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GROUP RINGS SATISFYING A POLYNOMIAL IDENTITY. II

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# GROUP RINGS SATISFYING A POLYNOMIAL IDENTITY II

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In an earlier paper we obtained necessary and sufficient conditions for the group ring K[G] to satisfy a polynomial identity. In this paper we obtain similar conditions for a twisted group ring  $K^t[G]$  to satisfy a polynomial identity. We also consider the possibility of K[G] having a polynomial part.

1. Twisted group rings. Let K be a field and let G be a (not necessarily finite) group. We let  $K^{t}[G]$  denote a twisted group ring of G over K. That is  $K^{t}[G]$  is an associative K-algebra with basis  $\{\bar{x} \mid x \in G\}$  and with multiplication defined by

$$\overline{x}\overline{y} = \gamma(x, y)\overline{xy}$$
,  $\gamma(x, y) \in K - \{0\}$ .

The associativity condition is equivalent to  $\overline{x}(\overline{y}\overline{z}) = (\overline{x}\overline{y})\overline{z}$  for all  $x, y, z \in G$  and this is equivalent to

$$\gamma(x, yz)\gamma(y, z) = \gamma(x, y)\gamma(xy, z)$$
.

We call the function  $\gamma: G \times G \to K - \{0\}$  the factor system of  $K^{\iota}[G]$ . If  $\gamma(x, y) = 1$  for all  $x, y \in G$  then  $K^{\iota}[G]$  is in fact the ordinary group ring K[G]. In this section we offer necessary and sufficient conditions for  $K^{\iota}[G]$  to satisfy a polynomial identity. The proof follows the one for K[G] given in [3] and we only indicate the suitable modifications needed. The following is Lemma 1.1 of [2].

LEMMA 1.1. If 
$$x \in G$$
, then in  $K^{t}[G]$  we have  
(i)  $1 = \gamma(1, 1)^{-1} \overline{1}$   
(ii)  $\overline{x}^{-1} = \gamma(x, x^{-1})^{-1} \gamma(1, 1)^{-1} \overline{x^{-1}}$   
 $= \gamma(x^{-1}, x)^{-1} \gamma(1, 1)^{-1} \overline{x^{-1}}$ .

PROPOSITION 1.2. Suppose  $K^t[G]$  satisfies a polynomial identity of degree n and set  $k = (n !)^2$ . Then G has a characteristic subgroup  $G_0$  such that  $[G: G_0] \leq (k + 1)!$  and such that for all  $x \in G_0$ 

$$[G: C_G(x)] \leq k^{4^{(k+1)!}}.$$

*Proof.* This is the twisted analog of Corollary 3.5 of [3]. We consider § 3 of [3] and observe that each of the prerequisite results for that corollary also has a twisted analog.

First Lemma 3.1 of [3] holds for  $K^{\iota}[G]$  with no change in the proof. Of course x must be replaced by  $\overline{x}$  in the formula

$$lpha_{_1} \overline{x} eta_{_1} + lpha_{_2} \overline{x} eta_{_2} + \, \cdots \, + \, lpha_t \overline{x} eta_t = \overline{x} \gamma \; .$$

Second Theorem 3.4 of [3] also holds for  $K^t[G]$  with no change in its statement. The proof is modified just slightly so that the inductive result to be proved is as follows. For each  $x_j, x_{j+1}, \dots, x_n \in G$ , then either  $f_j(\bar{x}_j, \bar{x}_{j+1}, \dots, \bar{x}_n) = 0$  or for some  $\mu \in \mathcal{M}_j, \ \mu(\bar{x}_j, \bar{x}_{j+1}, \dots, \bar{x}_n) =$  $a \ \bar{y}$  for some  $a \in K - \{0\}, \ y \in \mathcal{A}_k(G)$ . Then replacing x's suitably by  $\bar{x}'s$  the proof carries through as before. Finally Corollary 3.5 of [3] holds for  $K^t[G]$  since it is just a group theoretic consequence of Theorem 3.4 of [3].

Let  $K^{i}[G]$  be a twisted group ring and let H be a subgroup of G. Then by  $K^{i}[H]$  we mean that twisted group ring of H which is naturally contained in  $K^{i}[G]$ . Let  $JK^{i}[G]$  denote the Jacobson radical of  $K^{i}[G]$ .

**PROPOSITION 1.3.** Suppose  $K^t[G]$  satisfies a polynomial identity of degree n and suppose further that G' is finite and  $K^t[G']$  is central in  $K^t[G]$ . Then G has a subgroup  $Z \supseteq G'$  such that

 $[G: Z] \leq (n/2)^{2|G'|}$ 

with  $K^{t}[Z]/(JK^{t}[G'] \cdot K^{t}[Z])$  commutative.

*Proof.* Since  $K^t[G']$  is commutative,  $JK^t[G']$  is the intersection of the maximal two-sided ideals of  $K^t[G']$ . Moreover  $K^t[G']/JK^t[G']$ is a finite dimensional semisimple algebra and hence it has at most

 $\dim_{\kappa} K^{t}[G']/JK^{t}[G'] \leq |G'|$ 

maximal two-sided ideals. Thus we may write

$$JK^{t}[G'] = \bigcap_{i=1}^{m} I_{i}, \ m \leq |G'|$$

where  $I_i$  is a maximal two-sided ideal of  $K^{i}[G']$ .

Fix a subscript *i*. Then  $K^t[G']/I_i = F_i$ , some finite field extension of *K*. Now  $K^t[G']$  is central in  $K^t[G]$ , so  $I_i \cdot K^t[G]$  is an ideal in  $K^t[G]$ . It is now easy to see that  $K^t[G]/(I_i \cdot K^t[G])$  is an  $F_i$ -algebra with a basis consisting of the images of coset representatives for *G'* in *G*. Thus clearly  $K^t[G]/(I_i \cdot K^t[G])$  is isomorphic to some twisted group ring  $F_{i}^{t_i}[G/G']$ , and this twisted group ring inherits the polynomial identity satisfied by  $K^t[G]$ . Thus by Proposition 1.4 of [2], G/G' has a subgroup  $\overline{Z}_i$  with  $[G/G': \overline{Z}_i] \leq (n/2)^2$  and with  $F_{i}^{t_i}[Z_i]$  central in  $F_i^{t_i}[G/G']$ . Let  $Z_i$  be the complete inverse image

of  $\overline{Z}_i$  in G. Then  $Z_i \supseteq G'$ ,  $[G: Z_i] \leq (n/2)^2$  and for all  $\alpha$ ,  $\beta \in K^t[Z_i]$  we have  $\alpha\beta - \beta\alpha \in I_i \cdot K^t[G]$ .

Set  $Z = \bigcap_{i=1}^{m} Z_i$ . Then

$$[G\colon Z] \leq \varPi_1^m[G\colon Z_i] \leq (n/2)^{_{2m}} \leq (n/2)^{_{2|G'|}}$$
 .

Moreover for all  $\alpha, \beta \in K^{t}[Z]$  we have

$$\alpha\beta - \beta\alpha \in \bigcap_{i}^{m} I_{i} \cdot K^{t}[G] = JK^{t}[G'] \cdot K^{t}[G]$$

since  $K^{t}[G]$  is free over  $K^{t}[G']$ . Hence since  $K^{t}[G]$  is free over  $K^{t}[Z]$  we have

$$\alpha\beta - \beta\alpha \in K^{t}[Z] \cap (JK^{t}[G'] \cdot K^{t}[G]) = JK^{t}[G'] \cdot K^{t}[Z]$$

and the result follows.

We now come to our main result on twisted group rings satisfying a polynomial identity.

THEOREM 1.4. Let  $K^{t}[G]$  be a twisted group ring of G over K. Let  $G \supseteq A \supseteq B$  be subgroups of G with B finite and central in A and with  $K^{t}[A]/(JK^{t}[B] \cdot K^{t}[A])$  commutative.

(i) If  $[G: A] < \infty$  then  $K^t[G]$  satisfies a polynomial identity of degree  $n = 2[G: A] \cdot |B|$ .

(ii) If  $K^t[G]$  satisfies a polynomial identity of degree n, then there exists suitable A and B with  $[G: A] \cdot |B|$  bounded by some fixed function of n.

*Proof.* The proof of (i) is identical to the proof of Theorem 1.3 (i) of [3]. Observe that  $JK^{i}[B] \cdot K^{i}[A] = K^{i}[A] \cdot JK^{i}[B]$  is an ideal of  $K^{i}[A]$  by Lemma 1.2 of [1].

We now consider part (ii). Let  $K^{t}[G]$  satisfy a polynomial identity of degree n. Set

$$a = a(n) = (n !)^2, \qquad b = b(n) = a^{4^{(a+1)!}}.$$

Then by Proposition 1.2 G has a subgroup  $G_0$  with

$$[G: G_0] \leq (a+1)!, \quad G_0 = \varDelta_b(G_0)$$

where  $\Delta_k$  is defined in [3].

 $\mathbf{Set}$ 

$$c = c(n) = (b^4)^{b^4}$$
,  $d = d(n) = (n/2)^{2c}$ .

Then by Theorem 4.4 of [3],  $|G'_0| \leq c$ . Let  $G_1 = C_{G_0}(G'_0)$ . Then  $G'_1 \subseteq G'_0$  so  $G'_1$  is a finite central subgroup of  $G_1$ . Moreover

$$|G_1'| \leq c, \qquad [G_0:G_1] \leq c!.$$

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Let  $x \in G_1$ . Then conjugation by  $\overline{x}$  induces an automorphism of  $K^t[G_1]$ . Moreover since  $G_1'$  is central in  $G_1$  we have

$$ar{x}^{-1}ar{y}ar{x} = \lambda_x(y)ar{y}$$

for all  $y \in G'_1$ . It follows easily that  $\lambda_x$  is a linear character of  $G'_1$  into K, that is  $\lambda_x \in \text{Hom}(G'_1, K - \{0\})$ . In addition, it follows easily that the map  $x \to \lambda_x$  is in fact a group homomorphism

$$G_1 \longrightarrow \operatorname{Hom} (G'_1, K - \{0\})$$
.

Let  $G_2$  denote the kernel of this homomorphism. Then

$$[G_1:G_2] \leq |\operatorname{Hom} \left(G_1',\, K-\{0\}
ight)| \leq |G_1'| \leq c$$
 .

Set  $B = G'_2$ . Then  $B \subseteq G'_1$  so  $|B| \leq c$  and  $K^t[B]$  is central in  $K^t[G_2]$ . By Proposition 1.3,  $G_2$  has a subgroup  $A \supseteq B$  with

$$[G_2:A] \leq (n/2)^{2|B|} \leq d$$

and with  $K^{t}[A]/(JK^{t}[B] \cdot K^{t}[A])$  commutative. Since  $|B| \leq c$  and since

$$[G: A] = [G: G_0] [G_0: G_1] [G_1: G_2] [G_2: A] \leq (a+1)! \cdot c \cdot c \cdot d$$

the result follows.

It is interesting to interpret this result for various fields. If K has characteristic 0 and if B is a finite group, then  $K^t[B]$  is semisimple by Proposition 1.5 of [1]. Thus

COROLLARY 1.5. Let  $K^{t}[G]$  be a twisted group ring of G over K and let K have characteristic 0. Let A be an abelian subgroup of G with  $K^{t}[A]$  commutative.

(i) If  $[G: A] < \infty$  then  $K^{i}[G]$  satisfies a polynomial identity of degree n = 2 [G: A].

(ii) If  $K^{t}[G]$  satisfies a polynomial identity of degree n, then there exists such a group A with [G: A] bounded by some fixed function of n.

COROLLARY 1.6. Let  $K^{i}[G]$  be a twisted group ring of G over K and let K have characteristic p > 0. Let  $G \supseteq A \supseteq P$  be subgroups of G with P a finite p-group central in A and with  $K^{i}[A]/(JK^{i}[P] \cdot K^{i}[A])$  commutative.

(i) If  $[G:A] < \infty$  then  $K^{\iota}[G]$  satisfies a polynomial identity of degree  $n = 2[G:A] \cdot |P|$ .

(ii) If  $K^t[G]$  satisfies a polynomial identity of degree n, then there exists suitable A and P with  $[G:A] \cdot |P|$  bounded by some fixed function of n.

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*Proof.* Let B be given as in Theorem 1.4 and let P be its normal Sylow p-subgroup. Then P is also central in A. Moreover by Proposition 1.5 of [1]  $JK^t[B] = JK^t[P] \cdot K^t[B]$  so the result clearly follows.

Finally in the above if K is a perfect field of characteristic p, then by Lemma 2.1 of [1],  $K^t[P] \cong K[P]$  so  $K^t[P]/JK^t[P] = K$ . It then follows easily that

$$K^t[A]/(JK^t[P] \cdot K^t[A]) \cong K^{t'}[A/P]$$

is in fact some twisted group ring of A/P.

2. Generalized polynomial identities. Let E be an algebra A generalized polynomial over E is, roughly speaking, a over K. polynomial in the indeterminates  $\zeta_1, \zeta_2, \dots, \zeta_n$  in which elements of E are allowed to appear both as coefficients and between the indeterminates. We say that E satisfies a generalized polynomial identity if there exists a nonzero generalized polynomial  $f(\zeta_1, \zeta_2, \dots, \zeta_n)$  such that  $f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$  for all  $\alpha_1, \alpha_2, \dots, \alpha_n \in E$ . The problem here is precisely what does it mean for f to be nonzero. For example, suppose that the center of E is bigger than K and let  $\alpha$  be a central element not in K. Then E satisfies the identity  $f(\zeta_1) = \alpha \zeta_1 - \zeta_1 \alpha$ but surely this must be considered trivial. Again, suppose that E is not prime. Then we can choose nonzero  $\alpha, \beta \in E$  such that E satisfies the identity  $f(\zeta_1) = \alpha \zeta_1 \beta$  and this must also be considered trivial. We avoid these difficulties by restricting the allowable form of the polynomials.

We say that f is a multilinear generalized polynomial of degree n if

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} f^{\sigma}(\zeta_1, \zeta_2, \cdots, \zeta_n)$$

and

$$f^{\sigma}(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{j=1}^{a_{\sigma}} \alpha_{0 \sigma j} \zeta_{\sigma(1)} \alpha_{1 \sigma, j} \zeta_{\sigma(2)} \cdots \alpha_{n-1 \sigma, j} \zeta_{\sigma(n)} \alpha_{n, \sigma, j}$$

where  $\alpha_{i \sigma, j} \in E$  and  $a_{\sigma}$  is some positive integer. This form is of course motivated by Lemma 3.2 of [3]. The above f is said to be nondegenerate if for some  $\sigma \in S_n$ ,  $f^{\sigma}$  is not a polynomial identity satisfied by E. Otherwise f is degenerate.

In this section we will study group rings K[G] which satisfy nondegenerate multilinear generalized polynomial identities. Let  $\Delta = \Delta(G)$  denote the F. C. subgroup of G and let  $\theta: K[G] \to K[\Delta(G)]$ denote the natural projection. LEMMA 2.1. Suppose K[G] satisfies a nondegenerate multilinear generalized polynomial of degree n. Then K[G] satisfies a polynomial identity as given above with

$$\sum\limits_{j=1}^{a_1} heta(lpha_{\scriptscriptstyle 0,1,j}) \, heta\left(lpha_{\scriptscriptstyle 1,1,j}
ight) \, \cdots \, heta(lpha_{\scriptscriptstyle n,1,j}) 
eq 0 \; .$$

*Proof.* Let K[G] satisfy f as above. Since f is nondegenerate, by reordering the  $\zeta$ 's if necessary, we may assume that  $f^1(\zeta_1, \zeta_2, \dots, \zeta_n)$  is not an identity for K[G]. Thus since  $f^1$  is multilinear there exists  $x_1, x_2, \dots, x_n \in G$  with

$$0 \neq f^{1}(x_{1}, x_{2}, \dots, x_{n})$$
  
=  $\sum_{j=1}^{a_{1}} \alpha_{0,1,j} x_{1} \alpha_{1,1,j} x_{2} \cdots \alpha_{n-1,1,j} x_{n} \alpha_{n,1,j}$ .

If we replace  $\zeta_i$  in f by  $x_i\zeta_i$  we see clearly that K[G] satisfies a suitable f with

(\*) 
$$0 \neq \sum_{j=1}^{a_1} \alpha_{0,1,j} \alpha_{1,1,j} \cdots \alpha_{n,1,j}$$

For each i, j write

$$lpha_{i,i,j} = \sum_k eta_{ijk} y_k$$

where  $\beta_{ijk} \in K[\Delta]$  and  $\{y_k\}$  is a finite set of coset representatives for  $\Delta$  in G. We substitute this into (\*) above. It then follows easily that for some  $k_0, k_1, \dots, k_n$  we have

$$0
eq \sum_{j=1}^{a_1}eta_{0jk_0}y_{k_0}eta_{1jk_1}y_{k_1}\cdotseta_{njk_n}y_{k_n}$$
 .

Thus if  $z_i$  is defined by  $z_i = y_{k_0}y_{k_1}\cdots y_{k_{i-1}}$  and  $z_0 = 1$  then

$$0 \neq \sum_{j=1}^{a_1} eta_{0jk_0}^{z_0^{-1}} eta_{1jk_1}^{z_1^{-1}} \cdots eta_{njk_n}^{z_n^{-1}}$$
 .

Now  $\beta_{ijk_i} = \theta(\alpha_{i,1,j}y_{k_i}^{-1})$  so

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$$eta_{ijk_i}^{z_i^+} = heta(z_ilpha_{i,1,j}y_{k_i}^{-1}z_i^{-1}) = heta(z_ilpha_{i,1,j}z_{i+1}^{-1}) \;.$$

It therefore follows that if we replace  $\zeta_i$  in f by  $z_{i+1}^{-1}\zeta_i z_{i+1}$  and if, in addition, we multiply f on the left by  $z_0$  and on the right by  $z_{n+1}^{-1}$ , then this new multilinear generalized polynomial identity obtained has the required property.

LEMMA 2.2. Let  $\alpha_1, \alpha_2, \dots, \alpha_u, \beta_1, \beta_2, \dots, \beta_u \in K[G]$ . Suppose that for some integers k and t

 $|oldsymbol{\bigcup}\operatorname{Supp}lpha_i|=r,\;|oldsymbol{\bigcup}\operatorname{Supp}eta_i|=s$ 

and

$$(\bigcup_i \operatorname{Supp} \alpha_i) \cap \varDelta_k(G) \subseteq \varDelta_t(G)$$

with  $k \ge rst^r$ . Let T be a subset of G and suppose that for all  $x \in G$ -T we have

$$\alpha_1 x \beta_1 + \alpha_2 x \beta_2 + \cdots + \alpha_u x \beta_u = 0.$$

Then either [G:T] < (k+2)! or

$$\theta_k(\alpha_1)\beta_1 + \theta_k(\alpha_2)\beta_2 + \cdots + \theta_k(\alpha_u)\beta_u = 0$$
.

*Proof.* Let  $A = \bigcup_i \operatorname{Supp} \alpha_i$ ,  $B = \bigcup_i \operatorname{Supp} \beta_i$  and write

$$egin{array}{lll} A' &= A \cap arDelta_k = \{g_1, \, g_2, \, \cdots, \, g_n\} \ A'' &= A - arDelta_k = \{y_1, \, y_2, \, \cdots, \, y_m\} \ B &= \{z_1, \, z_2, \, \cdots, \, z_s\} \;. \end{array}$$

Here of course m + n = r. Set  $W = \bigcap_{i=1}^{n} C_{G}(g_{i})$ . Since by assumption  $A' \subseteq \Delta_{t}(G)$  we have clearly  $[G: W] \leq t^{r}$ . Observe that for all  $x \in W$ , x centralizes  $\theta_{k}(\alpha_{i})$ .

Suppose that

$$\gamma = \theta_k(\alpha_1)\beta_1 + \theta_k(\alpha_2)\beta_2 + \cdots + \theta_k(\alpha_u)\beta_u \neq 0$$

and let  $v \in \text{Supp } \gamma$ . If  $y_i$  is conjugate to  $vz_j^{-1}$  in G for some i, j choose  $h_{ij} \in G$  with  $h_{ij}^{-1}y_ih_{ij} = vz_j^{-1}$ .

Write  $\alpha_i = \theta_k(\alpha_i) + \alpha_i^{\prime\prime}$  and then write

$$lpha_i^{''} = \sum a_{ij} y_j, \; eta_i = \sum b_{ij} z_j$$
 .

Let  $x \in W$ -T. Then we must have

$$0 = x^{-1}\alpha_1 x \beta_1 + x^{-1}\alpha_2 x \beta_2 + \cdots + x^{-1}\alpha_u x \beta_u$$
  
=  $[\theta_k(\alpha_1)\beta_1 + \theta_k(\alpha_2)\beta_2 + \cdots + \theta_k(\alpha_u)\beta_u]$   
+  $[\alpha_1^{\prime x}\beta_1 + \alpha_2^{\prime x}\beta_2 + \cdots + \alpha_u^{\prime x}\beta_u]$ .

Since v occurs in the support of the first term it must also occur in the second and hence there exists  $y_i, z_j$  with  $v = y_i^z z_j$  or

$$x^{-1}y_ix = vz_j^{-1} = h_{ij}^{-1}y_ih_{ij}$$

Thus  $x \in C_G(y_i)h_{ij}$ . We have therefore shown that

$$W \subseteq T \cup \bigcup_{ij} C_{\scriptscriptstyle G}(y_i) h_{ij}$$
.

Let  $w_1, w_2, \dots, w_d$  be a complete set of coset representatives for W in G. Then  $d = [G: W] \leq t^r$  and the above yields

$$G = Tw_1 \cup Tw_2 \cup \cdots \cup Tw_d \cup S$$

where

$$S = igcup_{i,j,\mathfrak{c}} C_{\scriptscriptstyle G}(y_i) h_{ij} w_{\scriptscriptstyle \mathcal{C}}$$
 .

Now the number of cosets in the above union for S is at most

 $rsd \leq rst^r \leq k$ 

by assumption on k. Moreover  $y_i \notin A_k$  so  $[G: C_G(y_i)] > k$  for all *i*. Thus by Lemma 2.3 of [3]  $S \neq G$  and then Lemma 2.1 of [3] yields

$$[G: \widetilde{T}] \leq (k+1)!$$

where

$$\widetilde{T} = \bigcup_{c} T w_{c}$$
.

Thus

 $[G:T] \le (k+1)! \ d \le (k+1)! \ (k+2)$ 

and the result follows.

We will need the following group theoretic lemma.

LEMMA 2.3. Let G be a group. The following are equivalent

(i)  $[G: \varDelta(G)] < \infty$  and  $|\varDelta(G)'| < \infty$ .

(ii) There exists an integer k with  $[G: \Delta_k(G)] < \infty$ .

*Proof.* Suppose that G satisfies (i) and set  $n = [G:\Delta]$ ,  $m = |\Delta'|$ . If  $x \in \Delta$ , then by Theorem 4.4 (i) of [3],  $[\Delta: C_{\Delta}(x)] \leq m$  and hence  $[G: C_{G}(x)] \leq nm$ . Thus (ii) follows with k = mn.

Now suppose that (ii) holds. Since  $\Delta(G) \supseteq \Delta_k(G)$  and  $[G: \Delta_k] < \infty$ we conclude that  $[G: \Delta] < \infty$ . Now  $\Delta(G)$  is a subgroup of G so every right translate of  $\Delta_k$  in G is either entirely contained in  $\Delta$  or is disjoint from  $\Delta$ . This implies that  $[\Delta: \Delta_k] < \infty$  and say

$$arDelta=arDelta_k y_1\cuparDelta_k y_2\cup\cdots\cuparDelta_k y_r$$
 .

Since each  $y_i \in \Delta$  we can set  $n = \max_i [G: C(y_i)] < \infty$ . If  $x \in \Delta$  then  $x \in \Delta_k y_i$  for some *i* and this implies easily that  $[G: C(x)] \leq nk$ . Thus  $[\Delta: C_4(x)] \leq nk$  and by Theorem 4.4 (ii) of [3],  $|\Delta'| < \infty$ .

We now come to the main result of this section

THEOREM 2.4. Let K[G] be a group ring of G over K and sup-

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pose that K[G] satisfies a nondegegerate multilinear polynomial identity. Then  $[G: \Delta(G)] < \infty$  and  $|\Delta(G)'| < \infty$ .

*Proof.* By Lemma 2.1. we may assume that K[G] satisfies

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} \sum_{j=1}^{\alpha_\sigma} a_{0,\sigma,j} \zeta_{\sigma(1)} \alpha_{1\sigma,j} \zeta_{\sigma(2)} \cdots \alpha_{n-1\sigma,j} \zeta_{\sigma(n)} \alpha_{n,\sigma,j}$$

with

$$\sum_{j=1}^{a_1} \theta(\alpha_{0,1,j}) \ \theta(\alpha_{n,1,j}) \cdots \ \theta(\alpha_{n,1,j}) \neq 0$$

We first define a number of numerical parameters associated with f. Set

$$a = \sum_{\sigma \in S_n} \sum_{i=0}^n \sum_{j=1}^{a_\sigma} |\operatorname{Supp} \alpha_{i,\sigma | j}|$$

and

$$r_0 = s_0 = a^{n+1}$$
.

Now consider

$$U = \bigcup_{\sigma \in S_n} \bigcup_{j=1}^{a_\sigma} \bigcup_{i=0}^n \operatorname{Supp} \theta(lpha_{i,\sigma,j})$$
 .

Then U is a finite subset of  $\Delta(G)$  so there exists an integer b with  $U \subseteq \Delta_b(G)$ . Set

$$t = b^{n+1}$$
 and  $k = r_0 s_0 t^{r_0}$ .

We assume now that  $[G: \varDelta_k] = \infty$  and derive a contradiction.

For  $i = 0, 1, \dots, n$  define  $S^i \subseteq S_n$  by

$$S^{i} = \{\sigma \in S_{n} \mid \sigma(1) = 1, \ \sigma(2) = 2, \dots, \sigma(i) = i\}$$
.

Then  $S^{\circ} = S_n$ ,  $S^n = \langle 1 \rangle$  and  $S^i$  is just an embedding of  $S_{n-i}$  in  $S_n$ . We define the multilinear generalized polynomial  $f_i$  of degree n-i by

$$f_{i}(\zeta_{i+1}, \zeta_{i+2}, \dots, \zeta_{n})$$

$$= \sum_{\sigma \in S^{i}} \sum_{j=1}^{a_{\sigma}} \theta(\alpha_{0 \sigma, j}) \theta(\alpha_{1, \sigma, j}) \cdots \theta(\alpha_{i-1 \cdot \sigma, j}) \alpha_{i \cdot \sigma, j} \zeta_{\sigma(i+1)} \cdots \alpha_{n-1 \cdot \sigma, j} \zeta_{\sigma(n)} \alpha_{n \sigma \cdot j}.$$

Thus  $f_0 = f$  and

$$f_n = \sum_{j=1}^{a_1} \theta(\alpha_{0,1\cdot j}) \theta(\alpha_{1,1\cdot j}) \cdots \theta(\alpha_{n-1,1\cdot j}) \alpha_{n-1,j}$$

is a nonzero element of K[G] since

$$\theta(f_n) = \sum_{j=1}^{a_1} \theta(\alpha_{0,1,j}) \theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{n-1,1,j}) \theta(\alpha_{n,1,j}) \neq 0$$
.

Let  $\mathscr{M}$  be the set of monomial polynomials obtained as follows. For each  $\sigma$ , j we start with

$$\alpha_{0,\sigma,j}\zeta_{\sigma(1)}\alpha_{1,\sigma,j}\zeta_{\sigma(2)}\cdots\alpha_{n-1,\sigma,j}\zeta_{\sigma(n)}\alpha_{n,\sigma,j}$$

and we modify it by (1) deleting some but not all of the  $\zeta_i$ ; (2) replacing some of the  $\alpha_{i,\sigma,j}$  by  $\theta(\alpha_{i,\sigma,j})$ ; and (3) replacing some of the  $\alpha_{i,\sigma,j}$  by 1. Then  $\mathscr{M}$  consists of all such monomials obtained for all  $\sigma$ , j and clearly  $\mathscr{M}$  is a finite set. Note that  $\mathscr{M}$  may contain the zero monomial but it contains no nonzero constant monomial since in (1) we do not allow all the  $\zeta_i$  to be deleted.

For  $i = 0, 1, \dots, n$  define  $\mathcal{M}_i \subseteq \mathcal{M}$  by  $\mu \in \mathcal{M}_i$  if and only if  $\zeta_1, \zeta_2, \dots, \zeta_i$  do not occur as variables in  $\mu$ . Thus  $\mathcal{M}_n \subseteq \{0\}$  where 0 is the zero monomial.

Under the assumption that  $[G: \Delta_k] = \infty$  we prove by induction on  $i = 0, 1, \dots, n$  that for all  $x_{i+1}, x_{i+2}, \dots, x_n \in G$  either

$$f_i(x_{i+1}, x_{i+2}, \cdots, x_n) = 0$$

or there exists  $\mu \in \mathscr{M}_i$  with  $\operatorname{Supp} \mu(x_{i+1}, x_{i+2}, \dots, x_n) \cap \mathscr{A}_k \neq \emptyset$ . Since  $f_0 = f$  is an identity satisfied by K[G] the result for i = 0 is clear.

Suppose the inductive result holds for some i-1 < n. Fix  $x_{i+1}, x_{i+2}, \dots, x_n \in G$  and let  $x \in G$  play the role of the *i*th variable. Let  $\mu \in \mathscr{M}_i$ . If  $\operatorname{Supp} \mu(x_{i+1}, \dots, x_n) \cap \mathscr{A}_k \neq \emptyset$  we are done. Thus we may assume that  $\operatorname{Supp} \mu(x_{i+1}, \dots, x_n) \cap \mathscr{A}_k = \emptyset$  for all  $\mu \in \mathscr{M}_i$ . Set  $\mathscr{M}_{i-1} - \mathscr{M}_i = \mathscr{N}_{i-1}$ .

Now let  $\mu \in \mathcal{N}_{i-1}$  so that  $\mu$  involves the variable  $\zeta_i$ . Write  $\mu = \mu' \zeta_i \mu''$  where  $\mu'$  and  $\mu''$  are monomials in the variables  $\zeta_{i+1}, \dots, \zeta_n$ . Then Supp  $\mu(x, x_{i+1}, \dots, x_n) \cap \mathcal{A}_k \neq \emptyset$  implies that

$$x \in h^{\prime-1} \varDelta_k h^{\prime \prime-1} = \varDelta_k h^{\prime-1} h^{\prime \prime-1}$$

where  $h' \in \text{Supp } \mu'(x_{i+1}, \dots, x_n)$  and  $h'' \in \text{Supp } \mu''(x_{i+1}, \dots, x_n)$ . Thus it follows that for all  $x \in G - T$  where

$$T = \bigcup_{\mu \in N_{i-1} \atop h', h''} \varDelta_k h'^{-1} h''^{-1}$$

we have  $\operatorname{Supp} \mu(x, x_{i+1}, \dots, x_n) \cap \Delta_k = \emptyset$  for all  $\mu \in \mathcal{M}_{i-1}$ . Thus by the inductive result for i-1 we conclude that for all  $x \in G-T$  we have  $f_{i-1}(x, x_{i+1}, \dots, x_n) = 0$ . Note that T is a finite union of right translates of  $\Delta_k$ , a subset of G of infinite index.

Now clearly

$$\begin{split} f_{i-1}(x, x_{i+1}, \cdots, x_n) \\ &= \sum_{\sigma \in S^i} \sum_{j=1}^{a_{\sigma}} \theta(\alpha_{0,\sigma,j}) \theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-2,\sigma,j}) \alpha_{i-1,\sigma,j} x \alpha_{i,\sigma,j} x_{\sigma(i+1)} \cdots \alpha_{n-1,\sigma,j} x_{\sigma(n)} \alpha_{n,\sigma,j} \\ &+ \sum_{\mu \in \mathcal{M}_i} \mu(x_{i+1}, \cdots, x_n) x \eta(x_{i+1}, \cdots, x_n) \end{split}$$

where the  $\eta(\zeta_{i+1}, \dots, \zeta_n)$  are suitable monomials. Since

$$f_{i-1}(x, x_{i+1}, \cdots, x_n) = 0$$

for all  $x \in G - T$  we can apply Lemma 2.2. However we must first observe that the hypotheses are satisfied.

Let r and s be defined as in Lemma 2.2. Using the basic fact that

$$|\operatorname{Supp} \alpha \beta| \leq |\operatorname{Supp} \alpha| |\operatorname{Supp} \beta|$$

for any  $\alpha, \beta \in K[G]$  it follows easily that

$$r \leqq a^{n+1} = r_{\scriptscriptstyle 0}, \qquad s \leqq a^{n+1} = s_{\scriptscriptstyle 0}$$
 .

Now  $\mu \in \mathscr{M}_i$  implies that  $\operatorname{Supp} \mu(x_{i+1}, \dots, x_n) \cap \mathcal{A}_k = \emptyset$ . Therefore the only left hand factors of x which have some support in  $\mathcal{A}_k$  come from the first of the two sums above. Here we have

Supp 
$$\theta(\alpha_{i \sigma, j}) \subseteq U \subseteq \Delta_b$$

and  $(\mathcal{\Delta}_b)^{n+1} \subseteq \mathcal{\Delta}_{b^{n+1}} = \mathcal{\Delta}_t$ . Thus the intersection of the supports of these left hand factors with  $\mathcal{\Delta}_k$  is easily seen to be contained in  $\mathcal{\Delta}_t$ . Finally

$$k = r_{\scriptscriptstyle 0} s_{\scriptscriptstyle 0} t^{r_{\scriptscriptstyle 0}} \ge rst^r$$

so the lemma applies.

There are two possible conclusions from Lemma 2.2. The first is that  $[G:T] < \infty$ . Since T is a finite union of right translates of  $\Delta_k$  this yields  $[G:\Delta_k] < \infty$ , a contradiction by our assumption. Thus the second conclusion must hold. Since as we observed above

$$\theta_k(\mu(x_{i+1}, \cdots, x_n)) = 0$$

and clearly

$$\theta_{k} \left[ \theta(\alpha_{0,\sigma,j}) \theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-2,\sigma,j}) \alpha_{i-1,\sigma,j} \right] \\ = \theta(\alpha_{0,\sigma,j}) \theta(\alpha_{1,\sigma,j}) \cdots \theta(\alpha_{i-2,\sigma,j}) \theta(\alpha_{i-1,\sigma,j})$$

we therefore obtain

$$0 = \sum_{\sigma \in S^i} \sum_{j=1}^{\alpha_{\sigma}} \theta(\alpha_{0 \sigma j}) \theta(\alpha_{1 \sigma j}) \cdots \theta(\alpha_{i-1 \sigma j}) \alpha_{i \sigma j} x_{\sigma(i+1)} \cdots \alpha_{n-1,\sigma,j} x_{\sigma(n)} \alpha_{n,\sigma,j}$$
  
=  $f_i(x_{i+1}, x_{i+2}, \cdots, x_n)$ 

and the induction step is proved.

In particular, we conclude for i = n that either  $f_n = 0$  or there exists  $\mu \in \mathcal{M}_n$  with  $\operatorname{Supp} \mu \cap \varDelta_k \neq \emptyset$ . However  $f_n$  is known to be a a nonzero constant function and  $\mathcal{M}_n \subseteq \{0\}$ . Hence we have a contradiction and we must therefore have  $[G: \varDelta_k] < \infty$ . By Lemma 2.3 this yields  $[G: \varDelta(G)] < \infty$  and  $|\varDelta(G)'| < \infty$  so the result follows.

3. Polynomial parts. Let E be an algebra over K. We say that E has a polynomial part it and only if E has an idempotent e such that eEe satisfies a polynomial identity. In this section we obtain necessary and sufficient conditions for K[G] to have a polynomial part.

We first discuss some well known properties of the standard polynomial  $s_n$  of degree n. Here

$$s_n(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

Suppose A is a subset of  $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$  of size a. Then we let  $s_a(A)$  denote  $s_a$  evaluated at these variables. This is of course only determined up to a plus or minus sign.

LEMMA 3.1. Let  $a_1, a_2, \dots, a_r$  be fixed integers with

$$a_{\scriptscriptstyle 1}+a_{\scriptscriptstyle 2}+\cdots+a_{\scriptscriptstyle r}=n$$
 .

Then

$$s_n(\zeta_1,\,\zeta_2,\,\cdots,\,\zeta_n)=\sum_{A_1,\,A_2,\,\cdots,\,A_r}\pm s_{a_1}(A_1)s_{a_2}(A_2)\,\cdots\,s_{a_r}(A_r)$$

where  $A_1, A_2, \dots, A_r$  run through all subsets of  $\{\zeta_1, \zeta_2, \dots, \zeta_n\}$  with  $|A_i| = a_i$  and  $A_1 \cup A_2 \cup \dots \cup A_r = \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ .

*Proof.* Consider all those terms in the sum for  $s_n$  such that the first  $a_1$  variables come from  $A_1$ , the next  $a_2$  variables come from  $A_2$ , etc. Then the subsum of all such terms is easily seen to be

$$\pm s_{a_1}(A_1)s_{a_2}(A_2)\cdots s_{a_r}(A_r)$$
.

This clearly yields the result.

THEOREM 3.2. Let K[G] be a group ring of G over K which satisfies a polynomial identity. Then K[G] satisfies a standard polynomial identity.

*Proof.* If K has characteristic 0 then Theorem 1.1 of [3] and proof of (i) of that theorem show that K[G] satisfies a standard identity. If K has characteristic p > 0 then Theorem 1.3 of [3] and

a slight modification of the proof of (i) of that theorem show that K[G] satisfies

$$s_{2n}(\zeta_1, \zeta_2, \cdots, \zeta_{2n}) s_{2n}(\zeta_{2n+1}, \zeta_{2n+2}, \cdots, \zeta_{4n}) \cdots$$
  
 $\cdots s_{2n}(\zeta_{2(m-1)n+1}, \zeta_{2(m-1)n+2}, \cdots, \zeta_{2mn}).$ 

Of course it also satisfies this polynomial with all possible permutations of the 2mn variables. Thus by Lemma 3.1 K[G] satisfies  $s_{2mn}$ .

THEOREM 3.3. Let K[G] be a group ring of G over K. Then the following are equivalent.

(i)  $[G: \varDelta(G)] < \infty$  and  $|\varDelta(G)'| < \infty$ .

(ii) K[G] satisfies a nondegenerate multilinear generalized polynomial identity.

(iii) K[G] has polynomial part.

(iv) K[G] has a central idempotent e such that eK[G] satisfies a standard identity.

*Proof.* (iv)  $\Rightarrow$  (iii). This is obvious.

(iii)  $\Rightarrow$  (ii). Let *e* be an idempotent such that E = eK[G]e satisfies a polynomial identity. By Lemma 3.2 of [3], *E* satisfies an identity of the form

$$g(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} b_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$
.

If  $\alpha \in K[G]$  then of course  $e\alpha e \in E$ . This shows immediately that K[G] satisfies the multilinear generalized polynomial identity

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} b_{\sigma} e \zeta_{\sigma(1)} e \zeta_{\sigma(2)} e \cdots e \zeta_{\sigma(n)} e$$
.

Moreover f is nondegenerate since  $b_{\sigma} \neq 0$  for some  $\sigma$  and then

$$f^{\sigma}(1, 1, \dots, 1) = b_{\sigma} e \neq 0$$
.

(ii)  $\Rightarrow$  (i). This follows from Theorem 2.4.

(i)  $\Rightarrow$  (iv). Suppose first that K has characteristic 0. Let  $H = \Delta(G)'$  so that H is a finite normal subgroup of G. Set

$$e = rac{1}{|H|} \sum_{x \in H} x \in K[G]$$
 .

Then e is a central idempotent in K[G] and eK[G] is easily seen to be isomorphic to K[G/H]. Now G/H has an abelian subgroup  $\Delta(G)/H$ of finite index so by Theorem 3.2 and Theorem 1.1 of [3],

$$eK[G] \cong K[G/H]$$

satisfies a standard identity.

Now let K have characteristic p > 0 and let  $A = C_{A(G)}(\Delta(G)')$ . Then A is normal in G,  $[G:A] < \infty$  and  $A' \subseteq \Delta(G)'$  so A' is central in A. Let H be the normal p-compliment of A' and define e as above. Then again e is central in K[G] and  $eK[G] \cong K[G/H]$ . Since G/H has a p-abelian subgroup A/H of finite index it follows from Theorem 3.2 and Theorem 1.3 of [3] that K[G/H] satisfies a standard identity. This completes the proof of the theorem.

#### References

1. D. S. Passman, Radicals of twisted group rings, Proc. London Math. Soc., 20 (1970) 409-437.

2. \_\_\_\_, Linear identies in group rings II, Pacific J. Math., 36 (1971), 485-505.

3. \_\_\_\_, Group rings satisfying a polynomial identity, J. of Algebra. (to appear).

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