TWO REMARKS ON ELEMENTARY EMBEDDINGS OF THE UNIVERSE

THOMAS J. JECH
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The paper contains the following two observations: 1. The existence of the least submodel which admits a given elementary embedding $j$ of the universe. 2. A necessary and sufficient condition on a complete Boolean algebra $B$ that the Cohen extension $V^B$ admits $j$.

A function $j$ defined on the universe $V$ is an elementary embedding of the universe if there is a submodel $M$ such that for any formula $\varphi$,

(*) $\forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \iff M \models \varphi(jx_1, \ldots, jx_n)].$

Let $j$ be an elementary embedding of the universe. If $N$ is a submodel, let $j_N = j|N$ be the restriction of $j$ to $N$. $N$ admits $j$ if

(**) $N \models j_N$ is an elementary embedding of the universe.

If $B$ is a complete Boolean algebra, let $V^B$ be the Cohen extension of $V$ by $B$. $V^B$ admits $j$ if

(***) $V^B \models \text{there exists an elementary embedding } i \text{ of the universe such that } i \supseteq j$

**Theorem 1.** There is a submodel $L(j)$ which is the least submodel which admits $j$.

**Theorem 2.** The Cohen extension $V^B$ admits $j$ if and only if the identity mapping on $j''B$ can be extended to a $j(V)$ -- complete homomorphism of $j(B)$ onto $j''B$.

Before giving the proof, we have a few remarks. The underlying set theory is the axiomatic theory $BG$ of sets and classes of Bernays and Gödel [1]. The formula $\varphi$ in (*) is supposed to have only set variables. However, if for any class $C$ we let $j(C) = \bigcup_{a \in a} j(C \cap V_a)$, then (*) holds also for formulas having free class variables ("normal formulas" of [1].) Incidentally, "$j$ is an elementary embedding of the universe" is expressible in the language of $BG$ (viz.: $j$ is an $\varepsilon$-isomorphism and $\forall C_1 \forall C_2. [\mathcal{F}_i(jC_1, jC_2) = j(\mathcal{F}_i(C_1, C_2))]$ where $\mathcal{F}_i$ are the Gödel operations).

---

1 This was observed independently by K. Hrbáček, giving a different proof.
A submodel $M$ is a transitive class containing all ordinals which is a model of $GB$; the classes of $M$ are all those subclasses $C$ of $M$ which satisfy the condition $\forall \alpha (C \cap V_\alpha \in M)$. The submodel $M$ in (*) is unique and $M = j(V)$. It is a known fact that if $j$ is not the identity then there exists a measurable cardinal. And, as proved recently by Kunen [2], $j(V) \neq V$. On the other hand, if there exists a measurable cardinal, then there exists a nontrivial elementary of the universe (cf. Scott [6]).

The notion $L(j)$ differs somewhat from the notion of relative constructibility, introduced by Lévy [4]; in general, $L(j) \subseteq L[j]$.

A homomorphism is $C$-complete, if it preserves all Boolean sums $\sum_{i \in J} u_i$ where $\{u_i : i \in I\} \subseteq C$. As usual, $j''B$ is the algebra $\{j(u) : u \in B\}$; $j(B)$ is an algebra, $j(B) \supseteq j''B$, and $j(B)$ is not necessarily complete (although $jV$-complete).

A similar observation as our Theorem 2 was used recently by J. Silver in his result about extendable cardinals.

As a corollary of Theorem 2, we get the following theorem of Lévy and Solovay [5]: If $\kappa$ is measurable and $|B| < \kappa$, then $\kappa$ is measurable in $V^B$.2

Let $j$ be a fixed elementary embedding of the universe. First we prove Theorem 1.

Let $M$ be a submodel.

**Lemma 1.** If $j_M$ is a class of $M$ then $M$ admits $j$.

**Proof.** We must show that for any formula $\varphi$,

$$(\forall \bar{x} \in M) M \models (\varphi(\bar{x}) \rightarrow jM \models \varphi(j\bar{x})).$$

If $M \models \varphi(\bar{x})$, then since $M \models \varphi(\bar{x})$ is a normal formula, we have $jV \models (jM \models \varphi(j\bar{x}))$. However, $\models$ is absolute, so that $M \models (jM \models \varphi(j\bar{x}))$.

**Lemma 2.** If $j \cap M$ is a class of $M$ and if $M$ is closed under $j$ (i.e., $j''M \subseteq M$), then $M$ admits $j$.

**Proof.** It suffices to show that $j_M$ is a class of $M$. Obviously, $j_M \cap M = j \cap M$, and because $M$ is closed under $j$, we have $j_M \subseteq M$, and $j_M = j_M \cap M = j \cap M$.

Now we define the model $L(j)$:

(i) $L_0(j) = 0$,

(ii) $L_\alpha(j) = \bigcup_{\beta < \alpha} L_\beta(j)$ if $\alpha$ is a limit ordinal,
(iii) \( L_{\alpha+1}(j) = \text{Def} \langle L_\alpha(j), \varepsilon, j \cap L_\alpha(j) \rangle \) if \( \alpha \) is even,
(iv) \( L_{\alpha+1}(j) = L_\alpha(j) \cup [j'' L_\alpha(j) \cap \mathcal{P}(L_\alpha(j))] \) if \( \alpha \) is odd,
(v) \( L(j) = \bigcup_{\alpha \in \text{On}} L_\alpha(j) \).

(iii) means that \( L_{\alpha+1}(j) \) consists of all subsets of \( L_\alpha(j) \) which are definable in \( L_\alpha(j) \) from \( j \cap L_\alpha(j) \). \( \mathcal{P}(L_\alpha(j)) \) is the set of all subset of \( L_\alpha(j) \).

By standard methods it follows that \( L_\alpha(j) \) is a submodel. That \( L_\alpha(j) \) satisfies the axiom of choice is proved in Lemma 4.

**Lemma 3.** \( i = j \cap L(j) \) is a class of \( L(j) \) and
\[
L(j) = L(i) = L^{L(j)}(i).
\]

**Proof.** By induction on \( \alpha \), we prove
\[
L_\alpha(j) = L_\alpha(i) = L^{L(i)}(i).
\]
If \( \alpha \) is a limit ordinal or \( \alpha = \beta + 1 \) with \( \beta \) even, then the proof is standard. Let \( \beta \) be odd:
\[
\begin{align*}
x \in L_{\beta+1}(j) & \iff x \in L_\beta(j) \lor [x \subseteq L_\beta(j) \land x \in L(j) \land (\exists y \in L_\beta(j))[x = j(y)]] \\
& \iff x \in L_\beta(i) \lor [x \subseteq L_\beta(i) \land (\exists y \in L_\beta(i))[x = i(y)]] \\
& \iff x \in L_{\beta+1}(i) \\
& \iff x \in L_{\beta+1}^{L(i)}(i).
\end{align*}
\]

**Corollary.** \( L(j) \models V = L(i) \).

**Lemma 4.** \( L(j) \models \text{Axiom of Choice} \).

**Proof.** If \( V = L(i) \) then there is a well ordering of the universe, definable from \( i \); hence \( L(j) \models AC \).

**Lemma 5.** \( L(j) \) is closed under \( j \).

**Proof.** (a) If \( X \subseteq \text{On} \) and \( X \in L(j) \) then there exists \( \alpha \) such that \( X \in L_\alpha(j) \) and \( j(X) \subseteq \alpha \subseteq L_\alpha(j) \); hence \( j(X) \in L_{\alpha+1}(j) \) and so \( j(X) \in L(j) \). Similarly, if \( X \subseteq \text{On} \times \text{On} \).

(b) If \( X \in L(j) \) is arbitrary, then since \( L(j) \models AC \), there exists a well founded relation \( R \subseteq L(j) \) on ordinals which is isomorphic to \( TC([X]) \), the transitive closure of \( \{X\} \). Hence \( j(TC([X])) = TC([jX]) \) is isomorphic to \( j(R) \) which is well founded and by (a), \( jR \subseteq L(j) \); thus \( j(X) \in L(j) \).

**Lemma 6.** If \( M \) admits \( j \) then
Let $B$ be a complete Boolean algebra. The Cohen extension $V^B$ is the Boolean-valued model of Scott [7] or Vopěnka [8]. There is a natural embedding $x \mapsto \dot{x}$ of $V$ into $V^B$ and $C \mapsto \dot{C}$ can be defined also for classes, in a natural way (in (**), we should rather write $i \supseteq j$). More generally, if $M$ is a submodel satisfying the axiom of choice and if $B \in M$ is an $M$-complete Boolean algebra then $M^B$ is the Cohen extension of $M$ by $B$.

**Lemma 7.** The condition in Theorem 2 is necessary.

**Proof.** Let $i$ be such that

1. $V^B \models i$ is an elementary embedding of the universe and $i \supseteq \dot{j}$.

Let $G$ be the canonical generic ultrafilter on $\dot{B}$, i.e.,

$$G \in V^{(B)}, \quad \text{dom}(G) = \{ \dot{u} : u \in B \},$$

$$G(\dot{u}) = u \text{ for all } u \in B.$$

From (1) it follows that

3. $V^B \models i(G)$ is an $i(\dot{V})$-complete ultrafilter on $i(\dot{B})$, i.e.,

4. $V^B \models i(G)$ is a $(jV)^\ast$-complete ultrafilter on $(jB)^\ast$.

Let $f$ be the following function from $j(B)$ into $B$:

$$f(\nu) = [\dot{\nu} \in i(G)].$$

By (4), $f$ is a $(jV)$-complete homomorphism of $j(B)$ into $B$ and for all $u \in B$, $f(ju) = [\dot{(ju)}^\ast \in i(G)] = [i(\dot{u}) \in i(G)] = [\dot{u} \in G] = u$. If we let $h = j \circ f$ then $h$ is a $j(V)$-complete homomorphism of $j(B)$ onto $j''B$ and $h|j''B$ is the identity.

**Lemma 8.** The condition is sufficient.

**Proof.** Let $h$ be a $j(V)$-complete homomorphism of $j(B)$ onto $j''B$ such that $h(ju) = ju$ for all $u \in B$. We are supposed to find $i$ such that (1) holds. To simplify the considerations, assume that $G$ is some $V$-complete ultrafilter on $B$ and that $V[G]$ is the universe. (This is possible because

$$V^B \models \dot{V}[G] \text{ is the universe,}$$
where $G$ is the canonical generic ultrafilter defined in (2).
Let $i(G) = h_{-1}(j''G)$. We have $i(G) \supseteq j''G$, and

$$i(G)$$

is a $j(V)$-complete ultrafilter on $j(B)$.

Let $\pi_\sigma: V^B \to V[G]$ be the $G$-interpretation of $V^B$:

$$\pi_\sigma(0) = 0,$$

$$\pi_\sigma(x) = \{\pi_\sigma(y) : x(y) \in G\}.$$

Since $j(B) \in j(V)$ is an $j(V)$-complete Boolean algebra, $j(V)^{j(B)} = j(V^B)$ is the Cohen extension of $j(V)$ by $j(B)$; it follows from the definition of $i(G)$ that $i(G)$ is a $j(V)$-complete ultrafilter on $j(B)$. Let $\pi_{iG}, (jV)^{jB} \to (jV)[iG]$ be the $i(G)$-interpretation of $(jV)^{jB}$ and let

$$i(\pi_\sigma x) = \pi_{iG}(jx),$$

for all $x \in V^B$.

Now we claim that $i$ is a function, $i$ is an elementary embedding of $V[G]$ into $(jV)[iG]$ and that $i \supseteq j$. To prove that, note that for any formula $\varphi$ and for all $\vec{x} \in V^B$,

$$[\varphi(\vec{x})]_{j^B} = j[\varphi(\vec{x})]_{i^B};$$

This can be proved by induction on the rank of $\vec{x}$ and on the complexity of $\varphi$. In particular, if $\pi_{iG} x = \pi_{iG} y$, then $[x = y]_{j^B} \in G$, so that $[jx = jy]_{j''G} \subseteq i(G)$ and so $i(\pi_{iG} x) = \pi_{iG}(jx) = \pi_{iG}(jy) = i(\pi_{iG} y)$. Similarly, if $V[G] \models \varphi(\pi_{iG} x)$, then $(jV)[iG] \models \varphi(i(\pi_{iG} x))$. If $x \in V$, then $i(x) = i(\pi_{iG} x) = \pi_{iG}(jx) = j(x)$.

This completes the proof of Theorem 2.

REFERENCES

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