

Pacific Journal of Mathematics

5-DESIGNS IN AFFINE SPACES

WILLIAM O'BANNON ALLTOP

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W. O. ALLTOP

The n -dimensional affine group over $GF(2)$ is triply transitive on 2^n symbols. For $n \geq 4$, $4 \leq k \leq 2^{n-1}$, any orbit of k -subsets is a 3 - $(2^n, k, \lambda)$ design. In this paper a sufficient condition that such a design be a 4-design is given. It is also shown that such a 4-design must always be a 5-design. A 5-design on 256 varieties with block size 24 is constructed in this fashion.

We shall call (Ω, \mathcal{D}) a t - (v, k, λ) design whenever $|\Omega| = v$, \mathcal{D} is a family of k -subsets of Ω and every t -subset of Ω is contained in exactly λ members of \mathcal{D} . The design is nontrivial provided \mathcal{D} is a proper subfamily of Σ_k , the family of all k -subsets of Ω . If G is a nontrivial t -ply transitive group acting on Ω , then an orbit of k -subsets under G yields a t -design. The design is nontrivial if G is not k -homogeneous (transitive on unordered k -subsets). The first known 5-designs arose from orbits under the quintuply transitive Mathieu groups M_{12} and M_{24} . Other 5-designs on 12, 24, 36, 48 and 60 varieties have been discovered (see [2; 3; 4]). In [1] a 5-design on $2^n + 2$ varieties is constructed for every $n \geq 4$. Here we shall discuss 5-designs on 2^n varieties, giving one example for $n = 8$.

Let Ω be an n -dimensional vector space over $GF(2)$, $n \geq 4$. Let L be the linear group $GL(n, 2)$ acting doubly transitively on $\Omega - \{0\}$ and T the group of translations $t_\alpha: \omega \rightarrow \omega + \alpha$. The group $A = \langle L, T \rangle$ is the triply transitive affine group on Ω . Let Σ_4, Σ_5 denote the families of 4-, 5-subsets of Ω respectively. (Ω, \mathcal{S}_0) is a 3 - $(2^n, 4, 1)$ design where \mathcal{S}_0 is the family of quadruples $\{\omega_i\}$ satisfying

$$\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0.$$

\mathcal{S}_0 is the orbit of affine planes in Ω . \mathcal{S}_1 is also an orbit, where $\mathcal{S}_1 = \Sigma_4 - \mathcal{S}_0$. Thus, A decomposes Σ_4 into only two orbits. From the design parameters of (Ω, \mathcal{S}_0) one establishes that

$$|\mathcal{S}_0| = \frac{1}{4} \binom{2^n}{3}$$

$$|\mathcal{S}_1| = (2^{n-2} - 1) \binom{2^n}{3}.$$

Suppose $Q \in \mathcal{S}_0$. The stabilizer of Q in A is transitive on $\Omega - Q$. Thus, \mathcal{S}_0 is an orbit under A , where \mathcal{S}_0 consists of those members of Σ_5 which contain a member of \mathcal{S}_0 . Now suppose $R \in \Sigma_5 - \mathcal{S}_0$.

Clearly there exists a translate of R of the form

$$R_0 = \{0, \omega_1, \omega_2, \omega_3, \omega_4\} .$$

Since R_0 contains no member of \mathcal{S}_0 , the ω_i 's must be linearly independent in Ω considered as a vector space. Since L is transitive on linearly independent quadruples in $\Omega - \{0\}$, it follows that A must be transitive on the family \mathcal{F}_1 , where $\mathcal{F}_1 = \Sigma_5 - \mathcal{F}_0$. Therefore, A also decomposes Σ_5 into only two orbits. From our knowledge of $|\mathcal{S}_0|$ we can deduce that

$$\begin{aligned} |\mathcal{F}_0| &= (2^n - 4) |\mathcal{S}_0| , \\ |\mathcal{F}_1| &= \frac{1}{5}(2^n - 4)(2^n - 8) |\mathcal{S}_0| . \end{aligned}$$

Geometrically \mathcal{F}_0 consists of the 5-subsets which generate 3-dimensional affine subspaces of Ω , while the members of \mathcal{F}_1 generate 4-dimensional subspaces. This classification of orbits in Σ_4 and Σ_5 will provide the information needed to investigate 4- and 5-designs which arise from orbits under A .

Suppose Δ is a k -subset of Ω and let \mathcal{D} denote the orbit of Δ under A . Let σ_i, τ_i denote the number of members of $\mathcal{S}_i, \mathcal{F}_i$ contained in Δ respectively, $i = 0, 1$. Let λ_i, μ_i denote the number of members of \mathcal{D} containing a fixed member of $\mathcal{S}_i, \mathcal{F}_i$ respectively, $i = 0, 1$. If $\lambda_0 = \lambda_1$ ($\mu_0 = \mu_1$), then (Ω, \mathcal{D}) is a 4-design (5-design). The following equations relating the $\sigma_i, \tau_i, \lambda_i, \mu_i$ are the result of straightforward counting arguments:

- (1) $\sigma_i |\mathcal{D}| = \lambda_i |\mathcal{S}_i|$
- (2) $\tau_i |\mathcal{D}| = \mu_i |\mathcal{F}_i|$
- (3) $\tau_0 = \sigma_0(k - 4) .$

From (1) and the fact that

$$|\mathcal{S}_0|/|\mathcal{S}_1| = 1/(2^n - 4)$$

we see that (Ω, \mathcal{D}) is a 4-design if and only if

$$(4) \quad \sigma_1 = \sigma_0(2^n - 4) .$$

Likewise from (2) and the fact that

$$|\mathcal{F}_0|/|\mathcal{F}_1| = 5/(2^n - 8)$$

we see that (Ω, \mathcal{D}) is a 5-design if and only if

$$(5) \quad \tau_1 = \tau_0(2^n - 8)/5 .$$

Since $\sigma_1 = \binom{k}{4} - \sigma_0$ and $\tau_1 = \binom{k}{5} - \tau_0$, we can use (3) to express σ_1, τ_0, τ_1 in terms of σ_0 and k . Substituting accordingly for σ_1, τ_0, τ_1 in (4) and (5) we obtain

$$(4') \quad \binom{k}{4} - \sigma_0 = \sigma_0(2^n - 4)$$

$$(5') \quad \binom{k}{5} - \sigma_0(k - 4) = \sigma_0(k - 4)(2^n - 8)/5.$$

After simplifying the preceding equations we see that both (4') and (5') are equivalent to

$$(6) \quad \sigma_0 = \binom{k}{4} / (2^n - 3).$$

We have in effect proved the following

THEOREM. *(Ω, \mathcal{D}) is a 5-design whenever (Ω, \mathcal{D}) is a 4-design. A necessary and sufficient condition for this to take place is that $\sigma_0 = \binom{k}{4} / (2^n - 3)$.*

The first thing to note is that $2^n - 3$ must divide $\binom{k}{4}$ for such a 5-design to exist. This is not possible for $6 \leq k \leq 2^{n-1}$ if $2^n - 3$ is a prime power. Therefore, the first feasible value of n is eight. For $n = 8$, the values of $k \leq 2^7$ for which $2^n - 3$ divides $\binom{k}{4}$ are 23, 24, 25, 46, 47 and 69. We pursue the case $n = 8, k = 24$.

Our theorem tells us that for $|\Delta| = 24$, (Ω, \mathcal{D}) is a 5-design provided $\sigma_0 = 42$. We must select a 24-subset Δ which contains exactly 42 members of \mathcal{S}_0 . One example of such a Δ is the following. Let $(u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2)$ be a basis for the vector space Ω . We define 3-dimensional vector subspaces of Ω :

$$\begin{aligned} U_0 &= (u_1, u_2, u_3) \\ V_0 &= (v_1, v_2, v_3) \\ W_0 &= (u_1 + v_1, u_2 + v_2, u_3 + v_3). \end{aligned}$$

Now let $\Delta = U \cup V \cup W$, where

$$\begin{aligned} U &= U_0 + w_1 \\ V &= V_0 + w_2 \\ W &= W_0 + (w_1 + w_2). \end{aligned}$$

For this Δ it is clear that $\sigma_0 \geq 42$ since each of the 3-dimensional

affine subspaces U, V, W contains 14 members of \mathcal{S}_0 . Suppose Δ contains additional members of \mathcal{S}_0 . There exists $Q \in \mathcal{S}_0$ such that Q meets at least two members of $\{U, V, W\}$. In order to decrease the number of cases to be considered we investigate the action of the stabilizer of Δ on $\{U, V, W\}$. Let $x, y \in L$ be defined by

$$\begin{aligned}
 x: & \begin{cases} u_i \rightarrow v_i \rightarrow (u_i + v_i) \rightarrow u_i, & 1 \leq i \leq 3 \\ w_1 \rightarrow w_2 \rightarrow (w_1 + w_2) \rightarrow w_1 \end{cases} \\
 y: & \begin{cases} u_i \rightarrow v_i \rightarrow u_i, & 1 \leq i \leq 3 \\ w_1 \rightarrow w_2 \rightarrow w_1. \end{cases}
 \end{aligned}$$

Letting x^*, y^* denote the action of x, y on $\{U, V, W\}$, we have

$$\begin{aligned}
 x^*: & U \rightarrow V \rightarrow W \rightarrow U \\
 y^*: & U \rightarrow V \rightarrow U, \quad W \rightarrow W.
 \end{aligned}$$

Hence, $\langle x^*, y^* \rangle$ acts as the symmetric group S_3 on $\{U, V, W\}$. We must only consider the cases where the partition of Q induced by (U, V, W) is of the form $(2, 2, 0)$, $(3, 1, 0)$ or $(2, 1, 1)$. These three cases are easily seen to be impossible, so no such Q exists. It follows that $\sigma_0 = 42$, and we have a 5-design on 256 varieties with blocks of size 24.

One wonders in how many affine spaces Ω such 5-designs exist. Since 143 divides $2^n - 3$ whenever $n \equiv 28 \pmod{60}$, there are infinitely many values of n for which $2^n - 3$ is not a prime power. For fixed k, n , with $6 \leq k \leq 2^{n-1}$, let us consider the problem heuristically. Suppose we select Δ from Σ_k randomly, each member of Σ_k having probability $1/\binom{2^n}{k}$ of being selected. Now σ_0 is a random variable on the probability space Σ_k . The expectation of σ_0 is

$$E = \binom{k}{4} / (2^n - 3).$$

A 5-design of the type under consideration exists if and only if σ_0 achieves its expectation in Σ_k . When E is an integer, it does not seem unreasonable that σ_0 would achieve its expectation.

The author has not investigated the construction of designs in affine spaces over $GF(2)$ by using more than one orbit under A .

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