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This paper studies systems of linear homogeneous p. d. e. in two independent variables with constant coefficients. For such systems powerful algebraic tools are available to obtain results which may indicate patterns for more general systems. Linear isomorphism is defined, and necessary and sufficient conditions for linear isomorphism between two systems are found. This result is obtained from the infinite prolongation of the systems, and two systems are isomorphic if and only if their infinite prolongations are isomorphic. One unexpected result is the important role played by lower-order coefficients which do not appear in such classical notions as ellipticity, hyperbolicity or characteristics. The classification problem for these p. d. e. is reduced to a problem in linear algebra involving a finite number of relations.

Usually notions such as jets, exterior forms, linear bundles, etc., in C^∞ or C^ω categories are used to express our definitions. Since we are concerned here with linear objects we shall use this opportunity to state the definitions in terms of linear algebra.

Let $z = (z^1, \cdots, z^m)$ be a point in the real vector space V. We denote $\partial_j z = (\partial_j z^1, \cdots, \partial_j z^m)$ where $j = 1$ or 2. Let a and b be real m by m matrices. Consider systems Σ of the form $\partial_2 z = \partial_1 z a + zb$.

The total prolongation $P\Sigma$ of Σ is the system $\partial_2 z = z_1 a + b$, $z_1 = \partial_1 z$, $\partial_2 z_1 = \partial_1 z_1 a + z_1 b$, where $z_1 = (z_1^1, \cdots z_1^m)$. Let PV denote the real vector space of $2m$-tuples (z, z_1). Observe that the equation $\partial_2 z_1 = \partial_1 z_1 a + z_1 b$ is dictated by 'involutiveness' from $z_1 = \partial_1 z$, since $\partial_2 z_1 = \partial_{12} z = \partial_1(\partial_1 z a + zb)$. If $z = f(x^1, x^2)$ is any solution of Σ then $(z, z_1) = (f, \partial f/\partial x^1)$ is a solution of $P\Sigma$, and every solution of $P\Sigma$ arises in this way.

Similarly, $P^2\Sigma$ is the system on $P^2 V = \{(z, z_1, z_2)\}$, $\partial_2 z = z_1 a + zb$, $\partial_2 z_1 = z_2 a + z_1 b$, $\partial_2 z_2 = \partial_1 z_2 a + z_2 b$, $z_1 = \partial_1 z$, $z_2 = \partial_1 z_1$. Then $P^n\Sigma$ is defined similarly on $P^n V$. Finally, $P^\infty\Sigma$ is the system on $z_\infty = (z, z_1, z_2, \cdots)$

$$\partial_2 z_\infty = z_\infty \begin{pmatrix} b & 0 & 0 & 0 & 0 & \cdots \\ a & b & 0 & 0 & 0 & \cdots \\ 0 & a & b & 0 & 0 & \cdots \\ 0 & 0 & a & b & 0 & \cdots \\ 0 & 0 & 0 & a & b & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix} = z_\infty P ,$$

$$\partial_1 z_\infty = z_\infty \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ I & 0 & 0 & 0 & 0 & \cdots \\ 0 & I & 0 & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & 0 & \cdots \\ 0 & 0 & 0 & I & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix} = z_\infty Q \ ,$$

where a, b, 0, I denote m by m real matrices, 0 the zero and I the identity matrices.

We shall denote by Σ' a system $\partial_2 y = \partial_1 yc + yd$ where y is a point in the real vector space U and dimension $(U) = m'$.

DEFINITION 1. Two systems Σ and Σ' are *isomorphic* if: (1) There is a linear transformation $\pi: P^r V \to U$,

$$\pi(z, z_1, \cdots, z_r) = zk + z_1 k_1 + \cdots + z_r k_r \ ,$$

where the k_j are m by m' matrices so that the 'pull-back' of the equations $\partial_2 y = \partial_1 yc + yd$ in Σ' are in $P_r \Sigma$. That is,

$$\partial_2 zk + \cdots + \partial_2 z_r k_r = (\partial_1 zk + \cdots \ \partial_1 z_r k_r)c + (zk + \cdots + z_r k_r)d$$

are equations in $P^r \Sigma$. Now if $z = f(x^1, x^2)$ is any solution of Σ then $y = fk + \partial_1 f k_1 + \cdots + \partial_1^r f k_r$ is a solution of Σ'.

(2) Similarly, there is a linear transformation $\rho: P^S U \to V$ so that the pull-back of $\partial_2 z = \partial_1 za + zb$ is in $P^S \Sigma'$.

(3) Finally, we require coherence in the following sense. Letting $p^s \pi: P^{r+s} V \to P^s U$ be defined by

$$p^s \pi(z, z_1, \cdots, z_{r+s}) = (zk + \cdots + z_r k_r, z_1 k + \cdots + z_{r+1} k_r, \cdots, z_s k + \cdots + z_{r+s} k_r) \ ,$$

we require that $\rho \circ p^s \pi(z, z_1, \cdots, z_{r+s}) = z$. This guarantees that if $z = f(x^1, x^2)$ is a solution of Σ corresponding to the solution $y = g(x^1, x^2)$ of Σ' under π, then g will correspond to f under ρ. We also require that $\pi \circ p^r \rho(y, \cdots, y_{r+s}) = y$.

It is not difficult to generalize this to nonlinear systems using jet language. One may compare isomorphism with absolute equivalence. Using [2, Prop. 1] absolutely equivalent systems can be proved to be isomorphic.

DEFINITION 2. Two systems Σ and Σ' are *infinitely isomorphic* if there exists a matrix K with countably infinite columns and rows such that

(1) each column has all but a finite number of elements zero;

(2) K^{-1} exists satisfying (1);

(3) the transformation $y_\infty = z_\infty K$ transforms $P^\infty \Sigma'$ to $P^\infty \Sigma$, that is, $P^\infty \Sigma$ is the system $\partial_2 z_\infty = z_\infty K P' K^{-1}$, $\partial_1 z_\infty = z_\infty K Q' K^{-1}$ when $P^\infty \Sigma'$ is $\partial_2 y_\infty = y_\infty P'$, $\partial_1 y_\infty = y_\infty Q'$.

PROPOSITION 1. *If Σ and Σ' are isomorphic they are infinitely isomorphic.*

Proof. From $\pi: P^r V \to U$ in Defintion 1 we obtain

$$p^\infty \pi \colon P^\infty V \to P^\infty U$$

by

$$p^\infty \pi(z_\infty) = (zk + \cdots + z_r k_r, z_1 k + \cdots + z_{r+1} k_r, \cdots, z_i k + \cdots + z_{i+r} k_r, \cdots)$$

which corresponds to a matrix K with finite columns. Similarly $p^\infty \rho$ corresponds to a matrix H, and $HK = KH = I$ follows from condition (3) in Definition 1.

We now obtain an algebraic condition equivalent to infinite isomorphism.

THEOREM 1. *Two systems Σ and Σ' are infinitely isomorphic if and only if*

(1) $m = m'$,

(2) *there exist m by m matrices $k, k_1, \cdots, k_r$ such that*

$$\begin{aligned} bk &= kd\,, \\ bk_1 + ak &= kc + k_1 d\,, \\ bk_2 + ak_1 &= k_1 c + k_2 d\,, \\ &\;\vdots \\ bk_r + ak_{r-1} &= k_{r-1} c + k_r d\,, \\ ak_r &= k_r c\,, \end{aligned}$$

and (3) *the infinite matrix*

$$K = \begin{pmatrix} k & 0 & 0 & 0 & \cdots \\ k_1 & k & 0 & 0 & \cdots \\ \cdot & k_1 & k & 0 & \cdots \\ \cdot & \cdot & k_1 & k & \cdots \\ \cdot & \cdot & \cdot & k_1 & \cdots \\ k_r & \cdot & \cdot & \cdot & \\ 0 & k_r & \cdot & \cdot & \\ 0 & 0 & k_r & \cdot & \\ \cdot & \cdot & \cdot & k_r & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{pmatrix}$$

has a column-finite inverse.

Proof. Assume (1), (2) and (3). Then K defines a linear transformation between $P^\infty V$ and $P^\infty U$, which gives an infinite isomorphism, since the equations in (2) are equivalent to $PK = KP'$, and (1) implies $Q = Q'$ while it is easy to check that $QK = KQ$.

Conversely, if K is a matrix defining an isomorphism between $P^\infty \Sigma$ and $P^\infty \Sigma'$, then $PK = KP'$ and $QK = KQ'$. Let W be the real vector space of countably infinite column vectors whose components are all zero except possibly a finite number. These matrices P, Q, K can be regarded as linear transformations on W to itself.

Then K defines a linear transformation, K_0, on $W/Q'(W)$ to $W/Q(W)$ by $K_0(w + Q'(W)) = K(w) + Q(W)$, since if $w = Q'(v)$ then $K(w) = KQ'(v) = QK(v)$. This K_0 is onto because K is onto (it has an inverse), and K_0 is one-to-one because if

$$K(w) = Q(v) = QK(v_1) = KQ'(v_1) ,$$

then $w = Q'(v_1)$.

Since $W/Q(W)$ is isomorphic to $W/Q'(W)$, their dimensions are the same. But the dimension of $W/Q(W)$ is m, since and vector $(v_1, v_2, \cdots)^t$, where the v_i are 1 by m, may be expressed as

$$(v_1, 0, 0, \cdots)^t + Q(v_2, v_3, \cdots)^t .$$

Similarly for Q'. Hence $m = m'$.

From $QK = Q'K = KQ'$, if one partitions K into m by m submatrices, it is immediate that K must have the form in (3). Then relations (2) follow at once from $PK = KP'$.

THEOREM 2. *If $P^\infty \Sigma$ and $P^\infty \Sigma'$ are infinitely isomorphic, then Σ and Σ' are isomorphic.*

Proof. Let $k, k_1, \cdots, k_r$ be as in Theorem 1. Then $\pi\colon P^r V \to U$ defined by $\pi(z, z_1, \cdots, z_r) = zk + z_1k_1 + \cdots + z_rk_r$ satisfies

$$\begin{aligned}
&\partial_2(zk + z_1k_1 + \cdots + z_rk_r) - \partial_1(zk + \cdots + z_rk_r)c \\
&\qquad - (zk + \cdots + z_rk_r)d \\
&\quad \equiv (z_1a + zb)k + (z_2a + z_1b)k_1 + \cdots + (\partial_1 z_r a + z_r b)k_r \\
&\qquad - (z_1k + z_2k_1 + \cdots + z_rk_{r-1} + \partial_1 z_r k_r)c - (zk + z_1k_1 \\
&\qquad + \cdots + z_rk_r)d \qquad (\text{modulo } P^r\Sigma) \\
&\quad = z(bk - kd) + z_1(ak + bk_1 - kc - k_1d) \\
&\qquad + \cdots + z_r(ak_{r-1} + bk_r - k_{r-1}c - k_rd) + \partial_1 z_r(ak_r - k_rc) = 0 .
\end{aligned}$$

Thus condition (1) of Definition 1 is satisfied.

If $H = K^{-1}$ the analogous relations for H define $\rho: P^s U \to V$ which satisfies condition (2) of Definition 1. Finally, condition (3) follows from $HK = KH = I$.

COROLLARY. *In order that Σ and Σ' be isomorphic it is necessary that $m = m'$ and b be similar to d.*

Proof. The matrix K^{-1} exists only if k^{-1} exists, so $k^{-1}bk = d$.

This last observation shows the importance of the O-order coefficients, which is rather unexpected in light of the classical notions such as characteristics which are invariants but in which b plays no part [3, 4, 5].

Observe that if Σ and Σ' are isomorphic using the relations in Theorem 2, then from $ak_r = k_r c$ one sees that for any eigenvector v of c, $ak_r v = k_r cv = \lambda k_r v$, so $k_r v$ is an eigenvector of a belonging to the same eigenvalue provided $k_r v \neq 0$. The author conjectures that in all cases the eigenvalues of Σ and Σ' coincide.

EXAMPLE 1. If Σ is $\partial_2 z^1 = z^1$, $\partial_2 z^2 = \partial_1 z^1$ and Σ' is $\partial_2 y^1 = y^1$, $\partial_2 y^2 = 0$, then taking $r = 0$ and

$$k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we find Σ and Σ' isomorphic. The Cauchy problem for Σ with initial data $z^1(x^1, 0) = \varphi(x^1)$ and $z^2(x^1, 0) = \psi(x^1)$ is well-posed, for its solution is $z^1 = \varphi(x^1)\exp(x^2)$, $z^2 = \varphi'(x^1)\exp(x^2) + \psi(x^1) - \varphi'(x^1)$.

EXAMPLE 2. Now let Σ be $\partial_2 z^1 = z^2$, $\partial_2 z^2 = \partial_1 z^1$ and Σ' be

$$\partial_2 y^1 = y^2, \ \partial_2 y^2 = 0 \ .$$

These systems are not isomorphic, for Σ' has solutions

$$y^1 = x^2\psi(x^1) + \varphi(x^1), \ y^2 = \psi(x^1) \ ,$$

while Σ is equivalent to $\partial_{22} z^1 = \partial_1 z^1$, whose solution for $z^1(x^1, 0) = 0$, $z^2(x^1, 0) = \psi(x^1)$ requires ψ that be infinitely differentiable [1, p.27].

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