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MATRIX CHARACTERIZATIONS OF CIRCULAR-ARC GRAPHS

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A graph G is a circular-arc graph if there is a one-to-one correspondence between the vertices of G and a family of arcs on a circle such that two distinct vertices are adjacent when the corresponding arcs intersect. Circular-arc graphs are characterized as graphs whose adjacency matrix has the quasi-circular 1's property. Two interesting subclasses of circular-arc graphs are also discussed proper circular-arc graphs and graphs whose augmented adjacency matrix has the circular 1's property.

Given a finite family S of nonempty sets, the *intersection graph* $G(S)$ has vertices corresponding to the sets of S and two distinct vertices of $G(S)$ are adjacent if and only if the corresponding sets of S intersect. Suppose the graph G is isomorphic to $G(S)$ (i.e., there is a one-to-one correspondence between vertices which preserves adjacency). Then S is called an *intersection model* for G. If S is a family of arcs on a circle, G is called a *circular-arc graph*. See the example in Figure 1. If, in addition, no arc of S contains another arc, G is called a *proper circular-arc graph* (the graph in Figure 1a is not a proper circular-arc graph). Interval graphs and proper interval graphs are analogously defined. Lekkerkerker and Boland [10] have given a structure theorem for interval graphs (in the spirit of Kuratowski's famous characterization of planar graphs). Fulkerson and Gross [4] characterized interval graphs as graphs whose dominant clique-vertex incidence matrix ("dominant" means maximal) has the *consecutive 1's property for colums*, that is, the rows of this matrix can be permuted so that the 1's appear consecutively in each column. They also gave [4] an efficient algorithm to test whether a $(0, 1)$-matrix has the consecutive 1's property for columns. For other characterizations of interval graphs, see Gilmore and Hoffman [5] and Lekkerkerker and Boland [10]. The study of interval graphs was motivated by the central role they played in some work by Benzer [1, 2] concering the molecular substructure of genes. More recently, interval graphs have been applied to problems in archeology [8] and ecology [3]. Proper interval graphs have been characterized by Roberts [11, 12] with a structure theorem and as graphs whose augmented adjacency matrix (defined below) has the consecutive 1's property for columns. He showed that they were equivalent to the indifference graphs of mathematical psychology.

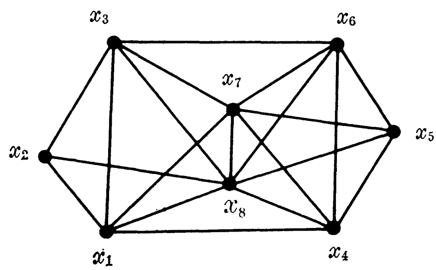

Figure 1a. G

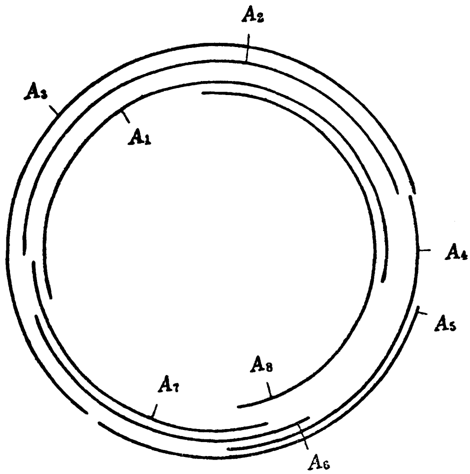

Figure 1b. Circular-arc Model of G

		C_1			C_2				
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
R_1	x_1	1	1	1	1	0	0	1	1
	x_2	1	1	1	0	0	0	0	1
	x_3	1	1	1	0	0	1	1	1
R_2	x_4	1	0	0	1	1	1	1	1
	x_5	0	0	0	1	1	1	1	1
	x_6	0	0	1	1	1	1	1	1
	x_7	1	0	1	1	1	1	1	1
	x_8	1	1	1	1	1	1	1	1

Figure 1c. $M^*(G)$

The problem of characterizing circular-arc graphs was posed in [6] and recently discussed in some detail by Klee [9]. Such graphs may be applicable in testing for circular arrangements of genetic molecules (see [9]). Circular-arc and proper circular-arc graphs also are of interest to workers in coding theory because of their relation to "circular" codes [7]. In this paper, we shall characterize circular-arc and proper circular-arc graphs in terms of their augmented adjacency matrices. In a forthcoming paper [14], we shall present structure theorems for proper circular-arc graphs and for the related unit circular-arc graphs.

We shall deal throughout with graphs in which the set of vertices is finite and the adjacency relation is symmetric and irreflexive (no loops). A *clique* is a subset of vertices in which every two (distinct) vertices are adjacent. Associated with a graph G is an *adjacency matrix* $M(G)$ defined with entry $(i, j) = 1$ if vertices x_i and x_j are adjacent, and $=0$ otherwise. Note that $M(G)$ is symmetric and has 0's on the main diagonal. We define $M^*(G)$, the *augmented adjacency matrix*, to be $M^*(G) = M(G) + I$, i.e., $M(G)$ with 1's added on the main diagonal. A (0, 1)-matrix is said to have the *circular* 1'*s property for columns* if the rows of M can be permuted so that the 1's in each column are circular, that is, appear in a circularly consecutive fashion (think of the matrix as wrapped around a cylinder; see the example in Figure 1c). The consecutive and circular 0's properties for columns are similarly defined. Note that a (0, 1)-matrix has the circular 1's property for columns if and only if it has the circular 0's property for columns.

THEOREM 1. *Let M_1 be a* (0, 1)*-matrix. Form the matrix M_2 from M_1 by complementing* (*interchanging* 0's *and* 1's) *those columns with a* 1 *in the first row of M_1. Then M_1 has the circular* 1'*s property for columns if and only if M_2 has the consecutive* 1'*s property for columns.*

Proof. If the rows of a matrix are arranged so that the 1's (and hence, the 0's) in each column are circular, then complementing any set of columns yields a matrix with this same property. However, a matrix whose first row is all 0's clearly has the consecutive 1's property for columns if and only if it has the circular 1's property for columns.

Fulkerson and Gross [4] have described an efficient algorithm to test whether a (0, 1)-matrix has the consecutive 1's property for columns and to obtain a desired row permutation when one exists. Using Theorem 1, their algorithm now solves the corresponding problem for circular 1's. The natural generalization of the matrix characterizations of interval and proper interval graphs mentioned

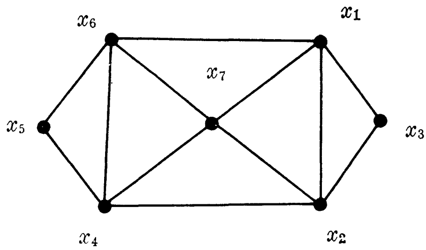

Figure 2a. G

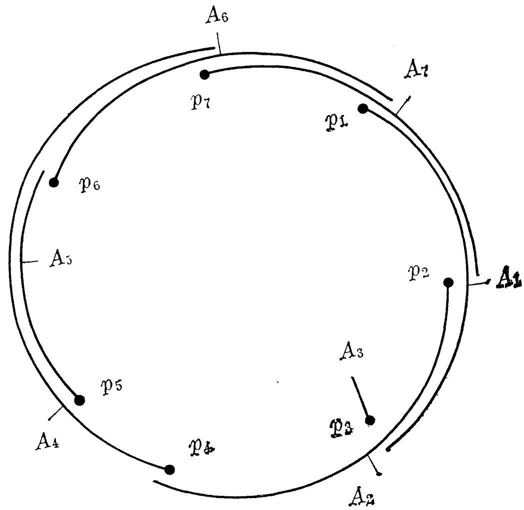

Figure 2b. Circular-arc Model of G

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	1	1	0	0	1	1
x_2	1	1	1	1	0	0	1
x_3	1	1	1	0	0	0	0
x_4	0	1	0	1	1	1	1
x_5	0	0	0	1	1	1	0
x_6	1	0	0	1	1	1	1
x_7	1	1	0	1	0	1	1

Figure 2c. $M^*(G)$ with U_i's and V_i's circled

above would use the circular 1's property for columns in characterizing circular-arc and proper circular-arc graphs. However, this approach seems to fail; for example, the graph in Figure 1a has an augmented adjacency matrix with the circular 1's property for columns, yet it is not a proper circular-arc graph. An additional condition is needed to characterize $M^*(G)$ when G is a proper circular-arc graph. And a weakened form of the circular 1's property for columns will be shown to characterize $M^*(G)$ when G is a circular-arc graph.

Let M be a symmetric $(0, 1)$-matrix with 1's on the main diagonal. Let U_i be the circular set of 1's in column i starting at the main diagonal and going down (and around) as far as possible, i.e., until a 0 is encountered. Let V_i be the analogous set of 1-entries in row i starting at the main diagonal and going right (thus entry (i, j) is in V_i if and only if entry (j, i) is in U_i). Then M is said have *quasi-circular* 1's if the U_i's and V_i's contain all the 1's in M (the matrix in Figure 2c has quasi-circular 1's but the matrix with circular 1's in Figure 1c does not).

THEOREM 2. *G is a circular-arc graph if and only if its vertices can be indexed so that $M^*(G)$ has quasi-circular 1's.*

Proof. Let S, a family of arcs on a circle, be an intersection model for the graph G. Without loss of generality, we may assume that S is chosen so that (a) none of its arcs equals the whole circle, (b) the arcs are closed (i.e., they contain their endpoints), and (c) no two arcs have a common counterclockwise endpoint. Starting with an arbitrary arc, index the arcs of S by the order in which their counterclockwise endpoints occur as one goes clockwise around the circle (see the example in Figure 2b). Let x_i be the vertex of G corresponding to arc A_i. We claim that with this indexing of the vertices, $M^*(G)$ now has quasi-circular 1's. Suppose x_i is adjacent to x_j and $i < j$. Then arcs A_i and A_j intersect. Either the counterclockwise endpoint of A_j is in A_i or vice versa. In the former case, for $i < k \leq j$, arc A_i intersects A_k and so $M^*(G)$ has a circular set of 1's in column i starting at the main diagonal and going down at least to entry (j, i). A similar argument holds in the latter case. This proves our claim.

Suppose the n vertices of G are indexed so that $M^*(G)$ has quasi-circular 1's. For $i = 1, 2, \cdots, n$, let p_i label the ith hour point on an n-hour clock (see example in Figure 2). Suppose U_i, the maximal circular set of 1's starting from the main diagonal in column i of $M^*(G)$, ends at entry (m_i, i). Then draw arc A_i with counterclockwise endpoint p_i and clockwise endpoint p_{m_i} (possibly $m_i = i$ and arc A_i is a point; if column i is all 1's, $m_i = i - 1$ or

$m_i = n$ if $i = 1$). Suppose that arcs A_i and A_j ($i \neq j$) intersect and that the counterclockwise endpoint of A_j is in A_i. This is equivalent to the fact that U_i extends at least as far as entry (j, i) and thus x_i is adjacent to x_j. Hence the A_i's form an intersection model for G.

Observe that the proof of Theorem 2 gives a simple algorithm for constructing a circular-arc model for G when $M^*(G)$ has quasi-circular 1's. Unfortunately, we currently lack an algorithm to test whether $M^*(G)$ can have quasi-circular 1's. However, there are two important subclasses of circular-arc graphs for which we have both an efficient test and an efficient construction of circular-arc models. These are proper circular-arc graphs and the graphs G such that $M^*(G)$ has the circular 1's property for columns. To show that the latter graphs are circular-arc graphs, we first need the following lemma.

LEMMA 3. *G is a circular-arc graph if $M^*(G)$ has the consecutive 0's property for columns.*

Proof. Let the vertices of G be indexed so that the 0's occur consecutively in each column (and, by symmetry, in each row) of $M^*(G)$. Let C_1 be the set of columns whose 0's are below the main diagonal and C_2 the set of columns with 0's above it (see Figure 1c). Let R_1 and R_2 be the corresponding sets of rows. If columns i and j are in C_1 with $i < j$, then the column j has all 1's above the main diagonal and hence entry (i, j) is 1. Thus the vertices corresponding to columns in C_1 form a clique K_1. Similarly the vertices corresponding to columns in C_2 form a clique K_2.

On a circle draw a set of 90° closed arcs such that (a) no two arcs have the same endpoints, and (b) the arcs have a common point of intersection. Identify successive rows of R_2 with successive arcs (in clockwise order). Now for each column in C_1, draw an arc which is the complement of the union of the arcs corresponding to rows in which the column has a 0 (see the example in Figure 1). The reader can easily check that we now have an intersection model for the vertices in K_1 and K_2. Any vertex that is not in K_1 or K_2 corresponds to a column of $M^*(G)$ that is all 1's. For any such vertex, we draw a 180° closed arc whose clockwise endpoint is a common intersection point of the arcs corresponding to rows of R_2 (see Figure 1).

THEOREM 4. *G is a circular-arc graph if $M^*(G)$ has the circular 1's property for columns.*

Proof. Let the vertices of G be indexed so that the 1's in each column of $M^*(G)$ are circular. If $M^*(G)$ now has quasi-circular 1's, we are finished by Theorem 2. If $M^*(G)$ does not have quasi-circular

1's, let entry (j, k) be a 1-entry not contained in the sets U_k or V_j (see the definition of quasi-circular 1's). By a cyclic permutation of rows and columns, if necessary, we can assume that this entry is in the last row (i.e., row j is the last row) and further that entry $(1, k)$

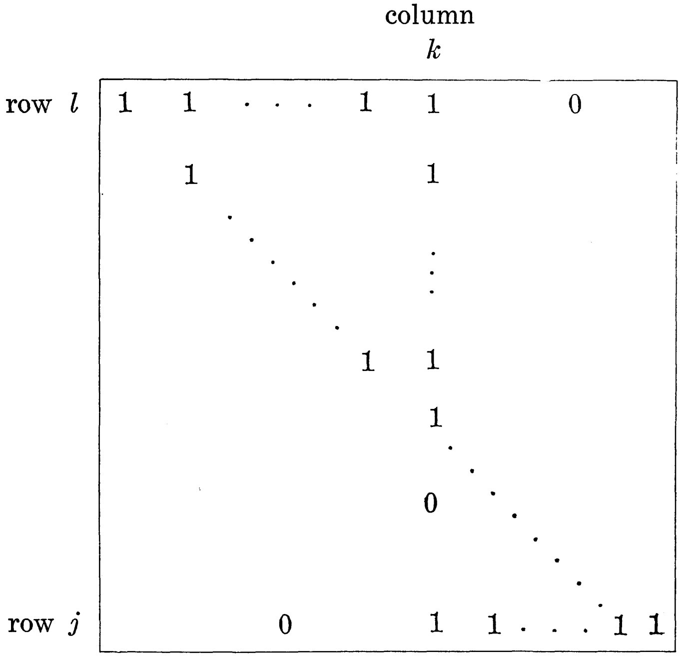

Figure 3: See text

is in V_1 (see Figure 3). Note that entry $(1, k)$ must be a 1-entry since the 1's in column k are circular, yet entry (j, k) is not in U_k. Since the 0's are circular in each row (and column) of $M^*(G)$, the 0's in row j (the last row) must all be to the left of entry (j, k) and the 0's (if any) in row 1 must all be to the right of entry $(1, k)$. Hence no column of $M^*(G)$ has 0's in both the first and last rows. Thus $M^*(G)$ has consecutive 0's in each column and this theorem follows from Lemma 3.

Using the Fulkerson-Gross algorithm and Theorem 1, we can test for an arrangement of $M^*(G)$ with circular 1's. If such an arrangement exists, we can construct a circular-arc model for G by the method in Theorem 2 or Lemma 3. The matrix in Figure 2c does not have the circular 1's property for columns and hence the converse of Theorem 4 is false.

A symmetric (0, 1)-matrix is said to have *circularly compatible* 1's if the 1's in each column are circular and if, after inverting and/or cyclicly permuting the order of the rows and (corresponding) columns, the last 1 (in cyclicly descending order) of the circular set in the second column is always at least as low as the last 1 of the circular set in the first column unless one of these columns is all 1's or all 0's (inversion and cyclic permutation do not affect the circularity of the 1's). See the example in Figure 5c.

LEMMA 5. *Suppose the n vertices of G are indexed so that the* 1*'s in each column of $M^*(G)$ are circular. If this arrangement of $M^*(G)$ does not have circularly compatible* 1*'s, then $M^*(G)$ has the consecutive* 0*'s property for columns and the vertices of G partition into two cliques.*

Proof. Assume no row (or column) is all 1's, for after cyclicly permuting to make such a row first, the 0's in each column are consecutive, and a vertex corresponding to such a row can be added to either of two cliques shown above to partition the other vertices. Suppose this arrangement of $M^*(G)$ does not have circularly compatible 1's. After inverting and/or cyclicly permuting the rows and columns of this arrangement, we can assume that if entries $(k, 1)$ and $(j, 2)$ are the last 1-entries of the circular set of 1's in the first and second columns, respectively, then $j < k$ (see Figure 4). Assume entry $(k, 2)$ is 0 (that is, the 0's in column 2 start in entry $(j + 1, 2)$ and extend down at least to entry $(k, 2)$), for otherwise columns 1 and 2 have 0's in different rows and cyclicly permuting column 2 to

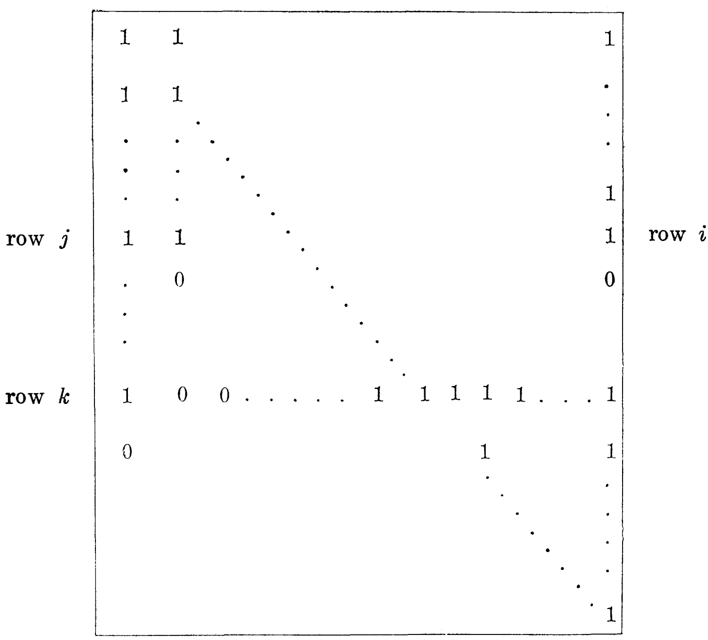

Figure 4. See text.

column 1 (and similarly with rows) yields consecutive 0's in each row and column, as desired. Then the 0's in row k occur between the first column and the main diagonal. Let entry (i, n) be the last 1 in the circular set of 1's in the last column. Assume $i < k$ or else cyclicly permute the last column (with a crresponding row permutation) to become the first column and repeat the permutation until $i < k$ (if $i \geqq k$, then after such a permutation, the last 1 of the circular set in column 1 is at least one entry lower; but no column is all 1's; and so eventually $i < k$). Since the 0's in row k are (still) to the left of the main diagonal, entry (k, n) is 1. Since $i < k$, the last column has all 1's from row k down. Then no row has 0's in both the first and last column. The vertices of G partition into two cliques by the argument in Lemma 3.

THEOREM 6. *G is a proper circular-arc graph if and only if there is an arrangement of $M^*(G)$ with circularly compatible* 1's.

Proof. Suppose S, a family of arcs on a circle such that no arc of S contains another, is an intersection model for G. Index the arcs of S as in the necessity proof in Theorem 2. This induces the desired arrangement of $M^*(G)$. If $M^*(G)$ is arranged with circularly compatible 1's, then construct a circular-arc model as prescribed in Theorem 4. In the resulting model, one arc can contain another only if they have a common endpoint. In such cases we slightly extend the shorter arc or shorten the longer arc. Details are left to the reader (see example in Figure 5).

We test for circularly compatible 1's as follows. First test G to see if there are two cliques which partition the vertices of G (this is equivalent to testing whether the complement of G is bipartite). If not, then it is sufficient by Lemma 5 to test for the circular 1's property for columns, and any arrangement of $M^*(G)$ with circular 1's must have circularly compatible 1's by Lemma 5. Suppose now that two such cliques exist. If an arrangement of $M^*(G)$ has circular 1's and the first and last rows correspond to vertices in different cliques, then the 0's must be consecutive in each column, i.e., $M^*(G)$ has the consecutive 0's property for columns. Now $M^*(G)$ has an arrangement with circularly compatible 1's if and only if M^c, the complement of $M^*(G)$ (obtained by interchanging 0's and 1's), has a corresponding arrangement with circularly compatible 1's and in addition, M^c has the consecutive 1's property for columns. A modification in the last step of the Fulkerson-Gross algorithm yields a test for the existence of an arrangement of M^c with both consecutive 1's and circularly compatible 1's (see the end of Chapter Two in [13]).

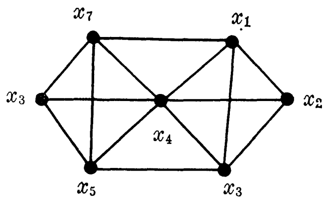

Figure 5a. G

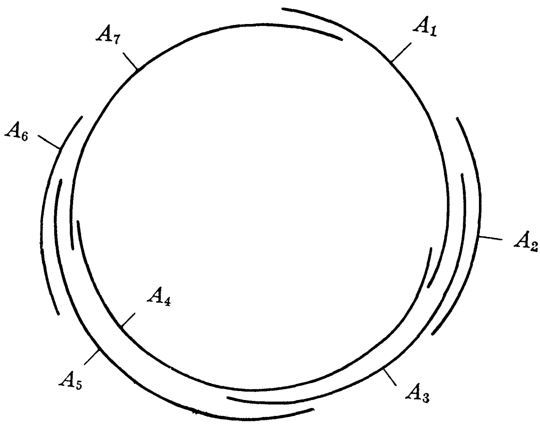

Figure 5b. "Proper" Circular-arc Model of G

	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1	1	1	1	1	0	0	1
x_2	1	1	1	1	0	0	0
x_3	1	1	1	1	1	0	0
x_4	1	1	1	1	1	1	1
x_5	0	0	1	1	1	1	1
x_6	0	0	0	1	1	1	1
x_7	1	0	0	1	1	1	1

Figure 5c. $M^*(G)$

REFERENCES

1. S. Benzer, *On the topology of the genetic fine structure*, Proc. Nat. Acad. Sci. U.S.A., **45** (1959), 1607-1620.

2. ———, *The fine structure of the gene*, Scientific American, **206** (1962) No. 1, 70-84.

3. J. Cohen, *Interxal graphs and food webs*, The RAND Corporation D-17696-PR.

4. D. R. Fulkerson, and O. A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math., **15** (1965), 835-855.

5. P. C. Gilmore, and A. J. Hoffman, *A characterization of comparability graphs and of internal graphs*, Canad. J. Math., **16** (1964), 539-548.

6. H. H. Hadwiger, Debrunner, and V. Klee, *Combinatorial Geometry in the Plane*, New York: Holt Rinehardt and Winston, 1964, p. 54.

7. L. Harper, Rockefeller University, personal communication.

8. D. Kendall, *Incidence matrices, Interval Graphs, and Seriation in Archeology*, Pacific J. Math., **28** (1969), 565-570.

9. V. Klee, *What are the intersection graphs of arcs in a circle*? Amer. Math. Monthly, **76** (1969) No. 7, 810-813.

10. C. G. Lekkerkerker, and J. Ch. Boland, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math., **51** (1962), 45-64.

11. F. S. Robert, *Representions of indifference relations*, Doctoral Dissertation, Stanford University, 1968.

12. ———, *Indifference Graphs*, in Proof Techniques in Graph Theory (Proc. 2nd Ann Arbor Graph Theory Conf.), F. Harary (ed.), New York: Academic Press, 1969.

13. A. C. Tucker, *Two characterizations of proper circular-arc graphs*, Stanford Operations Research Department Report No. 69-6.

14. ———, *Structure theorems for some circular graphs*, to appear.

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