GROUP RINGS SATISFYING A POLYNOMIAL IDENTITY. II

DONALD STEVEN PASSMAN
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D. S. PASSMAN

In an earlier paper we obtained necessary and sufficient conditions for the group ring $K[G]$ to satisfy a polynomial identity. In this paper we obtain similar conditions for a twisted group ring $K^t[G]$ to satisfy a polynomial identity. We also consider the possibility of $K[G]$ having a polynomial part.

1. Twisted group rings. Let $K$ be a field and let $G$ be a (not necessarily finite) group. We let $K^t[G]$ denote a twisted group ring of $G$ over $K$. That is $K^t[G]$ is an associative $K$-algebra with basis $\{x | x \in G\}$ and with multiplication defined by

$$\bar{x}y = \gamma(x, y)\bar{x}y, \quad \gamma(x, y) \in K - \{0\}.$$ 

The associativity condition is equivalent to $\bar{x}(y\bar{z}) = (\bar{x}y)\bar{z}$ for all $x, y, z \in G$ and this is equivalent to

$$\gamma(x, yz)\gamma(y, z) = \gamma(x, y)\gamma(xy, z).$$

We call the function $\gamma: G \times G \rightarrow K - \{0\}$ the factor system of $K^t[G]$. If $\gamma(x, y) = 1$ for all $x, y \in G$ then $K^t[G]$ is in fact the ordinary group ring $K[G]$. In this section we offer necessary and sufficient conditions for $K^t[G]$ to satisfy a polynomial identity. The proof follows the one for $K[G]$ given in [3] and we only indicate the suitable modifications needed. The following is Lemma 1.1 of [2].

**Lemma 1.1.** If $x \in G$, then in $K^t[G]$ we have

(i) \(1 = \gamma(1, 1)^{-1}\bar{1}\)

(ii) \(\bar{x}^{-1} = \gamma(x, x^{-1})^{-1}\gamma(1, 1)^{-1}\bar{x}^{-1} = \gamma(x^{-1}, x)^{-1}\gamma(1, 1)^{-1}\bar{x}^{-1}\).

**Proposition 1.2.** Suppose $K^t[G]$ satisfies a polynomial identity of degree $n$ and set $k = (n!)^2$. Then $G$ has a characteristic subgroup $G_0$ such that $[G: G_0] \leq (k + 1)!$ and such that for all $x \in G_0$

$$[G: C_0(x)] \leq k^{k(k+1)!}.$$ 

**Proof.** This is the twisted analog of Corollary 3.5 of [3]. We consider §3 of [3] and observe that each of the prerequisite results for that corollary also has a twisted analog.
First Lemma 3.1 of [3] holds for \( K'[G] \) with no change in the proof. Of course \( x \) must be replaced by \( \bar{x} \) in the formula\
\[
\alpha_1\bar{x}\beta_1 + \alpha_2\bar{x}\beta_2 + \cdots + \alpha_n\bar{x}\beta_n = \bar{x}\gamma.
\]
Second Theorem 3.4 of [3] also holds for \( K'[G] \) with no change in its statement. The proof is modified just slightly so that the inductive result to be proved is as follows. For each \( x_j, x_{j+1}, \ldots, x_n \in G \), then either \( f_j(\bar{x}_j, \bar{x}_{j+1}, \ldots, \bar{x}_n) = 0 \) or for some \( \mu \in \mathcal{L} \), \( \mu(\bar{x}_j, \bar{x}_{j+1}, \ldots, \bar{x}_n) = a \bar{y} \) for some \( a \in K - \{0\} \), \( y \in \mathcal{L} \). Then replacing \( x \)'s suitably by \( \bar{x} \)'s the proof carries through as before. Finally Corollary 3.5 of [3] holds for \( K'[G] \) since it is just a group theoretic consequence of Theorem 3.4 of [3].

Let \( K'[G] \) be a twisted group ring and let \( H \) be a subgroup of \( G \). Then by \( K'[H] \) we mean that twisted group ring of \( H \) which is naturally contained in \( K'[G] \). Let \( JK'[G] \) denote the Jacobson radical of \( K'[G] \).

**Proposition 1.3.** Suppose \( K'[G] \) satisfies a polynomial identity of degree \( n \) and suppose further that \( G' \) is finite and \( K'[G'] \) is central in \( K'[G] \). Then \( G \) has a subgroup \( Z \supseteq G' \) such that
\[
[G: Z] \leq (n/2)^{|G'|}
\]
with \( K'[Z]/(JK'[G'] \cdot K'[Z]) \) commutative.

**Proof.** Since \( K'[G'] \) is commutative, \( JK'[G'] \) is the intersection of the maximal two-sided ideals of \( K'[G'] \). Moreover \( K'[G']/JK'[G'] \) is a finite dimensional semisimple algebra and hence it has at most
\[
\dim_K K'[G']/JK'[G'] \leq |G'|
\]
maximal two-sided ideals. Thus we may write
\[
JK'[G'] = \bigcap_i I_i, \quad m \leq |G'|
\]
where \( I_i \) is a maximal two-sided ideal of \( K'[G'] \).

Fix a subscript \( i \). Then \( K'[G']/I_i = F_i \), some finite field extension of \( K \). Now \( K'[G'] \) is central in \( K'[G] \), so \( I_i \cdot K'[G] \) is an ideal in \( K'[G] \). It is now easy to see that \( K'[G']/(I_i \cdot K'[G]) \) is an \( F_i \)-algebra with a basis consisting of the images of coset representatives for \( G' \) in \( G \). Thus clearly \( K'[G']/(I_i \cdot K'[G]) \) is isomorphic to some twisted group ring \( F_i \cdot K'[G']/G' \), and this twisted group ring inherits the polynomial identity satisfied by \( K'[G] \). Thus by Proposition 1.4 of [2], \( G/G' \) has a subgroup \( \bar{Z}_i \) with \( [G/G': \bar{Z}_i] \leq (n/2)^2 \) and with \( F_i \cdot [Z_i] \) central in \( F_i \cdot [G'/G'] \). Let \( Z_i \) be the complete inverse image
of $Z_i$ in $G$. Then $Z_i \supseteq G'$, $[G: Z_i] \leq (n/2)^2$ and for all $\alpha, \beta \in K'[Z_i]$ we have $\alpha \beta - \beta \alpha \in I_i \cdot K'[G]$.

Set $Z = \bigcap^n Z_i$. Then

$$[G: Z] \leq \prod^n [G: Z_i] \leq (n/2)^{2m} \leq (n/2)^{\omega[G']}. $$

Moreover for all $\alpha, \beta \in K'[Z]$ we have

$$\alpha \beta - \beta \alpha \in \bigcap^n I_i \cdot K'[G] = JK'[G'] \cdot K'[G]$$

since $K'[G]$ is free over $K'[G']$. Hence since $K'[G]$ is free over $K'[Z]$ we have

$$\alpha \beta - \beta \alpha \in K'[Z] \cap (JK'[G'] \cdot K'[G]) = JK'[G'] \cdot K'[Z]$$

and the result follows.

We now come to our main result on twisted group rings satisfying a polynomial identity.

**Theorem 1.4.** Let $K'[G]$ be a twisted group ring of $G$ over $K$. Let $G \supseteq A \supseteq B$ be subgroups of $G$ with $B$ finite and central in $A$ and with $K'[A]/(JK'[B] \cdot K'[A])$ commutative.


(ii) If $K'[G]$ satisfies a polynomial identity of degree $n$, then there exists suitable $A$ and $B$ with $[G: A] \cdot |B|$ bounded by some fixed function of $n$.

**Proof.** The proof of (i) is identical to the proof of Theorem 1.3 (i) of [3]. Observe that $JK'[B] \cdot K'[A] = K'[A] \cdot JK'[B]$ is an ideal of $K'[A]$ by Lemma 1.2 of [1].

We now consider part (ii). Let $K'[G]$ satisfy a polynomial identity of degree $n$. Set

$$a = a(n) = (n!)^2, \quad b = b(n) = a^{(a+1)!}. $$

Then by Proposition 1.2 $G$ has a subgroup $G_0$ with

$$[G: G_0] \leq (a + 1)!, \quad G_0 = \Delta_0(G_0)$$

where $\Delta_k$ is defined in [3].

Set

$$c = c(n) = (b!)^{2^k}, \quad d = d(n) = (n/2)^{2c}. $$

Then by Theorem 4.4 of [3], $|G_0| \leq c$. Let $G_i = C_{\Delta_0}(G_0)$. Then

$$G_i \supseteq G_0,$$

so $G_i$ is a finite central subgroup of $G_i$. Moreover

$$|G_i| \leq c, \quad [G_0: G_i] \leq c!. $$
Let \( x \in G_i \). Then conjugation by \( x \) induces an automorphism of \( K'[G_i] \). Moreover since \( G'_i \) is central in \( G_i \) we have
\[
\bar{x}^{-1}y\bar{x} = \lambda_x(y)\bar{y}
\]
for all \( y \in G'_i \). It follows easily that \( \lambda_x \) is a linear character of \( G'_i \) into \( K \), that is \( \lambda_x \in \text{Hom}(G'_i, K - \{0\}) \). In addition, it follows easily that the map \( x \mapsto \lambda_x \) is in fact a group homomorphism
\[
G_{\lambda} \longrightarrow \text{Hom}(G'_i, K - \{0\}) .
\]
Let \( G_2 \) denote the kernel of this homomorphism. Then
\[
[G_i : G_2] \leq |\text{Hom}(G'_i, K - \{0\})| \leq |G'_i| \leq c .
\]
Set \( B = G'_2 \). Then \( B \subseteq G'_i \) so \( |B| \leq c \) and \( K'[B] \) is central in \( K'[G_2] \). By Proposition 1.3, \( G_2 \) has a subgroup \( A \supseteq B \) with
\[
[G_2 : A] \leq (n/2)^{|B|} \leq d
\]
and with \( K'[A]/(JK'[B] \cdot K'[A]) \) commutative. Since \( |B| \leq c \) and since
\[
[G : A] = [G : G_0] [G_0 : G_i] [G_i : G_2] [G_2 : A] \leq (a + 1)! \cdot c \cdot c \cdot d
\]
the result follows.

It is interesting to interpret this result for various fields. If \( K \) has characteristic 0 and if \( B \) is a finite group, then \( K'[B] \) is semisimple by Proposition 1.5 of [1]. Thus

**Corollary 1.5.** Let \( K'[G] \) be a twisted group ring of \( G \) over \( K \) and let \( K \) have characteristic 0. Let \( A \) be an abelian subgroup of \( G \) with \( K'[A] \) commutative.

(i) If \( [G : A] < \infty \) then \( K'[G] \) satisfies a polynomial identity of degree \( n = 2[G : A] \).

(ii) If \( K'[G] \) satisfies a polynomial identity of degree \( n \), then there exists such a group \( A \) with \( [G : A] \) bounded by some fixed function of \( n \).

**Corollary 1.6.** Let \( K'[G] \) be a twisted group ring of \( G \) over \( K \) and let \( K \) have characteristic \( p > 0 \). Let \( G \supseteq A \supseteq P \) be subgroups of \( G \) with \( P \) a finite \( p \)-group central in \( A \) and with \( K'[A]/(JK'[P] \cdot K'[A]) \) commutative.

(i) If \( [G : A] < \infty \) then \( K'[G] \) satisfies a polynomial identity of degree \( n = 2[G : A] \cdot |P| \).

(ii) If \( K'[G] \) satisfies a polynomial identity of degree \( n \), then there exists suitable \( A \) and \( P \) with \( [G : A] \cdot |P| \) bounded by some fixed function of \( n \).
Proof. Let $B$ be given as in Theorem 1.4 and let $P$ be its normal Sylow $p$-subgroup. Then $P$ is also central in $A$. Moreover by Proposition 1.5 of [1] $JK'[B] = JK'[P] \cdot K'[B]$ so the result clearly follows.

Finally in the above if $K$ is a perfect field of characteristic $p$, then by Lemma 2.1 of [1], $K'[P] \cong K[P]$ so $K'[P]/JK'[P] = K$. It then follows easily that

$$K'[A]/(JK'[P] \cdot K'[A]) \cong K'[A/P]$$

is in fact some twisted group ring of $A/P$.

2. Generalized polynomial identities. Let $E$ be an algebra over $K$. A generalized polynomial over $E$ is, roughly speaking, a polynomial in the indeterminates $\zeta_1, \zeta_2, \ldots, \zeta_n$ in which elements of $E$ are allowed to appear both as coefficients and between the indeterminates. We say that $E$ satisfies a generalized polynomial identity if there exists a nonzero generalized polynomial $f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ such that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$ for all $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$. The problem here is precisely what does it mean for $f$ to be nonzero. For example, suppose that the center of $E$ is bigger than $K$ and let $\alpha$ be a central element not in $K$. Then $E$ satisfies the identity $f(\zeta_1) = \alpha \zeta_1$ but surely this must be considered trivial. Again, suppose that $E$ is not prime. Then we can choose nonzero $\alpha, \beta \in E$ such that $E$ satisfies the identity $f(\zeta_1) = \alpha \zeta \beta$ and this must also be considered trivial. We avoid these difficulties by restricting the allowable form of the polynomials.

We say that $f$ is a multilinear generalized polynomial of degree $n$ if

$$f(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{\sigma \in S_n} f^\sigma(\zeta_1, \zeta_2, \ldots, \zeta_n)$$

and

$$f^\sigma(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{j=1}^{\delta} \alpha_{\sigma,j} \zeta_{\sigma(1)} \zeta_1 \alpha_{\sigma,j} \zeta_{\sigma(2)} \alpha_2 \ldots \alpha_{n-1} \alpha_{\sigma,j} \zeta_{\sigma(n)}$$

where $\alpha_{i,j} \in E$ and $\alpha_{\sigma}$ is some positive integer. This form is of course motivated by Lemma 3.2 of [3]. The above $f$ is said to be nondegenerate if for some $\sigma \in S_n$, $f^\sigma$ is not a polynomial identity satisfied by $E$. Otherwise $f$ is degenerate.

In this section we will study group rings $K[G]$ which satisfy nondegenerate multilinear generalized polynomial identities. Let $\Delta = \Delta(G)$ denote the F. C. subgroup of $G$ and let $\theta: K[G] \to K[\Delta(G)]$ denote the natural projection.
LEMMA 2.1. Suppose $K[G]$ satisfies a nondegenerate multilinear generalized polynomial of degree $n$. Then $K[G]$ satisfies a polynomial identity as given above with

$$\sum_{j=1}^{a_1} \theta(\alpha_{0,1,j}) \theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{n,1,j}) \neq 0.$$ 

Proof. Let $K[G]$ satisfy $f$ as above. Since $f$ is nondegenerate, by reordering the $\zeta$'s if necessary, we may assume that $f^1(\zeta_1, \zeta_2, \cdots, \zeta_n)$ is not an identity for $K[G]$. Thus since $f^1$ is multilinear there exists $x_1, x_2, \cdots, x_n \in G$ with

$$0 \neq f^1(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{a_1} \alpha_{0,1,j} x_1 \alpha_{1,1,j} x_2 \cdots \alpha_{n-1,1,j} x_n \alpha_{n,1,j}.$$ 

If we replace $\zeta_i$ in $f$ by $x_i \zeta_i$ we see clearly that $K[G]$ satisfies a suitable $f$ with

$$(*) \quad 0 \neq \sum_{j=1}^{a_1} \alpha_{0,1,j} \alpha_{1,1,j} \cdots \alpha_{n,1,j}.$$ 

For each $i, j$ write

$$\alpha_{i,1,j} = \sum_k \beta_{ijk} y_k$$

where $\beta_{ijk} \in K[\Delta]$ and $\{y_k\}$ is a finite set of coset representatives for $\Delta$ in $G$. We substitute this into (*) above. It then follows easily that for some $k_0, k_1, \cdots, k_n$ we have

$$0 \neq \sum_{j=1}^{a_1} \beta_{0jk_0} y_{k_0} \beta_{1jk_1} y_{k_1} \cdots \beta_{njk_n} y_{k_n}. $$

Thus if $z_i$ is defined by $z_i = y_{k_0} y_{k_1} \cdots y_{k_{i-1}}$ and $z_0 = 1$ then

$$0 \neq \sum_{j=1}^{a_1} \beta_{0jk_0}^{-1} \beta_{1jk_1}^{-1} \cdots \beta_{njk_n}^{-1}. $$

Now $\beta_{ijk_i} = \theta(\alpha_{i,1,j} y_{k_i}^{-1})$ so

$$\beta_{ijk_i}^{-1} = \theta(z_i \alpha_{i,1,j} y_{k_i}^{-1} z_i^{-1}) = \theta(z_i \alpha_{i,1,j} z_i^{-1}).$$

It therefore follows that if we replace $\zeta_i$ in $f$ by $z_i^{-1} \zeta_i z_{i+1}$ and if, in addition, we multiply $f$ on the left by $z_0$ and on the right by $z_n^{-1}$, then this new multilinear generalized polynomial identity obtained has the required property.

LEMMA 2.2. Let $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_n \in K[G]$. Suppose that for some integers $k$ and $t$
\[ | \bigcup_i \text{Supp } \alpha_i | = r, \quad | \bigcup_i \text{Supp } \beta_i | = s \]

and

\[(\bigcup_i \text{Supp } \alpha_i) \cap \Delta_k(G) \subseteq \Delta_i(G) \]

with \( k \geq r s t^r \). Let \( T \) be a subset of \( G \) and suppose that for all \( x \in G - T \) we have

\[ \alpha_1 x \beta_1 + \alpha_2 x \beta_2 + \cdots + \alpha_s x \beta_s = 0. \]

Then either \([G: T] < (k + 2)!\) or

\[ \theta_k(\alpha_1) \beta_1 + \theta_k(\alpha_2) \beta_2 + \cdots + \theta_k(\alpha_s) \beta_s = 0. \]

**Proof.** Let \( A = \bigcup_i \text{Supp } \alpha_i, B = \bigcup_i \text{Supp } \beta_i \) and write

\[ A' = A \cap \Delta_k = \{g_1, g_2, \cdots, g_n\} \]

\[ A'' = A - \Delta_k = \{y_1, y_2, \cdots, y_m\} \]

\[ B = \{z_1, z_2, \cdots, z_s\}. \]

Here of course \( m + n = r \). Set \( W = \bigcap_i C_G(g_i) \). Since by assumption \( A' \subseteq \Delta_i(G) \) we have clearly \([G: W] \leq t^r \). Observe that for all \( x \in W \), \( x \) centralizes \( \theta_k(\alpha_i) \).

Suppose that

\[ \gamma = \theta_k(\alpha_1) \beta_1 + \theta_k(\alpha_2) \beta_2 + \cdots + \theta_k(\alpha_s) \beta_s \neq 0 \]

and let \( v \in \text{Supp } \gamma \). If \( y_i \) is conjugate to \( v z_i^{-1} \) in \( G \) for some \( i, j \) choose \( h_{ij} \in G \) with \( h_{ij}^{-1} y_i h_{ij} = v z_j^{-1} \).

Write \( \alpha_i = \theta_k(\alpha_i) + \alpha_i' \) and then write

\[ \alpha_i' = \sum a_{ij} y_j, \quad \beta_i = \sum b_{ij} z_j. \]

Let \( x \in W - T \). Then we must have

\[ 0 = x^{-1} \alpha_1 x \beta_1 + x^{-1} \alpha_2 x \beta_2 + \cdots + x^{-1} \alpha_s x \beta_s = [\theta_k(\alpha_1) \beta_1 + \theta_k(\alpha_2) \beta_2 + \cdots + \theta_k(\alpha_s) \beta_s] + [\alpha_i' \beta_1 + \alpha_i' \beta_2 + \cdots + \alpha_i' \beta_s]. \]

Since \( v \) occurs in the support of the first term it must also occur in the second and hence there exists \( y_i, z_j \) with \( v = y_i^j z_j \) or

\[ x^{-1} y_i x = v z_j^{-1} = h_{ij}^{-1} y_i h_{ij}. \]

Thus \( x \in C_G(y_i) h_{ij} \). We have therefore shown that

\[ W \subseteq T \cup \bigcup_{i,j} C_G(y_i) h_{ij}. \]
Let $w_1, w_2, \ldots, w_d$ be a complete set of coset representatives for $W$ in $G$. Then $d = [G: W] \leq t^r$ and the above yields
\[ G = Tw_1 \cup Tw_2 \cup \cdots \cup Tw_d \cup S \]
where
\[ S = \bigcup_{i,j,e} C_0(y_i) h_i w_e. \]
Now the number of cosets in the above union for $S$ is at most
\[ rsd \leq rst^r \leq k \]
by assumption on $k$. Moreover $y_i \in A_k$ so $[G : C_0(y_i)] > k$ for all $i$. Thus by Lemma 2.3 of [3] $S \neq G$ and then Lemma 2.1 of [3] yields
\[ [G : T^*] \leq (k + 1)! \]
where
\[ T^* = \bigcup_e Tw_e. \]
Thus
\[ [G : T] \leq (k + 1)! \quad d \leq (k + 1)! \quad (k + 2) \]
and the result follows.

We will need the following group theoretic lemma.

**Lemma 2.3.** Let $G$ be a group. The following are equivalent

(i) $[G : \Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$.

(ii) There exists an integer $k$ with $[G : \Delta_k(G)] < \infty$.

**Proof.** Suppose that $G$ satisfies (i) and set $n = [G : \Delta]$, $m = |\Delta'|$. If $x \in \Delta$, then by Theorem 4.4 (i) of [3], $[\Delta : C_\Delta(x)] \leq m$ and hence $[G : C_0(x)] \leq nm$. Thus (ii) follows with $k = mn$.

Now suppose that (ii) holds. Since $\Delta(G) \supseteq \Delta_k(G)$ and $[G : \Delta_k] < \infty$ we conclude that $[G : \Delta] < \infty$. Now $\Delta(G)$ is a subgroup of $G$ so every right translate of $\Delta_k$ in $G$ is either entirely contained in $\Delta$ or is disjoint from $\Delta$. This implies that $[\Delta : \Delta_k] < \infty$ and say
\[ \Delta = \Delta_k y_1 \cup \Delta_k y_2 \cup \cdots \cup \Delta_k y_r. \]
Since each $y_i \in \Delta$ we can set $n = \max_i [G : C(y_i)] < \infty$. If $x \in \Delta$ then $x \in \Delta_k y_i$ for some $i$ and this implies easily that $[G : C(x)] \leq nk$. Thus $[\Delta : C_\Delta(x)] \leq nk$ and by Theorem 4.4 (ii) of [3], $|\Delta'| < \infty$.

We now come to the main result of this section

**Theorem 2.4.** Let $K[G]$ be a group ring of $G$ over $K$ and sup-
pose that $K[G]$ satisfies a nondegenerate multilinear polynomial identity. Then $[G: Δ(G)] < ∞$ and $|Δ(G)'| < ∞$.

Proof. By Lemma 2.1, we may assume that $K[G]$ satisfies

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{s \in S_n} \sum_{j=1}^{a_s} a_{0, s, j} \zeta_{\sigma(1)} \alpha_{1, s, j} \zeta_{\sigma(2)} \cdots \zeta_{n-1, s, j} \zeta_{\sigma(n)} \alpha_{n, s, j}$$

with

$$\sum_{j=1}^{a_s} \theta(\alpha_{0,1,j}) \theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{n,1,j}) \neq 0.$$

We first define a number of numerical parameters associated with $f$. Set

$$a = \sum_{s \in S_n} \sum_{j=1}^{a_s} \sum_{i=1}^{n} |\text{Supp} \alpha_{i,s,j}|$$

and

$$r_0 = s_0 = a^{n+1}.$$

Now consider

$$U = \bigcup_{s \in S_n} \bigcup_{j=1}^{a_s} \bigcup_{i=0}^{n} \text{Supp} \theta(\alpha_{i,s,j}).$$

Then $U$ is a finite subset of $Δ(G)$ so there exists an integer $b$ with $U \subseteq Δ_0(G)$. Set

$$t = b^{n+1} \quad \text{and} \quad k = r_0 s_0 t^{b_0}.$$

We assume now that $[G: Δ_k] = ∞$ and derive a contradiction.

For $i = 0, 1, \cdots, n$ define $S^i \subseteq S_n$ by

$$S^i = \{s \in S_n \mid \sigma(1) = 1, \sigma(2) = 2, \cdots, \sigma(i) = i\}.$$

Then $S^0 = S_n$, $S^n = \langle 1 \rangle$ and $S^i$ is just an embedding of $S_{n-i}$ in $S_n$. We define the multilinear generalized polynomial $f_i$ of degree $n-i$ by

$$f_i(\zeta_{i+1}, \zeta_{i+2}, \cdots, \zeta_n)$$

$$= \sum_{s \in S_i} \sum_{j=1}^{a_s} \theta(\alpha_{0,i,j}) \theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{i-1,1,j}) \alpha_{i,s,j} \zeta_{\sigma(i+1)} \cdots \zeta_{n-1, s, j} \zeta_{\sigma(n)} \alpha_{n, s, j}.$$

Thus $f_0 = f$ and

$$f_n = \sum_{j=1}^{a_0} \theta(\alpha_{0,1,j}) \theta(\alpha_{1,1,j}) \cdots \theta(\alpha_{n-1,1,j}) \alpha_{n, 1, j}$$

is a nonzero element of $K[G]$ since
\[ \theta(f_n) = \sum_{j=1}^{a_1} \theta(\alpha_{0,j})\theta(\alpha_{1,j}) \cdots \theta(\alpha_{n-1,j})\theta(\alpha_{n,j}) \not= 0. \]

Let \( \mathcal{M} \) be the set of monomial polynomials obtained as follows. For each \( \sigma, j \) we start with
\[ \alpha_{0,\sigma,j} \alpha_{1,\sigma,j} \alpha_{2,\sigma,j} \cdots \alpha_{n-1,\sigma,j} \alpha_{n,\sigma,j} \]
and we modify it by (1) deleting some but not all of the \( \zeta_i \); (2) replacing some of the \( \alpha_{i,\sigma,j} \) by \( \theta(\alpha_{i,\sigma,j}) \); and (3) replacing some of the \( \alpha_{i,\sigma,j} \) by 1. Then \( \mathcal{M} \) consists of all such monomials obtained for all \( \sigma, j \) and clearly \( \mathcal{M} \) is a finite set. Note that \( \mathcal{M} \) may contain the zero monomial but it contains no nonzero constant monomial since in (1) we do not allow all the \( \zeta_i \) to be deleted.

For \( i = 0, 1, \ldots, n \) define \( \mathcal{M}_i \subseteq \mathcal{M} \) by \( \mu \in \mathcal{M}_i \) if and only if \( \zeta_1, \zeta_2, \ldots, \zeta_i \) do not occur as variables in \( \mu \). Thus \( \mathcal{M}_n \subseteq \{0\} \) where 0 is the zero monomial.

Under the assumption that \( [G: \Delta_k] = \infty \) we prove by induction on \( i = 0, 1, \ldots, n \) that for all \( x_{i+1}, x_{i+2}, \ldots, x_n \in G \) either
\[ f_i(x_{i+1}, x_{i+2}, \ldots, x_n) = 0 \]
or there exists \( \mu \in \mathcal{M}_i \) with \( \text{Supp} \mu(x_{i+1}, x_{i+2}, \ldots, x_n) \cap \Delta_k \not= \emptyset \). Since \( f_0 = f \) is an identity satisfied by \( K[G] \) the result for \( i = 0 \) is clear.

Suppose the inductive result holds for some \( i - 1 < n \). Fix \( x_{i+1}, x_{i+2}, \ldots, x_n \in G \) and let \( x \in G \) play the role of the \( i \)th variable. Let \( \mu \in \mathcal{M}_i \). If \( \text{Supp} \mu(x_{i+1}, \ldots, x_n) \cap \Delta_k \not= \emptyset \) we are done. Thus we may assume that \( \text{Supp} \mu(x_{i+1}, \ldots, x_n) \cap \Delta_k = \emptyset \) for all \( \mu \in \mathcal{M}_i \). Set \( \mathcal{M}_{i-1} = \mathcal{M}_i \).

Now let \( \mu \in \mathcal{M}_{i-1} \) so that \( \mu \) involves the variable \( \zeta_i \). Write \( \mu = \mu'\zeta_i'\mu'' \) where \( \mu' \) and \( \mu'' \) are monomials in the variables \( \zeta_{i+1}, \ldots, \zeta_n \). Then \( \text{Supp} \mu(x, x_{i+1}, \ldots, x_n) \cap \Delta_k \not= \emptyset \) implies that
\[ x \in h' \Delta_k h'^{-1} \]
where \( h' \in \text{Supp} \mu'(x_{i+1}, \ldots, x_n) \) and \( h'' \in \text{Supp} \mu''(x_{i+1}, \ldots, x_n) \). Thus it follows that for all \( x \in G - T \) where
\[ T = \bigcup_{\mu \in \mathcal{M}_{i-1}} \Delta_k h'^{-1} h''^{-1} \]
we have \( \text{Supp} \mu(x, x_{i+1}, \ldots, x_n) \cap \Delta_k = \emptyset \) for all \( \mu \in \mathcal{M}_{i-1} \). Thus by the inductive result for \( i - 1 \) we conclude that for all \( x \in G - T \) we have \( f_{i-1}(x, x_{i+1}, \ldots, x_n) = 0 \). Note that \( T \) is a finite union of right translates of \( \Delta_k \), a subset of \( G \) of infinite index.

Now clearly
\[ f_{i-1}(x_i, x_{i+1}, \ldots, x_n) \]

\[ = \sum_{\sigma \in S^i} \sum_{j=1}^{a^i} \theta(\alpha_0, \sigma, j) \theta(\alpha_1, \sigma, j) \cdots \theta(\alpha_{i-2}, \sigma, j) \alpha_{i-1, \sigma, j} x_{\sigma_1(\sigma+1)} \cdots x_{\sigma_{i-1}(\sigma+1)} \alpha_{i-1, \sigma, j} x_{\sigma_i(\sigma+1)} \cdots x_{\sigma_n(\sigma+1)} \alpha_{n, \sigma, j} \]

\[ + \sum_{\mu \in \mathcal{M}} x_{\sigma_i+1} \cdots x_{\sigma_n} x_{\eta_i(\sigma_i+1)} \cdots x_{\sigma_n} \]

where the \( \eta_i(\sigma_i+1, \ldots, \sigma_n) \) are suitable monomials. Since

\[ f_{i-1}(x_i, x_{i+1}, \ldots, x_n) = 0 \]

for all \( x \in G - T \) we can apply Lemma 2.2. However we must first observe that the hypotheses are satisfied.

Let \( r \) and \( s \) be defined as in Lemma 2.2. Using the basic fact that

\[ |\text{Supp} \alpha \beta| \leq |\text{Supp} \alpha| + |\text{Supp} \beta| \]

for any \( \alpha, \beta \in K[G] \) it follows easily that

\[ r \leq a^{n+1} = r_0, \quad s \leq a^{n+1} = s_0. \]

Now \( \mu \in \mathcal{M} \) implies that \( \text{Supp} \mu(x_{i+1}, \ldots, x_n) \cap \Delta_k = \emptyset \). Therefore the only left hand factors of \( x \) which have some support in \( \Delta_k \) come from the first of the two sums above. Here we have

\[ \text{Supp} \theta(\alpha_{i, \sigma, j}) \subseteq U \subseteq \Delta_b \]

and \( (\Delta_b)^{n+1} \subseteq \Delta_k^{n+1} = \Delta_i \). Thus the intersection of the supports of these left hand factors with \( \Delta_k \) is easily seen to be contained in \( \Delta_i \). Finally

\[ k = r_0 s_0 t r_0 \geq rst \]

so the lemma applies.

There are two possible conclusions from Lemma 2.2. The first is that \( [G: T] < \infty \). Since \( T \) is a finite union of right translates of \( \Delta_k \) this yields \( [G: \Delta_k] < \infty \), a contradiction by our assumption. Thus the second conclusion must hold. Since as we observed above

\[ \theta_k(\mu(x_{i+1}, \ldots, x_n)) = 0 \]

and clearly

\[ \theta_k[\theta(\alpha_0, \sigma, j) \theta(\alpha_1, \sigma, j) \cdots \theta(\alpha_{i-2}, \sigma, j) \alpha_{i-1, \sigma, j}] = \theta(\alpha_0, \sigma, j) \theta(\alpha_1, \sigma, j) \cdots \theta(\alpha_{i-2}, \sigma, j) \theta(\alpha_{i-1, \sigma, j}) \]

we therefore obtain

\[ 0 = \sum_{\sigma \in S^i} \sum_{j=1}^{a^i} \theta(\alpha_0, \sigma, j) \theta(\alpha_1, \sigma, j) \cdots \theta(\alpha_{i-1, \sigma, j}) \alpha_{i, \sigma, j} x_{\sigma_1(\sigma+1)} \cdots x_{\sigma_{i-1}(\sigma+1)} \alpha_{n, \sigma, j} \]

\[ = f_i(x_{i+1}, x_{i+2}, \ldots, x_n) \]

and the induction step is proved.
In particular, we conclude for \( i = n \) that either \( f_n = 0 \) or there exists \( \mu \in \mathcal{M}_n \) with \( \text{Supp} \mu \cap \Delta_k \neq \emptyset \). However \( f_n \) is known to be a nonzero constant function and \( \mathcal{M}_n \subseteq \{0\} \). Hence we have a contradiction and we must therefore have \( |G: \mathcal{A}| < \infty \). By Lemma 2.3 this yields \( |G: \Delta(G)| < \infty \) and \( |\Delta(G)'| < \infty \) so the result follows.

3. Polynomial parts. Let \( E \) be an algebra over \( K \). We say that \( E \) has a polynomial part if and only if \( E \) has an idempotent \( e \) such that \( eEe \) satisfies a polynomial identity. In this section we obtain necessary and sufficient conditions for \( K[G] \) to have a polynomial part.

We first discuss some well known properties of the standard polynomial \( s_n \) of degree \( n \). Here

\[
s_n(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{\sigma \in S_n} (-1)^{\sigma} \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}.
\]

Suppose \( A \) is a subset of \( \{\zeta_1, \zeta_2, \ldots, \zeta_n\} \) of size \( a \). Then we let \( s_a(A) \) denote \( s_n \) evaluated at these variables. This is of course only determined up to a plus or minus sign.

**Lemma 3.1.** Let \( a_1, a_2, \ldots, a_r \) be fixed integers with

\[
a_1 + a_2 + \cdots + a_r = n.
\]

Then

\[
s_n(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{A_1, A_2, \ldots, A_r} \pm s_{a_1}(A_1) s_{a_2}(A_2) \cdots s_{a_r}(A_r)
\]

where \( A_1, A_2, \ldots, A_r \) run through all subsets of \( \{\zeta_1, \zeta_2, \ldots, \zeta_n\} \) with \( |A_i| = a_i \) and \( A_1 \cup A_2 \cup \cdots \cup A_r = \{\zeta_1, \zeta_2, \cdots, \zeta_n\} \).

**Proof.** Consider all those terms in the sum for \( s_n \) such that the first \( a_1 \) variables come from \( A_1 \), the next \( a_2 \) variables come from \( A_2 \), etc. Then the subsum of all such terms is easily seen to be

\[
\pm s_{a_1}(A_1) s_{a_2}(A_2) \cdots s_{a_r}(A_r).
\]

This clearly yields the result.

**Theorem 3.2.** Let \( K[G] \) be a group ring of \( G \) over \( K \) which satisfies a polynomial identity. Then \( K[G] \) satisfies a standard polynomial identity.

**Proof.** If \( K \) has characteristic 0 then Theorem 1.1 of [3] and proof of (i) of that theorem show that \( K[G] \) satisfies a standard identity. If \( K \) has characteristic \( p > 0 \) then Theorem 1.3 of [3] and
a slight modification of the proof of (i) of that theorem show that $K[G]$ satisfies

$$s_{2n}(\zeta_1, \zeta_2, \ldots, \zeta_{2n})s_{2n}(\zeta_{2m+1}, \zeta_{2m+2}, \ldots, \zeta_{4m})\cdots$$

$$\cdots s_{2n}(\zeta_{2(m-1)m+1}, \zeta_{2(m-1)m+2}, \ldots, \zeta_{2mn}).$$

Of course it also satisfies this polynomial with all possible permutations of the $2mn$ variables. Thus by Lemma 3.1 $K[G]$ satisfies $s_{2mn}$.

**Theorem 3.3.** Let $K[G]$ be a group ring of $G$ over $K$. Then the following are equivalent.

(i) $[G: \Delta(G)] < \infty$ and $|\Delta(G)'| < \infty$.


(iii) $K[G]$ has polynomial part.

(iv) $K[G]$ has a central idempotent $e$ such that $eK[G]$ satisfies a standard identity.

**Proof.** (iv) $\Rightarrow$ (iii). This is obvious.

(iii) $\Rightarrow$ (ii). Let $e$ be an idempotent such that $E = eK[G]e$ satisfies a polynomial identity. By Lemma 3.2 of [3], $E$ satisfies an identity of the form

$$g(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{\sigma \in S_n} b_{\sigma \xi_{\sigma(1)} \xi_{\sigma(2)} \ldots \xi_{\sigma(n)}}.$$ 

If $e \in K[G]$ then of course $eae \in E$. This shows immediately that $K[G]$ satisfies the multilinear generalized polynomial identity

$$f(\zeta_1, \zeta_2, \ldots, \zeta_n) = \sum_{\sigma \in S_n} b_{\sigma e^{\xi_{\sigma(1)}} e^{\xi_{\sigma(2)}} e \ldots e^{\xi_{\sigma(n)}} e}.$$ 

Moreover $f$ is nondegenerate since $b_\sigma \neq 0$ for some $\sigma$ and then

$$f^a(1, 1, \ldots, 1) = b_\sigma e \neq 0.$$ 

(ii) $\Rightarrow$ (i). This follows from Theorem 2.4.

(i) $\Rightarrow$ (iv). Suppose first that $K$ has characteristic 0. Let $H = \Delta(G)'$ so that $H$ is a finite normal subgroup of $G$. Set

$$e = \frac{1}{|H|} \sum_{x \in H} x \in K[G].$$

Then $e$ is a central idempotent in $K[G]$ and $eK[G]$ is easily seen to be isomorphic to $K[G/H]$. Now $G/H$ has an abelian subgroup $\Delta(G)/H$ of finite index so by Theorem 3.2 and Theorem 1.1 of [3],

$$eK[G] \cong K[G/H].$$
satisfies a standard identity.

Now let $K$ have characteristic $p > 0$ and let $A = C_{A(G)}(A(G'))$. Then $A$ is normal in $G$, $[G : A] < \infty$ and $A' \subseteq A(G)'$ so $A'$ is central in $A$. Let $H$ be the normal $p$-compliment of $A'$ and define $e$ as above. Then again $e$ is central in $K[G]$ and $eK[G] \cong K[G/H]$. Since $G/H$ has a $p$-abelian subgroup $A/H$ of finite index it follows from Theorem 3.2 and Theorem 1.3 of [3] that $K[G/H]$ satisfies a standard identity. This completes the proof of the theorem.

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