x})$, then $(jV)[iG] \models \varphi(i(\pi_\gamma \bar{x}))$. If $x \in V$, then $i(x) = i(\pi_\gamma \bar{x}) = \pi_{i\gamma}(j\bar{x}) = j(x)$.

This completes the proof of Theorem 2.

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