Pacific Journal of Mathematics

COHOMOLOGY OF GROUP GERMS AND LIE ALGEBRAS

Stanisław Sławomir Świerczkowski

Vol. 39, No. 2 June 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 2, 1971

COHOMOLOGY OF GROUP GERMS AND LIE ALGEBRAS

S. ŚWIERCZKOWSKI

Let π be a continuous representation of a Lie group G in a finite dimensional real vector space V. Denote by $H_{\square}(G,V)$ the cohomology with empty supports in the sense of Sze-tsen Hu. If L is the Lie algebra of G, π induces an L-module structure on V and there is the associated cohomology $H(L,V)$ of Chevalley-Eilenberg. Our main result is the construction of an isomorphism $H_{\square}(G, V) \simeq H(L, V)$.

This is preceded by a closer analysis of $H_{\square}(G, V)$. It is clear from the definition that to know $H_{\square}(G, V)$, it suffices to know an arbitrary neighbourhood of 1 in G and its action on V. The totality of neighbourhoods of 1 in G may be regarded as an object of a more fine nature than a local group; we call it a group germ. More precisely, a group germ is defined as a group object in the category Γ of topological germs [18]. The Eilenberg-MacLane definition [3] of the cohomology of an abstract group is carried over from the category of sets to Γ (i.e., from groups to group germs). Thus for any group germs g, a, where a is abelian, and any g-action on a, we have cohomology groups $H(g, a)$. It turns out that $H_{\square}(G, V) \simeq H(g, a)$ for a suitable choice of g and a, in all dimensions > 1. To cope with dim 0 and 1 it seems convenient to introduce the concept of an action of a group germ g on an abelian topological group A and associate with this a cohomology $H(g, A)$. This is only a slight modification of the previous $H(g, a)$, so that both cohomologies coincide in dimensions >1 and $H^1(g, A)$ is a quotient of $H^1(g, a)$, if a is suitably related to A. ($H^0(g, A)$ is the subgroup of g-stable elements of A and $H^0(g, a)$ is always trivial). One now has $H_{\square}(G, V) \simeq H(g, V)$ in all dimensions, for a group germ g corresponding to G.

We are grateful to W.T. van Est for his comments on an earlier version of this paper which have resulted in many improvements.

1. Group germs. Let T be the category of pointed topological spaces. For $A, B \in T$ write $A \simeq B$ if and only if there is a $C \in T$ which is an open subspace of both A and B. Denote by $[A]$ the equivalence class of A. For morphisms $f: A \to B, f': A' \to B'$ in T write $f \simeq f'$ if and only if $A \cong A'$, $B \simeq B'$ and there is a $C \in T$ which is an open subspace of both A and A' such that $f \mid C = f' \mid C$. Denote the equivalence class of $f: A \to B$ by $[f]: [A] \to [B]$. There is now precisely one category Γ whose objects are the equivalence classes $[A]$, the morphisms are the equivalence classes $[f]\colon [A] \to [B]$, and such that $A \mapsto [A]$, $f \mapsto [f]$ is a functor $T \to \Gamma$. Γ will be called the category of *topological germs*. (For a similar definition see [18]).

LEMMA. *The functor $T \to \Gamma$ preserves zero objects and finite products.*

We omit the straightforward verification. As a conclusion, all finite products exist in Γ. Let S be a zero object in T, i.e., a one-point set, and denote the zero object [S] in Γ by e. Any morphism in Γ which factorizes through e will be denoted by 0.

DEFINITION. A group object in Γ will be called a group germ. The category of group germs will be denoted by $Gr\Gamma$.

We recall the definitions. A group object in Γ is an object $g \in \Gamma$ together with morphisms $\mu\colon g \times g \to g$, $\nu\colon g \to g$ such that $\mu(\mu \times 1) = \mu(1 \times \mu)$ (i.e., associativity), $\nu^2 = id$ and

$$\begin{array}{ccccc} g \times e & \xrightarrow[1\times 0]{} & g \times g & \xleftarrow[0\times 1]{} & e \times g \\ & \searrow^{\pi_1} & \downarrow{\scriptstyle\mu} & \swarrow_{\pi_2} & \\ & & g & & \end{array} \quad , \quad \begin{array}{ccccc} g & \xrightarrow[(\nu,1)]{} & g \times g & \xleftarrow[(1,\nu)]{} & g \\ & \searrow^{0} & \downarrow{\scriptstyle\mu} & \swarrow_{0} & \\ & & g & & \end{array}$$

(π_i are the product projections; all diagrams drawn are assumed to commute). A morphism $g \to g'$ in $Gr\Gamma$ is a $\varphi\colon g \to g'$ in Γ such that $\mu'(\varphi \times \varphi) = \varphi\mu$ and $\nu'\varphi = \varphi\nu$.

Let Λ be the category of local topological groups. Following ([8], p. 393) we mean by a local topological group an abstract local group in the sense of Malcev [15] together with a topology on the set Q of its elements such that the map $(x, y) \mapsto xy^{-1}$ is continuous on the domain of its definition and that domain is open in $Q \times Q$. A morphism $Q \to Q'$ in Λ is an $f\colon Q \to Q'$ in T such that $f(x)f(y)$ is defined whenever xy is defined, and if defined, $f(x)f(y) = f(xy)$.

Define a functor $U\colon \Lambda \to Gr\Gamma$ as follows. Given $Q \in \Lambda$, let $j(x) = x^{-1}$ and $\varphi(x, y) = xy$, the domain of φ being an open subspace D of $Q \times Q$, so that $[D] = [Q] \times [Q]$ (cf. Lemma). Let UQ be the topological germ $[Q]$ together with the morphisms $\nu = [j]\colon [Q] \to [Q]$, $\mu = [\varphi]\colon [Q] \times [Q] \to [Q]$ in Γ. Then $UQ \in Gr\Gamma$. For a morphism f in Λ put $Uf = [f]$.

PROPOSITION. *For each $g \in Gr\Gamma$ there exists a $Q \in \Lambda$ such that $g = UQ$.*

Proof. Suppose $g = [A]$, $A \in T$ and denote the base point of A by 1. The definition of a group object in Γ implies the existence of open neighbourhoods P, V, W of 1 in A such that $P \subset V \subset W$ and

(i) there exists $\varphi: W \times W \to A$ such that $\mu = [\varphi]$,

(ii) there exists $j: V \to W$ such that $\nu = [j]$,

(iii) $\varphi(j(x), x) = \varphi(x, j(x)) = 1$, $\varphi(x, 1) = \varphi(1, x) = x$ and both $\varphi(x, \varphi(y, z))$, $\varphi(\varphi(x, y), z)$ are defined and equal for all $x, y, z \in V$,

(iv) $j(P) \subset V$ and $P \xrightarrow[j|P]{} V \xrightarrow[j]{} P$ is the identity on P.

Put $Q = P \cap j^{-1}(P)$. Then $j(Q) \subset Q$ and $j^2 =$ identity on Q. Define $x^{-1} = j(x)$. For any $x, y \in Q$ say that xy is defined if and only if $\varphi(x, y) \in Q$, and if this is so, put $xy = \varphi(x, y)$. Then $Q \in \Lambda$ and $g = UQ$.

2. Cohomology of group germs. Let $\tau: g \times g \to g \times g$ be the transposition morphism of the product. Call $g \in \mathrm{Gr}\Gamma$ abelian if $g \times g \xrightarrow[\tau]{} g \times g \xrightarrow[\mu]{} g$ equals μ. Note that for such g and any $b \in \Gamma$, $\hom_\Gamma(b, g)$ has a structure of an abelian group (obtained by applying the functor $\hom_\Gamma(b, -)$: $\Gamma \to$ Sets to the diagrams defining g).

Given $a, g \in Gr\Gamma$, where a is abelian, call $\alpha: g \times a \to a$ a *g-action* on a if

$$\begin{array}{ccccc} g \times a \times a & \xrightarrow[(1,1)\times 1\times 1]{} & g \times g \times a \times a & \xrightarrow[1\times\tau\times 1]{} & g \times a \times g \times a \\ \Big\downarrow{\scriptstyle 1\times\mu} & & & & \Big\downarrow{\scriptstyle \alpha\times\alpha} \\ g \times a & \xrightarrow[\alpha]{} & a & \xleftarrow[\mu]{} & a \times a \end{array},$$

$$\begin{array}{ccc} g \times g \times a & \xrightarrow[\mu\times 1]{} & g \times a \\ \Big\downarrow{\scriptstyle 1\times\alpha} & & \Big\downarrow{\scriptstyle \alpha} \\ g \times a & \xrightarrow[\alpha]{} & a \end{array}, \qquad \begin{array}{ccc} a & & \\ \Big\downarrow{\scriptstyle (0,1)} & \searrow{\scriptstyle 1} & \\ g \times a & \xrightarrow[\alpha]{} & a \end{array}.$$

Given such g-action, put $\Phi^n = \hom_\Gamma(g^n, a)$, where $g^n = g \times \cdots \times g$ ($n \geqq 1$ times). Define $\delta_i: \Phi^n \to \Phi^{n+1}$; $i = 0, \cdots, n+1$, by putting for each $\varphi \in \Phi^n$,

$$\delta_0\varphi: g \times g^n \xrightarrow[1\times\varphi]{} g \times a \xrightarrow[\alpha]{} a,$$

$$\delta_i\varphi: g^{i-1} \times g^2 \times g^{n-i} \xrightarrow[1\times\mu\times 1]{} g^n \xrightarrow[\varphi]{} a;\ i = 1, \cdots, n,$$

$$\delta_{n+1}\varphi: g^n \times g \xrightarrow[\pi_1]{} g^n \xrightarrow[\varphi]{} a, \quad (\pi_1 = \text{first projection}).$$

Then each δ_i is a morphism of abelian groups. (This is easily shown for $i > 0$; for $i = 0$ one needs the first diagram in the definition of a g- action). Now let $\delta\varphi = \sum_{0\leqq i\leqq n+1}(-1)^i\delta_i\varphi$. By direct verification (or by the proof of the Theorem in § 4) one sees that $\delta^2 = 0$.

DEFINITION. For any g-action on a, $H(g, a)$ will denote the cohomology of $0 \longrightarrow \Phi^1 \xrightarrow[\delta]{} \Phi^2 \xrightarrow[\delta]{} \cdots$.

REMARK. It is not hard to see that for any g-action on a one can find $Q, A \in \Lambda$, A abelian, and a Q-action on A in the sense of ([12], p. 40) such that $g = UQ$, $a = UA$ and $\alpha = [m]$, where $m(x, p) = xp$ whenever the latter is defined for $x \in Q, P \in A$. Moreover $H(g, a) \simeq H_L(Q, A) =$ the local cohomology defined in ([12], p. 42).

3. Cohomology with coefficients in a group. Suppose that there are given $Q \in \Lambda$, an abelian topological group A and a morphism $m: Q \times A \to A$ in T. Then m will be called a Q-action on A if, denoting $m(x, p)$ by xp,

(i) $x(p_1 + p_2) = xp_1 + xp_2$ for all $x \in Q$; $p_1, p_2 \in A$,

(ii) $x_1(x_2p) = (x_1x_2)p$ whenever x_1x_2 is defined in Q,

(iii) $1p = p$ for all $p \in A$.

Call such Q-action m on A equivalent to a Q'-action m' on A if and only if there is an $S \in \Lambda$ such that S is an open local subgroup of both Q and Q' and $m \mid S \times A = m' \mid S \times A$. An equivalence class of Q-actions will be called a g-action, where g is the common value of UQ for all Q-actions in that class. Any Q-action in the class will be called a representative of the g-action.

Given any g-action on A, put $a = UA$ and let $\alpha: g \times a \to a$ be equal to $[m]: [Q] \times [A] \to [A]$ where $m: Q \times A \to A$ is any of its representatives. Then α is a g-action on a. Define $\delta^0: A \to \Phi^1$, where $\Phi^1 = \hom_\Gamma(g, a)$, as follows. For $m: Q \times A \to A$ as above, consider the map $A \to \hom_T(Q, A)$ assigning to $p \in A$ the map $Q \to A$ given by $x \mapsto m(x, p) - p$, for all $x \in Q$. The image of $Q \mapsto A$ under the functor $T \to \Gamma$ is in Φ^1; denote it by $\delta^0 p$. Then δ^0 is a morhism of abelian groups depending only on the g-action on A. Moreover one verifies easily that $\delta\delta^0 = 0$, where $\delta: \Phi^1 \to \Phi^2$ was defined in § 2.

DEFINITION. For any g-action on A, $H(g, A)$ will denote the cohomology of $\Phi: 0 \longrightarrow A \xrightarrow[\delta^0]{} \Phi^1 \xrightarrow[\delta]{} \Phi^2 \xrightarrow[\delta]{} \cdots$.

There is a description of $H(g, A)$ using the local group cohomology of W. T. van Est. For $Q \in \Lambda$, an abelian topological group A and a Q-action m on A, let $H(Q, A)$ be the cohomology defined as in [8] (or, in terms of cotriads, in [19]), but based on continuous cochains. Any Q'-action m' on A such that $Q' \subset Q$ and $m \mid Q' \times A = m'$ will be called *contained in m*. If this is so, the restriction of cochains yields a map $H(Q, A) \to H(Q', A)$.

PROPOSITION. *For any g-action on A,* $H(g, A) = \lim_{\rightarrow} H(Q, A)$, *the direct limit being taken over the partially ordered by inclusion* (and directed) *set of all Q-actions on A representing the g-action.*

4. Cohomology of enlargeable group germs. A group germ g will be called *enlargeable* if and only if there exists a group $G \in \Lambda$ such that $g = UG$. Such G will be called an enlargement of g.

LEMMA. *Suppose g is an enlargeable group germ and there is given a g-action on an abelian topological group A. Then there exists an enlargement G of g and a G-action on A which represents the g-action.*

Proof. Suppose $m: Q \times A \to A$, where $Q \in \Lambda$, represents the g-action. Replacing Q by a sufficiently small neighbourhood of 1, if needed, we may assume that Q is enlargeable (i.e., Q is a local subgroup of a group; [8], p. 393). Let G be the abstract group with the following presentation by generators and relations: Q is the set of generators and for $x_1, \cdots, x_n \in Q$, $x_1 x_2 \cdots x_n = 1$ is a defining relation if and only if this equality holds in the local group Q, after a suitable placement of brackets. The enlargeability of Q implies that the obvious map $Q \to G$ is injective; we use it to identify Q with a subset of G. The topology on Q defines now a fundamental system of neighbourhoods in G ([2], Chapter 2, § II) making G into a topological group with the open subset Q. For each $x \in Q$, define $\pi^m(x): A \to A$ by $\pi^m(x)p = m(x, p)$, for all $p \in A$. Then $\pi^m: Q \to \operatorname{Aut}(A)$ is a morphism of the abstract local group Q into the automorphism group of A. The construction of G implies that there is a group morphism $\pi: G \to \operatorname{Aut}(A)$ such that $\pi \mid Q = \pi^m$. If $x \in G$, then $x = x_1 x_2 \cdots x_k$: $x_1, \cdots, x_k \in Q$, whence $\pi(x) = \pi^m(x_1) \cdots, \pi^m(x_k): A \to A$ is continuous. The continuity of m is now easily seen to imply that the action $m_0: G \times A \to A$ given by $m_0(x, p) = \pi(x)p$ is continuous. It evidently represents the g-action.

Given topological groups G, A, where A is abelian, and a G-action on A, let $H_\square(G, A)$ denote the corresponding cohomology with empty supports ([12], p. 42 and below).

THEOREM. *Suppose g is an enlargeable group germ and there is given a g-action on a finite dimentional real vector space V. Then for any enlargement G of g and any G-action on V representing the g-action, $H(g, V) \simeq H_\square(G, V)$.*

Proof. Recall first $H_\square(G, V)$. Suppose $m: G \times V \to V$ is the G-action. Define $\pi: G \to GL(V)$ by $\pi(x)p = m(x, p)$. Denote by C the complex of V-valued, continuous, inhomogenous cochains on G. That is, $C = \bigoplus_{n \geq 0} C^n$, where $C^0 = V$ and C^n is the set of continuous maps from $G \times \cdots \times G$ (n times) to V, made into an abelian group by the addition in V. $\delta: C^0 \to C^1$ is defined by $(\delta p)(x_1) = \pi(x_1)p - p$ for all $p \in C^0$, and $\delta: C^n \to C^{n+1}$, $(n \geqq 1)$, by

$$(\delta f)(x_1, \cdots, x_{n+1}) = \pi(x_1)f(x_2, \cdots, x_{n+1}) + \sum_{1 \leqq i \leqq n} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) + (-1)^{n+1} f(x_1, \cdots, x_n)$$

for all $f \in C^n$. Call $f \in C^n$ *locally trivial* if there is a neighbourhood Q of 1 in G such that $f(x_1, \cdots, x_n) = 0$ whenever all $x_1, \cdots, x_n$ are in Q. The locally trivial cochains form a subcomplex C_l of C. Let $\bar{C}$ be the quotient complex C/C_l. Its cohomology is by definition $H_\square(G, V)$.

Consider now, for each $n \geqq 1$, the map $C^n \to \Phi^n$ (see Definition, § 3) given by $f \mapsto [f]$. Let $C^0 \to \Phi^0$ be the identity. All these maps are morphisms of abelian groups and they define a cochain map of C into Φ. Since $G \times \cdots \times G$ is completely regular at 1 ([16], p. 29), each $C^n \to \Phi^n$ is an epimorphism. Clearly its kernel is C_l^n. Therefore the cochain map $C \to \Phi$ induces an isomorphism $\bar{C} \to \Phi$.

REMARK. The cohomology of C has been discussed in [4]-[7], [9], [11], [12] and [17].

5. Cohomology of Lie group germs. A local topological group Q will be called a local Lie group if the space Q admits an analytic manifold structure such that the map $(x, y) \mapsto xy^{-1}$ is analytic on the open submanifold of $Q \times Q$ on which it is defined. Any such manifold structure on Q is unique ([10], p. 107).

Let $g \in Gr\Gamma$. We shall call g a Lie group germ if $g = UQ$ for some local Lie group Q. The Lie algebra of any such Q will be called the Lie algebra of g; it is easy to see that the latter is well defined.

Given a Lie algebra L and an L-module V which is a finite dimensional real vector space, let $H(L, V)$ denote the Chevalley-Eilenberg cohomology [1].

THEOREM 1. *If g is a Lie group germ with Lie algebra L, then for every g-action on a finite dimensional vector space V, $H(g, V) \simeq H(L, V)$.*

Here the L-module structure of V is defined by the g-action as follows. Let $m: Q \times V \to V$, where Q is a local Lie group, be a representative of the g-action. Define $\pi^m: Q \to GL(V)$ by $\pi^m(x)p = m(x, p)$. Then π^m is a morphism of local Lie groups, thus it is differentiable ([10], p. 107). Its differential at $1 \in Q$ defines a morphism of their Lie algebras $\pi_0^m: L \to gl(V)$, ([10], p. 102) which does not depend on the choice of Q. Thus V becomes an L-module.

Since a Lie group germ is known to be enlargeable, it follows from the considerations in § 4 that, under the assumptions of Theorem 1, there is a Lie group G with a continuous representation $\pi: G \to GL(V)$ such that $H(g, V) \simeq H_{\square}(G, V)$. Thus Theorem 1 will follow if we show.

THEOREM 2. *Given a Lie group G and $\pi: G \to GL(V)$ a continuous representation in a finite dimentional real vector space V, let $\pi_0 : L \to g(V)$ be the corresponding morphism of Lie algebras, making V into an L-module. Then $H_{\square}(G, V) \simeq H(L, V)$.*

6. Smooth cohomology with empty supports. For the proof of Theorem 2 we shall need to know that the definition of $H_{\square}(G, V)$, as given in § 4, yields the same cohomology if smooth (i.e., indefinitely differentiable) cochains are used instead of continuous ones. Thus let ${}_dC \subset C$ be the subcomplex of smooth cochains and put ${}_dC_l = {}_dC \cap C_l$, ${}_d\bar{C} = {}_dC/{}_dC_l$.

PROPOSITION. $$H({}_d\bar{C}) \simeq H(\bar{C}) .$$

Proof. We shall modify a construction due to G. D. Mostow ([17], p. 33) so that it becomes applicable modulo the locally trivial cochains.

Let K be the complex of V-valued, continuous, homogeneous cochains on G with homogeneous coboundary ($K^n = F^n(G, V)$ in the notation of [17]). Let K_l be the subcomplex of locally trivial cochains and put $\bar{K} = K/K_l$. Denote by ${}_dK \subset K$ the subcomplex of smooth cochains and put ${}_dK_l = {}_dK \cap K_l$. Then ${}_dK \subset K$ induces a cochain map γ of ${}_d\bar{K} = {}_dK/{}_dK_l$ into $\bar{K}$. The standard isomorphism $K \simeq C$ ([3], p. 54) obviously carries K_l and ${}_dK$ into C_l and ${}_dC$ respectively. Hence it will suffice to prove that $H(\gamma): H({}_d\bar{K}) \to H(\bar{K})$ is an isomorphism.

Let $\mathscr{U}$ denote the family of neighbourhoods of 1 in G, and choose a sequence $\varphi_0, \varphi_1, \varphi_2, \cdots$ of real valued smooth functions on G with compact supports and Haar integral 1 such that for every $Q \in \mathscr{U}$ there is a φ_i whose support is contained in Q. For every i, define a cochain map $\alpha_i: K \to {}_dK$ by

$$\begin{aligned}(\alpha_i f)(x_0, \cdots, x_n) &= \int_G \cdots \int_G f(x_0\xi_0, \cdots, x_n\xi_n)\varphi_i(\xi_0)\cdots\varphi_i(\xi_n)d\xi_0\cdots d\xi_n \\ &= \int_G \cdots \int_G f(\xi_0, \cdots, \xi_n)\varphi_i(x_0^{-1}\xi_0)\cdots\varphi_i(x_n^{-1}\xi_n)d\xi_0\cdots d\xi_n\end{aligned}$$

for $f \in K^n$; $n \geqq 0$. Also define maps $u_i: K \to K$ of degree -1 by

$$(u_i f)(x_0, \cdots, x_{n-1})$$
$$= \sum_{j=1}^{n} (-1)^j \int_G \cdots \int_G f(x_0, \cdots, x_{j-1}, x_{j-1}\xi_j, \cdots, x_{n-1}\xi_n)\varphi_i(\xi_j) \cdots \varphi_i(\xi_n) d\xi_j \cdots d\xi_n$$

for $f \in K^n$; $n \geqq 1$, and by $u_i f = 0$ for $f \in K^0$.

It is easy to see that if $f \in K_l$, then there is an i such that $\alpha_i f$ and $u_i f$ are in K_l. One verifies the identities

$$(*) \qquad f - \alpha_i f = \delta u_i f + u_i \delta f ; \qquad i = 0, 1, 2, \cdots$$

(see [5], § 4).

For $f \in K$, let $\bar{f}$ be its image in $\bar{K}$, and if $\bar{f}$ is a cocycle, let $\{f\} \in H(\bar{K})$ be its class.

To prove that $H(\gamma)$ is epimorphic, suppose that there is given a cocycle $\bar{f} \in \bar{K}$. Then $\delta f \in K_l$, whence for a suitable $i, f - \alpha_i f - \delta u_i f \in K_l$. Therefore $\{f\} = \{\alpha_i f\}$. But $\alpha_i f \in {}_dK$.

To show that $H(\gamma)$ is monomorphic, suppose that $f \in {}_dK$ is such that $\{f\} = 0$. Then there are $h \in K, g \in K_l$ such that $f - \delta h = g$. Hence $(*)$ implies

$f = \alpha_i \delta h + \alpha_i g + \delta u_i f + u_i \delta g = \delta(\alpha_i h + u_i f) + (\alpha_i + u_i \delta) g$. Thus, for suitable $i, f - \delta(\alpha_i h + u_i f) \in K_l$, and since $\alpha_i h + u_i f \in {}_dK$, it follows that the cohomology class of f in $H({}_d\bar{K})$ is zero.

7. A spectral sequence. Suppose G, π, V and L satisfy the assumptions of Theorem 2. By the result of § 6, Theorem 2 will follow if we show that $H({}_d\bar{C}) \simeq H(L, V)$. We shall consider a bicomplex F, similar to the one defined in [4], § 10, and we shall show that the quotient complex $\bar{F}$ obtained by factoring out the locally trivial cochains is such that

(i) the initial term of the first spectral sequence is

$$^0E_1^s = H^s({}_d\bar{C}) \quad \text{and} \quad {}^rE_1^s = 0 \quad \text{for all} \quad r > 0 ,$$

(ii) the initial term of the second spectral sequence is

$$^rE_1^0 = H^r(L, V) \quad \text{and} \quad {}^rE_1^s = 0 \quad \text{for all} \quad s > 0 .$$

As well known, this implies $H({}_d\bar{C}) \simeq H(L, V)$.

We begin by defining $F = \bigoplus_{r,s \geqq 0} {}^rF^s$. Let $L_1, \cdots, L_r$ be r copies of L and $G_1, \cdots, G_s$, s copies of G. Then, for $r, s \geqq 1, {}^rF^s$ is the vector space of all smooth maps

$$L_1 \times \cdots \times L_r \times G_1 \times \cdots \times G_s \to V$$

which are r-linear and alternating in the first r variables. For every $s \geqq 1, {}^0F^s$ is the subspace of ${}_dC^s$ composed of those cochains f which satisfy the following local normalization condition: for each $f \in {}^0F^s$, there is a $Q \in \mathscr{U}$ such that $f(x_1, \cdots, x_s) = 0$ whenever $x_1, \cdots, x_s \in Q$ and at least one x_i equals 1. ${}^rF^0$ is, for each $r \geqq 1$, the space of V-valued r-linear alternating functions on L, and ${}^0F^0 = V$.

For each $x \in G$, let $\rho_x: G \to G$ be the right translation $y \mapsto yx$. Denote by ρ_x^* the induced map on the tangent bundle. We shall identify L with the tangent space to G at 1. For each $X \in L$, $\widetilde{X}$ will denote the right invariant vector field (i.e., satisfying $\rho_x^* \widetilde{X} = \widetilde{X}$ for all x) taking at 1 the value X.

Occasionally an $f \in {}^rF^s$ will be interpreted as a differential form on G, depending on the parameter $(x_2, \cdots, x_s) \in G \times \cdots \times G$ which, for fixed value of the parameter, takes at $\widetilde{X}_1 \cdots, \widetilde{X}_r$ and $x_1 \in G$ the value $f(X_1, \cdots, X_r, x_1, \cdots, x_s)$. The morphisms

$$d_1: {}^rF^s \longrightarrow {}^{r+1}F^s, \; d_2: {}^rF^s \longrightarrow {}^rF^{s+1}$$

are now defined as follows.

If $f \in {}^nF^0$, let $d_1 f$ be given by the formula

$$\begin{aligned}(d_1 f)(X_1, \cdots, X_{n+1}) &= \frac{1}{n+1} \sum (-1)^{i+1} \pi_0(X_i) f(X_1, \cdot\cdot\hat{\ }\cdot\cdot, X_{n+1}) \\ &+ \frac{1}{n+1} \sum (-1)^{i+j} f([X_i, X_j], X_1 \cdot\hat{\cdot}\cdot, X_{n+1})\end{aligned}$$

for every $X_1, \cdots, X_{n+1} \in L$.

Let $f \in {}^rF^s$; $s \geqq 1$. For any fixed $x_2, \cdots, x_s \in G$ consider the differential form ω_f for which identically

$$\omega_f(\widetilde{X}_1, \cdots, \widetilde{X}_r; x_1) = \pi(x_1^{-1}) f(X_1, \cdots, X_r, x_1, \cdots, x_s) .$$

Let $d_1 f$ be the $(r+1)$-form whose value at x_1 is $\pi(x_1) d\omega_f$, d being the exterior derivative ([10], p. 21). One sees easily that $d_1 f \in {}^{r+1}F^s$.

Let $d_2: {}^0F^s \to {}^0F^{s+1}$ be the coboundary δ of § 4. Finally, let $d_2: {}^rF^s \to {}^rF^{s+1}$; $r \geqq 1$, be given by

$$\begin{aligned}&(d_2 f)(X_1, \cdots, X_r, x_1, \cdots, x_{s+1}) \\ &\quad = \sum (-1)^i f(X_1, \cdots, X_r, x_1, \cdots, x_i x_{i+1}, \cdots, x_{s+1}) \\ &\quad\quad + (-1)^{s+1} f(X_1, \cdots, X_r, x_1, \cdots, x_s) .\end{aligned}$$

This completes the definition of F.

One has $d_1 d_2 = d_2 d_1$ and $d_1^2 = d_2^2 = 0$ ([4], § 10). Moreover the complex

$${}^rF: 0 \longrightarrow {}^rF^0 \xrightarrow[d_2]{} {}^rF^1 \xrightarrow[d_2]{} \cdots$$

has for $r \geqq 1$ a contracting homotopy $u: {}^rF^{s+1} \to {}^rF^s$ given by

$$(uf)(X_1, \cdots, X_r, x_1, \cdots, x_s) = -f(X_1, \cdots, X_r, 1, x_1, \cdots, x_s)$$

([4], §9).

Call a bicochain $f \in {}^rF^s$ locally trivial if there exists a $Q \in \mathscr{U}$ such that $f(X_1, \cdots, X_r, x_1, \cdots, x_s) = 0$ for all $X_1, \cdots, X_r \in L, x_1, \cdots, x_s \in Q$. Let $\bar{F}$ be the quotient of F by the sub-bicomplex of locally trivial cochains. Then $\bar{F}$ is a bicomplex with operators $\bar{d}_1, \bar{d}_2$ induced by d_1, d_2. We shall show that it has the properties (i), (ii) stated at the beginning of this section.

For each r let ${}^r\bar{F}$ be the complex $0 \to {}^r\bar{F}^0 \to {}^r\bar{F}^1 \to \cdots$ with coboundary $\bar{d}_2$, and let for each s, $\bar{F}^s$ be defined similarly.

To obtain (i), one shows first that the inclusion ${}^0F \subset {}_dC$ induces an isomorphism $H({}^0\bar{F}) \to H({}_d\bar{C})$. This is a consequence of the two facts

(a) if $f \in {}_dC$ and δf is locally trivial, then f is cohomologous in ${}_dC$ to some $h \in {}^0F$,

(b) if $f \in {}^0F$ and $f - \delta g$ is locally trivial for some $g \in {}_dC$, then there exists an $h \in {}^0F$ such that $f - \delta h$ is locally trivial.

The proof of (a) and (b) is easily obtained from that of Lemmas 6.1 and 6.2 in [3], p. 62. One concludes that ${}^0E_1^s = H^s({}_d\bar{C})$, for the first spectral sequence. Since each ${}^r\bar{F}, r \geqq 1$, has a contracting homotopy $\bar{u}$ induced by u, ${}^rE_1^s = 0$ for $r \geqq 1$.

To prove (ii) observe first that $\bar{F}^0 = F^0$ and $H(F^0) = H(L, V)$, by definition. Hence ${}^rE_1^0 = H^r(L, V)$ for the second spectral sequence.

It remains to show that for each $s \geqq 1$, $\bar{F}^s$ is an acyclic complex. Let $f \in {}^rF^s$ be such that $d_1 f$ is locally trivial. Thus there is a $Q \in \mathscr{U}$ such that for each $x_2, \cdots, x_s \in Q$ the $(r+1)$-form $d\omega_f$ vanishes identically on Q. We may assume that Q is diffeomorphic to a Euclidean ball.

For $r = 0$, the condition $d\omega_f = 0$ on Q implies that $\pi(x_1^{-1})f(x_1, \cdots, x_s)$ does not depend on x_1 when $x_1, \cdots, x_s \in Q$. Consequently, by the local normalization condition, f is locally trivial. Hence $\bar{d}_1: {}^0\bar{F}^s \to {}^1\bar{F}^s$ is a monomorphism.

For $r \geqq 1$, and any $x_2, \cdots, x_s \in G$, the restriction $\omega_f \mid Q$ is a closed r-form on Q. Hence the Poincarè lemma ([13], p. 87) implies the existence of an $(r-1)$-form μ on Q such that $d\mu = \omega_f$. The proof of Poincarè lemma shows that μ depends smoothly on the parameter $(x_2, \cdots, x_s) \in Q \times \cdots \times Q$ (where smoothness is understood in the sense of [7], §1). Let φ be a smooth real-valued function on G, identically equal to 1 in some neighbourhood of the identity and vanishing outside some neighbourhood of the identity whose closure is contained in Q. For each $x_2, \cdots, x_s \in G$, let h be the $(r-1)$-form on G which at $x_1 \in G$ takes the value $\varphi(x_1)\varphi(x_2) \cdots \varphi(x_s)\pi(x_1)\mu$ when $x_1, \cdots, x_s \in Q$ and 0 otherwise.

Recalling the interpretation of ${}^{r}F^{s}$ as the space of r-forms depending on the parameter $(x_2, \cdots, x_s) \in G \times \cdots \times G$, we see readily that $h \in {}^{r-1}F^{s}$. Moreover the construction guarantees that $f - d_1 h$ is locally trivial. Thus $\bar{F}^{s}$ is exact at ${}^{r}\bar{F}^{s}$ and the proof of Theorem 2 is complete.

8. Explicit form of the isomorphism. We shall describe the isomorphism $H({}_d\bar{C}) \simeq H(L, V)$, i.e., $H({}^0\bar{F}) \simeq H(\bar{F}^0)$. Let Tot F be the total complex of F ([14], p. 340). For $f \in {}^0F^n$, $n \geqq 1, 1 \leqq j \leqq n$ and $X \in L$ denote by $\partial_j(X)f \in {}^0F^{n-1}$ the derivative in the direction X with respect to the jth variable at $x_j = 1$. Define maps $\tau^{n,r}\colon {}^0F^n \to {}^rF^{n-r}$; $r = 0, 1, \cdots, n$ by $\tau^{n,0} =$ identity, and for $r \geqq 1$

$$(\tau^{n,r}f)(X_1, \cdots, X_r, x_{r+1}, \cdots, x_n) = (\sum \operatorname{sgn}(i_1, \cdots, i_r)\partial_1(X_{i_1}) \cdots \partial_r(X_{i_r})f)(x_{r+1}, \cdots, x_n) ,$$

where $\sum$ ranges over all permutations of $(1, \cdots, r)$. It is shown in [4], p. 500 that the maps $\tau^n = \sum_{0 \leqq r \leqq n} \tau^{n,r}\colon {}^0F^n \to (\text{Tot } F)^n$ define a cochain map $\tau\colon {}^0F \to \text{Tot } F$. Let $\bar{\tau}\colon {}^0\bar{F} \to \text{Tot } \bar{F}$ be induced by τ. Denote by $\bar{p}_1, \bar{p}_2$ the projections $\text{Tot } \bar{F} \to \bar{F}^0$, $\text{Tot } \bar{F} \to {}^0\bar{F}$. These are evidently cochain maps and from the behaviour (i), (ii) of the spectral sequences it follows that $H(\bar{p}_1)$, $H(\bar{p}_2)$ are isomorphisms. Now $\bar{p}_2\bar{\tau}$ is the identity, thus $H(\bar{\tau})\colon H({}^0\bar{F}) \to H(\text{Tot } \bar{F})$ is an isomorphism, whence the same is true about $H(\bar{p}_1\bar{\tau})\colon H({}^0\bar{F}) \to H(\bar{F}^0)$. Clearly $\bar{p}_1\bar{\tau} \mid {}^0\bar{F}^n = \bar{\tau}^{n,n}$.

References

1. C. Chevalley and S, Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc., **63** (1948), 85-124.

2. C. Chevalley, *Theory of Lie groups*, Vol. 1. Princeton Univ. Press, Princeton 1946.

3. S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups*, Ann. of Math., **48** (1947). 51-78.

4. W. T. van Est, *Group cohomology and Lie algebra cohomology in Lie groups*, Indag. Math., **15** (1953), 484-504.

5. ———, *On the algebraic cohomology concepts in Lie groups I*, Indag. Math., **17** (1955), 225-233.

6. ———, *Une application d'une méthode de Cartan-Leray*, Indag. Math., **17** (1955), 542-545.

7. ———, *A generalization of the Cartain-Leray spectral sequence I*, Indag. Math., **20** (1958), 399-405.

8. ———, *Local and global groups I*, Indag. Math., **24** (1962), 391-408.

9. I. M. Gel'fand D. B. Fuks, *On classifying spaces for principal fiberings with Hausdorff bases*, Soviet Math. Dokl., **9** (1968), No. 4, 851-854.

10. S. Helgason, *Differential geometry and symmetric spaces*, Acad. Press, New York 1962.

11, G. Hochschild and G. D. Mostow, *Cohomology of Lie groups*, Illinois J. Math., **6** (1962), 367-401.

12. Sze-tsen Hu, *Cohomology theory in topological groups*, Michigan Math. J., **1** (1952), 11-59,

13. S. Lang, *Introduction to differentiable manifolds*, Interscience. New York-London 1962.
14. Saunders MacLane, *Homology*, Springer 1963.
15. A. I. Malcev, *Sur les groupes topologiques locaux et complets*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 606-608.
16. D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience, New York, 1955.
17. G. D. Mostow, *Cohomology of topological groups and solvmanifolds*, Ann. of Math., **73** (1961), 20-48.
18. A. Robinson, *Germs, Applications of Model Theory to Algebra,* Analysis and Probability, (Edited by W. A. J. Luxemburg), Holt, Reinhart, Winston 1969.
19. S. Świerczkowski, *On the cohomology of local groups*, J. Austral Math. Soc., **12** (1971), 249-255.

Received February 3, 1970. This work was partially supported by a grant from the Netherlands Organization for the Advancement of Pure Research, Z. W. O.

AUSTRALIAN NATIONAL UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 2 June, 1971