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Let π be a continuous representation of a Lie group G in a finite dimensional real vector space V. Denote by $H_\square(G, V)$ the cohomology with empty supports in the sense of Sze-tsen Hu. If L is the Lie algebra of G, π induces an L-module structure on V and there is the associated cohomology $H(L, V)$ of Chevalley-Eilenberg. Our main result is the construction of an isomorphism $H_\square(G, V) \simeq H(L, V)$.

This is preceded by a closer analysis of $H_\square(G, V)$. It is clear from the definition that to know $H_\square(G, V)$, it suffices to know an arbitrary neighbourhood of 1 in G and its action on V. The totality of neighbourhoods of 1 in G may be regarded as an object of a more fine nature than a local group; we call it a group germ. More precisely, a group germ is defined as a group object in the category Γ of topological germs [18]. The Eilenberg-MacLane definition [3] of the cohomology of an abstract group is carried over from the category of sets to Γ (i.e., from groups to group germs). Thus for any group germs g, a, where a is abelian, and any g-action on a, we have cohomology groups $H(g, a)$. It turns out that $H_\square(G, V) \simeq H(g, a)$ for a suitable choice of g and a, in all dimensions > 1. To cope with dim 0 and 1 it seems convenient to introduce the concept of an action of a group germ g on an abelian topological group A and associate with this a cohomology $H(g, A)$. This is only a slight modification of the previous $H(g, a)$, so that both cohomologies coincide in dimensions >1 and $H^1(g, A)$ is a quotient of $H^1(g, a)$, if a is suitably related to A. ($H^0(g, A)$ is the subgroup of g-stable elements of A and $H^0(g, a)$ is always trivial). One now has $H_\square(G, V) \simeq H(g, V)$ in all dimensions, for a group germ g corresponding to G.

We are grateful to W.T. van Est for his comments on an earlier version of this paper which have resulted in many improvements.

1. Group germs.

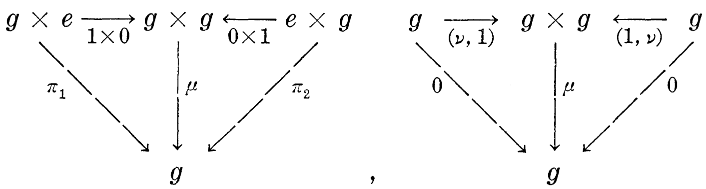
Let T be the category of pointed topological spaces. For $A, B \in T$ write $A \simeq B$ if and only if there is a $C \in T$ which is an open subspace of both A and B. Denote by $[A]$ the equivalence class of A. For morphisms $f: A \rightarrow B, f': A' \rightarrow B'$ in T write $f \simeq f'$ if and only if $A \cong A'$, $B \simeq B'$ and there is a $C \in T$ which is an open subspace of both A and A' such that $f \mid C = f' \mid C$. Denote the equivalence class of $f: A \rightarrow B$ by $[f]: [A] \rightarrow [B]$. There is now precisely one category Γ whose objects are the equivalence classes $[A]$, the morphisms are the equivalence classes $[f]: [A] \to [B]$, and such that $A \mapsto [A]$, $f \mapsto [f]$ is a functor $T \to \Gamma$. Γ will be called the category of *topological germs*. (For a similar definition see [18]).

LEMMA. *The functor $T \to \Gamma$ preserves zero objects and finite products.*

We omit the straightforward verification. As a conclusion, all finite products exist in Γ. Let S be a zero object in T, i.e., a one-point set, and denote the zero object [S] in Γ by e. Any morphism in Γ which factorizes through e will be denoted by 0.

DEFINITION. A group object in Γ will be called a group germ. The category of group germs will be denoted by $Gr\Gamma$.

We recall the definitions. A group object in Γ is an object $g \in \Gamma$ together with morphisms $\mu: g \times g \to g$, $\nu: g \to g$ such that $\mu(\mu \times 1) = \mu(1 \times \mu)$ (i.e., associativity), $\nu^2 = id$ and



(π_i are the product projections; all diagrams drawn are assumed to commute). A morphism $g \to g'$ in $Gr\Gamma$ is a $\varphi: g \to g'$ in Γ such that $\mu'(\varphi \times \varphi) = \varphi\mu$ and $\nu'\varphi = \varphi\nu$.

Let Λ be the category of local topological groups. Following ([8], p. 393) we mean by a local topological group an abstract local group in the sense of Malcev [15] together with a topology on the set Q of its elements such that the map $(x, y) \mapsto xy^{-1}$ is continuous on the domain of its definition and that domain is open in $Q \times Q$. A morphism $Q \to Q'$ in Λ is an $f: Q \to Q'$ in T such that $f(x)f(y)$ is defined whenever xy is defined, and if defined, $f(x)f(y) = f(xy)$.

Define a functor $U: \Lambda \to Gr\Gamma$ as follows. Given $Q \in \Lambda$, let $j(x) = x^{-1}$ and $\varphi(x, y) = xy$, the domain of φ being an open subspace D of $Q \times Q$, so that $[D] = [Q] \times [Q]$ (cf. Lemma). Let UQ be the topological germ $[Q]$ together with the morphisms $\nu = [j]: [Q] \to [Q]$, $\mu = [\varphi]: [Q] \times [Q] \to [Q]$ in Γ. Then $UQ \in Gr\Gamma$. For a morphism f in Λ put $Uf = [f]$.

PROPOSITION. *For each $g \in Gr\Gamma$ there exists a $Q \in \Lambda$ such that $g = UQ$.*

Proof. Suppose $g = [A]$, $A \in T$ and denote the base point of A by 1. The definition of a group object in Γ implies the existence of open neighbourhoods P, V, W of 1 in A such that $P \subset V \subset W$ and

(i) there exists $\varphi: W \times W \to A$ such that $\mu = [\varphi]$,

(ii) there exists $j: V \to W$ such that $\nu = [j]$,

(iii) $\varphi(j(x), x) = \varphi(x, j(x)) = 1$, $\varphi(x, 1) = \varphi(1, x) = x$ and both $\varphi(x, \varphi(y, z))$, $\varphi(\varphi(x, y), z)$ are defined and equal for all $x, y, z \in V$,

(iv) $j(P) \subset V$ and $P \xrightarrow[j|P]{} V \xrightarrow[j]{} P$ is the identity on P.

Put $Q = P \cap j^{-1}(P)$. Then $j(Q) \subset Q$ and $j^2 =$ identity on Q. Define $x^{-1} = j(x)$. For any $x, y \in Q$ say that xy is defined if and only if $\varphi(x, y) \in Q$, and if this is so, put $xy = \varphi(x, y)$. Then $Q \in \Lambda$ and $g = UQ$.

2. Cohomology of group germs. Let $\tau: g \times g \to g \times g$ be the transposition morphism of the product. Call $g \in Gr\Gamma$ abelian if $g \times g \xrightarrow[\tau]{} g \times g \xrightarrow[\mu]{} g$ equals μ. Note that for such g and any $b \in \Gamma$, $\hom_\Gamma(b, g)$ has a structure of an abelian group (obtained by applying the functor $\hom_\Gamma(b, -): \Gamma \to$ Sets to the diagrams defining g).

Given $a, g \in Gr\Gamma$, where a is abelian, call $\alpha: g \times a \to a$ a *g-action* on a if

$$\begin{array}{ccccc}
g \times a \times a & \xrightarrow[(1,1)\times 1 \times 1]{} & g \times g \times a \times a & \xrightarrow[1 \times \tau \times 1]{} & g \times a \times g \times a \\
\Big\downarrow {\scriptstyle 1 \times \mu} & & & & \Big\downarrow {\scriptstyle \alpha \times \alpha} \\
g \times a & \xrightarrow[\alpha]{} & a & \xleftarrow[\mu]{} & a \times a
\end{array} ,$$

$$\begin{array}{ccc}
g \times g \times a & \xrightarrow[\mu \times 1]{} & g \times a \\
\Big\downarrow {\scriptstyle 1 \times \alpha} & & \Big\downarrow {\scriptstyle \alpha} \\
g \times a & \xrightarrow[\alpha]{} & a
\end{array} , \qquad
\begin{array}{ccc}
a & & \\
\Big\downarrow {\scriptstyle (0,1)} & \searrow {\scriptstyle 1} & \\
g \times a & \xrightarrow[\alpha]{} & a
\end{array} .$$

Given such g-action, put $\Phi^n = \hom_\Gamma(g^n, a)$, where $g^n = g \times \cdots \times g$ ($n \geqq 1$ times). Define $\delta_i: \Phi^n \to \Phi^{n+1}$; $i = 0, \cdots, n+1$, by putting for each $\varphi \in \Phi^n$,

$$\delta_0\varphi: g \times g^n \xrightarrow[1 \times \varphi]{} g \times a \xrightarrow[\alpha]{} a ,$$

$$\delta_i\varphi: g^{i-1} \times g^2 \times g^{n-i} \xrightarrow[1 \times \mu \times 1]{} g^n \xrightarrow[\varphi]{} a; \; i = 1, \cdots, n ,$$

$$\delta_{n+1}\varphi: g^n \times g \xrightarrow[\pi_1]{} g^n \xrightarrow[\varphi]{} a , \quad (\pi_1 = \text{first projection}).$$

Then each δ_i is a morphism of abelian groups. (This is easily shown for $i > 0$; for $i = 0$ one needs the first diagram in the definition of a g- action). Now let $\delta\varphi = \sum_{0 \leqq i \leqq n+1} (-1)^i \delta_i \varphi$. By direct verification (or by the proof of the Theorem in § 4) one sees that $\delta^2 = 0$.

DEFINITION. For any g-action on a, $H(g, a)$ will denote the cohomology of $0 \longrightarrow \Phi^1 \xrightarrow[\delta]{} \Phi^2 \xrightarrow[\delta]{} \cdots$.

REMARK. It is not hard to see that for any g-action on a one can find $Q, A \in \Lambda$, A abelian, and a Q-action on A in the sense of ([12], p. 40) such that $g = UQ$, $a = UA$ and $\alpha = [m]$, where $m(x, p) = xp$ whenever the latter is defined for $x \in Q, P \in A$. Moreover $H(g, a) \simeq H_L(Q, A) =$ the local cohomology defined in ([12], p. 42).

3. Cohomology with coefficients in a group. Suppose that there are given $Q \in \Lambda$, an abelian topological group A and a morphism $m: Q \times A \rightarrow A$ in T. Then m will be called a Q-action on A if, denoting $m(x, p)$ by xp,

(i) $x(p_1 + p_2) = xp_1 + xp_2$ for all $x \in Q$; $p_1, p_2 \in A$,

(ii) $x_1(x_2 p) = (x_1 x_2)p$ whenever $x_1 x_2$ is defined in Q,

(iii) $1p = p$ for all $p \in A$.

Call such Q-action m on A equivalent to a Q'-action m' on A if and only if there is an $S \in \Lambda$ such that S is an open local subgroup of both Q and Q' and $m \mid S \times A = m' \mid S \times A$. An equivalence class of Q-actions will be called a g-action, where g is the common value of UQ for all Q-actions in that class. Any Q-action in the class will be called a representative of the g-action.

Given any g-action on A, put $a = UA$ and let $\alpha: g \times a \rightarrow a$ be equal to $[m]: [Q] \times [A] \rightarrow [A]$ where $m: Q \times A \rightarrow A$ is any of its representatives. Then α is a g-action on a. Define $\delta^0: A \rightarrow \Phi^1$, where $\Phi^1 = \hom_\Gamma(g, a)$, as follows. For $m: Q \times A \rightarrow A$ as above, consider the map $A \rightarrow \hom_\Gamma(Q, A)$ assigning to $p \in A$ the map $Q \rightarrow A$ given by $x \mapsto m(x, p) - p$, for all $x \in Q$. The image of $Q \mapsto A$ under the functor $T \rightarrow \Gamma$ is in Φ^1; denote it by $\delta^0 p$. Then δ^0 is a morhism of abelian groups depending only on the g-action on A. Moreover one verifies easily that $\delta\delta^0 = 0$, where $\delta: \Phi^1 \rightarrow \Phi^2$ was defined in § 2.

DEFINITION. For any g-action on A, $H(g, A)$ will denote the cohomology of $\Phi: 0 \longrightarrow A \xrightarrow[\delta^0]{} \Phi^1 \xrightarrow[\delta]{} \Phi^2 \xrightarrow[\delta]{} \cdots$.

There is a description of $H(g, A)$ using the local group cohomology of W. T. van Est. For $Q \in \Lambda$, an abelian topological group A and a Q-action m on A, let $H(Q, A)$ be the cohomology defined as in [8] (or, in terms of cotriads, in [19]), but based on continuous cochains. Any Q'-action m' on A such that $Q' \subset Q$ and $m \mid Q' \times A = m'$ will be called *contained in m.* If this is so, the restriction of cochains yields a map $H(Q, A) \rightarrow H(Q', A)$.

PROPOSITION. *For any g-action on A*, $H(g, A) = \lim_{\rightarrow} H(Q, A)$, *the direct limit being taken over the partially ordered by inclusion* (and directed) *set of all Q-actions on A representing the g-action.*

4. Cohomology of enlargeable group germs. A group germ g will be called *enlargeable* if and only if there exists a group $G \in \Lambda$ such that $g = UG$. Such G will be called an enlargement of g.

LEMMA. *Suppose g is an enlargeable group germ and there is given a g-action on an abelian topological group A. Then there exists an enlargement G of g and a G-action on A which represents the g-action.*

Proof. Suppose $m: Q \times A \to A$, where $Q \in \Lambda$, represents the g-action. Replacing Q by a sufficiently small neighbourhood of 1, if needed, we may assume that Q is enlargeable (i.e., Q is a local subgroup of a group; [8], p. 393). Let G be the abstract group with the following presentation by generators and relations: Q is the set of generators and for $x_1, \cdots, x_n \in Q$, $x_1x_2\cdots x_n = 1$ is a defining relation if and only if this equality holds in the local group Q, after a suitable placement of brackets. The enlargeability of Q implies that the obvious map $Q \to G$ is injective; we use it to identify Q with a subset of G. The topology on Q defines now a fundamental system of neighbourhoods in G ([2], Chapter 2, § II) making G into a topological group with the open subset Q. For each $x \in Q$, define $\pi^m(x): A \to A$ by $\pi^m(x)p = m(x, p)$, for all $p \in A$. Then $\pi^m: Q \to \mathrm{Aut}\,(A)$ is a morphism of the abstract local group Q into the automorphism group of A. The construction of G implies that there is a group morphism $\pi: G \to \mathrm{Aut}\,(A)$ such that $\pi \mid Q = \pi^m$. If $x \in G$, then $x = x_1x_2\cdots x_k$: $x_1, \cdots, x_k \in Q$, whence $\pi(x) = \pi^m(x_1)\cdots, \pi^m(x_k): A \to A$ is continuous. The continuity of m is now easily seen to imply that the action $m_0: G \times A \to A$ given by $m_0(x, p) = \pi(x)p$ is continuous. It evidently represents the g-action.

Given topological groups G, A, where A is abelian, and a G-action on A, let $H_\Box(G, A)$ denote the corresponding cohomology with empty supports ([12], p. 42 and below).

THEOREM. *Suppose g is an enlargeable group germ and there is given a g-action on a finite dimentional real vector space V. Then for any enlargement G of g and any G-action on V representing the g-action, $H(g, V) \simeq H_\Box(G, V)$.*

Proof. Recall first $H_\Box(G, V)$. Suppose $m: G \times V \to V$ is the G-action. Define $\pi: G \to GL(V)$ by $\pi(x)p = m(x, p)$. Denote by C the complex of V-valued, continuous, inhomogenous cochains on G. That is, $C = \bigoplus_{n\geq 0} C^n$, where $C^0 = V$ and C^n is the set of continuous maps from $G \times \cdots \times G$ (n times) to V, made into an abelian group by the addition in V. $\delta: C^0 \to C^1$ is defined by $(\delta p)(x_1) = \pi(x_1)p - p$ for all $p \in C^0$, and $\delta: C^n \to C^{n+1}$, $(n \geqq 1)$, by

$$(\delta f)(x_1, \cdots, x_{n+1}) = \pi(x_1)f(x_2, \cdots, x_{n+1}) + \sum_{1 \leqq i \leqq n} (-1)^i f(x_1, \cdots, x_i x_{i+1}, \cdots, x_{n+1}) + (-1)^{n+1} f(x_1, \cdots, x_n)$$

for all $f \in C^n$. Call $f \in C^n$ *locally trivial* if there is a neighbourhood Q of 1 in G such that $f(x_1, \cdots, x_n) = 0$ whenever all $x_1, \cdots, x_n$ are in Q. The locally trivial cochains form a subcomplex C_l of C. Let $\bar{C}$ be the quotient complex C/C_l. Its cohomology is by definition $H_\square(G, V)$.

Consider now, for each $n \geqq 1$, the map $C^n \to \Phi^n$ (see Definition, § 3) given by $f \mapsto [f]$. Let $C^0 \to \Phi^0$ be the identity. All these maps are morphisms of abelian groups and they define a cochain map of C into Φ. Since $G \times \cdots \times G$ is completely regular at 1 ([16], p. 29), each $C^n \to \Phi^n$ is an epimorphism. Clearly its kernel is C_l^n. Therefore the cochain map $C \to \Phi$ induces an isomorphism $\bar{C} \to \Phi$.

REMARK. The cohomology of C has been discussed in [4]-[7], [9], [11], [12] and [17].

5. Cohomology of Lie group germs. A local topological group Q will be called a local Lie group if the space Q admits an analytic manifold structure such that the map $(x, y) \mapsto xy^{-1}$ is analytic on the open submanifold of $Q \times Q$ on which it is defined. Any such manifold structure on Q is unique ([10], p. 107).

Let $g \in Gr\Gamma$. We shall call g a Lie group germ if $g = UQ$ for some local Lie group Q. The Lie algebra of any such Q will be called the Lie algebra of g; it is easy to see that the latter is well defined.

Given a Lie algebra L and an L-module V which is a finite dimensional real vector space, let $H(L, V)$ denote the Chevalley-Eilenberg cohomology [1].

THEOREM 1. *If g is a Lie group germ with Lie algebra L, then for every g-action on a finite dimensional vector space V, $H(g, V) \simeq H(L, V)$.*

Here the L-module structure of V is defined by the g-action as follows. Let $m: Q \times V \to V$, where Q is a local Lie group, be a representative of the g-action. Define $\pi^m: Q \to GL(V)$ by $\pi^m(x)p = m(x, p)$. Then π^m is a morphism of local Lie groups, thus it is differentiable ([10], p. 107). Its differential at $1 \in Q$ defines a morphism of their Lie algebras $\pi_0^m: L \to gl(V)$, ([10], p. 102) which does not depend on the choice of Q. Thus V becomes an L-module.

Since a Lie group germ is known to be enlargeable, it follows from the considerations in § 4 that, under the assumptions of Theorem 1, there is a Lie group G with a continuous representation $\pi: G \to GL(V)$ such that $H(g, V) \simeq H_{\square}(G, V)$. Thus Theorem 1 will follow if we show.

THEOREM 2. *Given a Lie group G and $\pi: G \to GL(V)$ a continuous representation in a finite dimentional real vector space V, let $\pi_0 : L \to g(V)$ be the corresponding morphism of Lie algebras, making V into an L-module. Then $H_{\square}(G, V) \simeq H(L, V)$.*

6. Smooth cohomology with empty supports. For the proof of Theorem 2 we shall need to know that the definition of $H_{\square}(G, V)$, as given in § 4, yields the same cohomology if smooth (i.e., indefinitely differentiable) cochains are used instead of continuous ones. Thus let ${}_dC \subset C$ be the subcomplex of smooth cochains and put ${}_dC_l = {}_dC \cap C_l$, ${}_d\bar{C} = {}_dC/{}_dC_l$.

PROPOSITION. $$H({}_d\bar{C}) \simeq H(\bar{C}) .$$

Proof. We shall modify a construction due to G. D. Mostow ([17], p. 33) so that it becomes applicable modulo the locally trivial cochains.

Let K be the complex of V-valued, continuous, homogeneous cochains on G with homogeneous coboundary ($K^n = F^n(G, V)$ in the notation of [17]). Let K_l be the subcomplex of locally trivial cochains and put $\bar{K} = K/K_l$. Denote by ${}_dK \subset K$ the subcomplex of smooth cochains and put ${}_dK_l = {}_dK \cap K_l$. Then ${}_dK \subset K$ induces a cochain map γ of ${}_d\bar{K} = {}_dK/{}_dK_l$ into $\bar{K}$. The standard isomorphism $K \simeq C$ ([3], p. 54) obviously carries K_l and ${}_dK$ into C_l and ${}_dC$ respectively. Hence it will suffice to prove that $H(\gamma): H({}_d\bar{K}) \to H(\bar{K})$ is an isomorphism.

Let $\mathscr{U}$ denote the family of neighbourhoods of 1 in G, and choose a sequence $\varphi_0, \varphi_1, \varphi_2, \cdots$ of real valued smooth functions on G with compact supports and Haar integral 1 such that for every $Q \in \mathscr{U}$ there is a φ_i whose support is contained in Q. For every i, define a cochain map $\alpha_i: K \to {}_dK$ by

$$\begin{aligned}(\alpha_i f)(x_0, \cdots, x_n) &= \int_G \cdots \int_G f(x_0\xi_0, \cdots, x_n\xi_n)\varphi_i(\xi_0)\cdots\varphi_i(\xi_n)d\xi_0\cdots d\xi_n \\ &= \int_G \cdots \int_G f(\xi_0, \cdots, \xi_n)\varphi_i(x_0^{-1}\xi_0)\cdots\varphi_i(x_n^{-1}\xi_n)d\xi_0\cdots d\xi_n\end{aligned}$$

for $f \in K^n$; $n \geqq 0$. Also define maps $u_i: K \to K$ of degree -1 by

$$(u_i f)(x_0, \cdots, x_{n-1}) = \sum_{j=1}^{n} (-1)^j \int_G \cdots \int_G f(x_0, \cdots, x_{j-1}, x_{j-1}\xi_j, \cdots, x_{n-1}\xi_n)\varphi_i(\xi_j) \cdots \varphi_i(\xi_n) d\xi_j \cdots d\xi_n$$

for $f \in K^n$; $n \geqq 1$, and by $u_i f = 0$ for $f \in K^0$.

It is easy to see that if $f \in K_l$, then there is an i such that $\alpha_i f$ and $u_i f$ are in K_l. One verifies the identities

$$(*) \qquad f - \alpha_i f = \delta u_i f + u_i \delta f\,; \qquad i = 0, 1, 2, \cdots$$

(see [5], § 4).

For $f \in K$, let $\bar{f}$ be its image in $\bar{K}$, and if $\bar{f}$ is a cocycle, let $\{f\} \in H(\bar{K})$ be its class.

To prove that $H(\gamma)$ is epimorphic, suppose that there is given a cocycle $\bar{f} \in \bar{K}$. Then $\delta f \in K_l$, whence for a suitable $i, f - \alpha_i f - \delta u_i f \in K_l$. Therefore $\{f\} = \{\alpha_i f\}$. But $\alpha_i f \in {}_dK$.

To show that $H(\gamma)$ is monomorphic, suppose that $f \in {}_dK$ is such that $\{f\} = 0$. Then there are $h \in K, g \in K_l$ such that $f - \delta h = g$. Hence $(*)$ implies

$f = \alpha_i \delta h + \alpha_i g + \delta u_i f + u_i \delta g = \delta(\alpha_i h + u_i f) + (\alpha_i + u_i \delta) g$. Thus, for suitable $i, f - \delta(\alpha_i h + u_i f) \in K_l$, and since $\alpha_i h + u_i f \in {}_dK$, it follows that the cohomology class of f in $H({}_d\bar{K})$ is zero.

7. **A spectral sequence.** Suppose G, π, V and L satisfy the assumptions of Theorem 2. By the result of § 6, Theorem 2 will follow if we show that $H({}_d\bar{C}) \simeq H(L, V)$. We shall consider a bicomplex F, similar to the one defined in [4], § 10, and we shall show that the quotient complex $\bar{F}$ obtained by factoring out the locally trivial cochains is such that

(i) the initial term of the first spectral sequence is

$${}^0E_1^s = H^s({}_d\bar{C}) \quad \text{and} \quad {}^rE_1^s = 0 \quad \text{for all} \quad r > 0\,,$$

(ii) the initial term of the second spectral sequence is

$${}^rE_1^0 = H^r(L, V) \quad \text{and} \quad {}^rE_1^s = 0 \quad \text{for all} \quad s > 0\,.$$

As well known, this implies $H({}_d\bar{C}) \simeq H(L, V)$.

We begin by defining $F = \bigoplus_{r,s \geqq 0} {}^rF^s$. Let $L_1, \cdots, L_r$ be r copies of L and $G_1, \cdots, G_s$, s copies of G. Then, for $r, s \geqq 1$, ${}^rF^s$ is the vector space of all smooth maps

$$L_1 \times \cdots \times L_r \times G_1 \times \cdots \times G_s \to V$$

which are r-linear and alternating in the first r variables. For every $s \geqq 1$, ${}^0F^s$ is the subspace of ${}_dC^s$ composed of those cochains f which satisfy the following local normalization condition: for each $f \in {}^0F^s$, there is a $Q \in \mathscr{U}$ such that $f(x_1, \cdots, x_s) = 0$ whenever $x_1, \cdots, x_s \in Q$ and at least one x_i equals 1. ${}^rF^0$ is, for each $r \geqq 1$, the space of V-valued r-linear alternating functions on L, and ${}^0F^0 = V$.

For each $x \in G$, let $\rho_x: G \to G$ be the right translation $y \mapsto yx$. Denote by ρ_x^* the induced map on the tangent bundle. We shall identify L with the tangent space to G at 1. For each $X \in L$, $\tilde{X}$ will denote the right invariant vector field (i.e., satisfying $\rho_x^*\tilde{X} = \tilde{X}$ for all x) taking at 1 the value X.

Occasionally an $f \in {}^rF^s$ will be interpreted as a differential form on G, depending on the parameter $(x_2, \cdots, x_s) \in G \times \cdots \times G$ which, for fixed value of the parameter, takes at $\tilde{X}_1 \cdots, \tilde{X}_r$ and $x_1 \in G$ the value $f(X_1, \cdots, X_r, x_1, \cdots, x_s)$. The morphisms

$$d_1: {}^rF^s \longrightarrow {}^{r+1}F^s,\ d_2: {}^rF^s \longrightarrow {}^rF^{s+1}$$

are now defined as follows.

If $f \in {}^nF^0$, let $d_1 f$ be given by the formula

$$\begin{aligned}(d_1 f)(X_1, \cdots, X_{n+1}) = {} & \frac{1}{n+1} \sum (-1)^{i+1} \pi_0(X_i) f(X_1, \cdots\hat{}\cdots, X_{n+1}) \\ & + \frac{1}{n+1} \sum (-1)^{i+j} f([X_i, X_j], X_1 \cdot\hat{\cdot}\cdot, X_{n+1})\end{aligned}$$

for every $X_1, \cdots, X_{n+1} \in L$.

Let $f \in {}^rF^s$; $s \geqq 1$. For any fixed $x_2, \cdots, x_s \in G$ consider the differential form ω_f for which identically

$$\omega_f(\tilde{X}_1, \cdots, \tilde{X}_r; x_1) = \pi(x_1^{-1}) f(X_1, \cdots, X_r, x_1, \cdots, x_s)\ .$$

Let $d_1 f$ be the $(r+1)$-form whose value at x_1 is $\pi(x_1) d\omega_f$, d being the exterior derivative ([10], p. 21). One sees easily that $d_1 f \in {}^{r+1}F^s$.

Let $d_2: {}^0F^s \to {}^0F^{s+1}$ be the coboundary δ of § 4. Finally, let $d_2: {}^rF^s \to {}^rF^{s+1}$; $r \geqq 1$, be given by

$$\begin{aligned}(d_2 f)&(X_1, \cdots, X_r, x_1, \cdots, x_{s+1}) \\ &= \sum (-1)^i f(X_1, \cdots, X_r, x_1, \cdots, x_i x_{i+1}, \cdots, x_{s+1}) \\ &\quad + (-1)^{s+1} f(X_1, \cdots, X_r, x_1, \cdots, x_s)\ .\end{aligned}$$

This completes the definition of F.

One has $d_1 d_2 = d_2 d_1$ and $d_1^2 = d_2^2 = 0$ ([4], § 10). Moreover the complex

$${}^rF: 0 \longrightarrow {}^rF^0 \xrightarrow[d_2]{} {}^rF^1 \xrightarrow[d_2]{} \cdots$$

has for $r \geqq 1$ a contracting homotopy $u: {}^rF^{s+1} \to {}^rF^s$ given by

$$(uf)(X_1, \cdots, X_r, x_1, \cdots, x_s) = -f(X_1, \cdots, X_r, 1, x_1, \cdots, x_s)$$

([4], § 9).

Call a bicochain $f \in {}^rF^s$ locally trivial if there exists a $Q \in \mathscr{U}$ such that $f(X_1, \cdots, X_r, x_1, \cdots, x_s) = 0$ for all $X_1, \cdots, X_r \in L, x_1, \cdots, x_s \in Q$. Let $\bar{F}$ be the quotient of F by the sub-bicomplex of locally trivial cochains. Then $\bar{F}$ is a bicomplex with operators $\bar{d}_1$, $\bar{d}_2$ induced by d_1, d_2. We shall show that it has the properties (i), (ii) stated at the beginning of this section.

For each r let ${}^r\bar{F}$ be the complex $0 \to {}^r\bar{F}^0 \to {}^r\bar{F}^1 \to \cdots$ with coboundary $\bar{d}_2$, and let for each s, $\bar{F}^s$ be defined similarly.

To obtain (i), one shows first that the inclusion ${}^0F \subset {}_dC$ induces an isomorphism $H({}^0\bar{F}) \to H({}_d\bar{C})$. This is a consequence of the two facts

(a) if $f \in {}_dC$ and δf is locally trivial, then f is cohomologous in ${}_dC$ to some $h \in {}^0F$,

(b) if $f \in {}^0F$ and $f - \delta g$ is locally trivial for some $g \in {}_dC$, then there exists an $h \in {}^0F$ such that $f - \delta h$ is locally trivial.

The proof of (a) and (b) is easily obtained from that of Lemmas 6.1 and 6.2 in [3], p. 62. One concludes that ${}^0E_1^s = H^s({}_d\bar{C})$, for the first spectral sequence. Since each ${}^r\bar{F}, r \geqq 1$, has a contracting homotopy $\bar{u}$ induced by u, ${}^rE_1^s = 0$ for $r \geqq 1$.

To prove (ii) observe first that $\bar{F}^0 = F^0$ and $H(F^0) = H(L, V)$, by definition. Hence ${}^rE_1^0 = H^r(L, V)$ for the second spectral sequence.

It remains to show that for each $s \geqq 1$, $\bar{F}^s$ is an acyclic complex. Let $f \in {}^rF^s$ be such that $d_1 f$ is locally trivial. Thus there is a $Q \in \mathscr{U}$ such that for each $x_2, \cdots, x_s \in Q$ the $(r+1)$-form $d\omega_f$ vanishes identically on Q. We may assume that Q is diffeomorphic to a Euclidean ball.

For $r = 0$, the condition $d\omega_f = 0$ on Q implies that $\pi(x_1^{-1})f(x_1, \cdots, x_s)$ does not depend on x_1 when $x_1, \cdots, x_s \in Q$. Consequently, by the local normalization condition, f is locally trivial. Hence $\bar{d}_1$: ${}^0\bar{F}^s \to {}^1\bar{F}^s$ is a monomorphism.

For $r \geqq 1$, and any $x_2, \cdots, x_s \in G$, the restriction $\omega_f \mid Q$ is a closed r-form on Q. Hence the Poincarè lemma ([13], p. 87) implies the existence of an $(r-1)$-form μ on Q such that $d\mu = \omega_f$. The proof of Poincarè lemma shows that μ depends smoothly on the parameter $(x_2, \cdots, x_s) \in Q \times \cdots \times Q$ (where smoothness is understood in the sense of [7], § 1). Let φ be a smooth real-valued function on G, identically equal to 1 in some neighbourhood of the identity and vanishing outside some neighbourhood of the identity whose closure is contained in Q. For each $x_2, \cdots, x_s \in G$, let h be the $(r-1)$-form on G which at $x_1 \in G$ takes the value $\varphi(x_1)\varphi(x_2) \cdots \varphi(x_s)\pi(x_1)\mu$ when $x_1, \cdots, x_s \in Q$ and 0 otherwise.

Recalling the interpretation of ${}^rF^s$ as the space of r-forms depending on the parameter $(x_2, \cdots, x_s) \in G \times \cdots \times G$, we see readily that $h \in {}^{r-1}F^s$. Moreover the construction guarantees that $f - d_1h$ is locally trivial. Thus $\bar{F}^s$ is exact at ${}^r\bar{F}^s$ and the proof of Theorem 2 is complete.

8. Explicit form of the isomorphism. We shall describe the isomorphism $H({}_d\bar{C}) \simeq H(L, V)$, i.e., $H({}^0\bar{F}) \simeq H(\bar{F}^0)$. Let Tot F be the total complex of F ([14], p. 340). For $f \in {}^0F^n, n \geqq 1, 1 \leqq j \leqq n$ and $X \in L$ denote by $\partial_j(X)f \in {}^0F^{n-1}$ the derivative in the direction X with respect to the jth variable at $x_j = 1$. Define maps $\tau^{n,r}: {}^0F^n \to {}^rF^{n-r}$; $r = 0, 1, \cdots, n$ by $\tau^{n,0} =$ identity, and for $r \geqq 1$

$$\begin{aligned}(\tau^{n,r}f)&(X_1, \cdots, X_r, x_{r+1}, \cdots, x_n)\\ &= (\textstyle\sum \operatorname{sgn}(i_1, \cdots, i_r)\partial_1(X_{i_1}) \cdots \partial_r(X_{i_r})f)(x_{r+1}, \cdots, x_n) ,\end{aligned}$$

where $\sum$ ranges over all permutations of $(1, \cdots, r)$. It is shown in [4], p. 500 that the maps $\tau^n = \sum_{0 \leqq r \leqq n} \tau^{n,r}: {}^0F^n \to (\text{Tot } F)^n$ define a cochain map $\tau: {}^0F \to \text{Tot } F$. Let $\bar{\tau}: {}^0\bar{F} \to \text{Tot } \bar{F}$ be induced by τ. Denote by $\bar{p}_1, \bar{p}_2$ the projections $\text{Tot } \bar{F} \to \bar{F}^0$, $\text{Tot } \bar{F} \to {}^0\bar{F}$. These are evidently cochain maps and from the behaviour (i), (ii) of the spectral sequences it follows that $H(\bar{p}_1), H(\bar{p}_2)$ are isomorphisms. Now $\bar{p}_2\bar{\tau}$ is the identity, thus $H(\bar{\tau}): H({}^0\bar{F}) \to H(\text{Tot } \bar{F})$ is an isomorphism, whence the same is true about $H(\bar{p}_1\bar{\tau}): H({}^0\bar{F}) \to H(\bar{F}^0)$. Clearly $\bar{p}_1\bar{\tau} \mid {}^0\bar{F}^n = \bar{\tau}^{n,n}$.

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