5-DESIGNS IN AFFINE SPACES

WILLIAM O’BANNON ALLTOP
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The \( n \)-dimensional affine group over \( GF(2) \) is triply transitive on \( 2^n \) symbols. For \( n \geq 4, 4 \leq k \leq 2^{n-1} \), any orbit of \( k \)-subsets is a \( 3-(2^n, k, \lambda) \) design. In this paper a sufficient condition that such a design be a \( 4 \)-design is given. It is also shown that such a \( 4 \)-design must always be a \( 5 \)-design. A \( 5 \)-design on 256 varieties with block size 24 is constructed in this fashion.

We shall call \((\Omega, \mathcal{D})\) a \( t-(v, k, \lambda) \) design whenever \( |\Omega| = v \), \( \mathcal{D} \) is a family of \( k \)-subsets of \( \Omega \) and every \( t \)-subset of \( \Omega \) is contained in exactly \( \lambda \) members of \( \mathcal{D} \). The design is nontrivial provided \( \mathcal{D} \) is a proper subfamily of \( \Sigma_k \), the family of all \( k \)-subsets of \( \Omega \). If \( G \) is a nontrivial \( t \)-ply transitive group acting on \( \Omega \), then an orbit of \( k \)-subsets under \( G \) yields a \( t \)-design. The design is nontrivial if \( G \) is not \( k \)-homogeneous (transitive on unordered \( k \)-subsets). The first known \( 5 \)-designs arose from orbits under the quintuply transitive Mathieu groups \( M_{12} \) and \( M_{24} \). Other \( 5 \)-designs on 12, 24, 36, 48 and 60 varieties have been discovered (see [2; 3; 4]). In [1] a \( 5 \)-design on \( 2^n + 2 \) varieties is constructed for every \( n \geq 4 \). Here we shall discuss \( 5 \)-designs on \( 2^n \) varieties, giving one example for \( n = 8 \).

Let \( \Omega \) be an \( n \)-dimensional vector space over \( GF(2) \), \( n \geq 4 \). Let \( L \) be the linear group \( GL(n, 2) \) acting doubly transitively on \( \Omega \setminus \{0\} \) and \( T \) the group of translations \( t_\alpha: \omega \to \omega + \alpha \). The group \( A = \langle L, T \rangle \) is the triply transitive affine group on \( \Omega \). Let \( \Sigma_4, \Sigma_5 \) denote the families of \( 4-, 5 \)-subsets of \( \Omega \) respectively. \((\Omega, \mathcal{H})\) is a \( 3-(2^n, 4, 1) \) design where \( \mathcal{H} \) is the family of quadruples \( \{\omega_i\} \) satisfying

\[
\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0.
\]

\( \mathcal{H} \) is the orbit of affine planes in \( \Omega \). \( \mathcal{H}_1 \) is also an orbit, where \( \mathcal{H}_1 = \Sigma_4 - \mathcal{H} \). Thus, \( A \) decomposes \( \Sigma_4 \) into only two orbits. From the design parameters of \((\Omega, \mathcal{H})\) one establishes that

\[
|\mathcal{H}| = \frac{1}{4} \binom{2^n}{3}
\]

\[
|\mathcal{H}_1| = (2^{n-2} - 1) \binom{2^n}{3}.
\]

Suppose \( Q \in \mathcal{H} \). The stabilizer of \( Q \) in \( A \) is transitive on \( \Omega \setminus Q \). Thus, \( \mathcal{H}_0 \) is an orbit under \( A \), where \( \mathcal{H}_0 \) consists of those members of \( \Sigma_5 \) which contain a member of \( \mathcal{H} \). Now suppose \( R \in \Sigma_5 - \mathcal{H}_0 \).
Clearly there exists a translate of $R$ of the form

$$R_0 = \{0, \omega_1, \omega_2, \omega_3, \omega_4\}.$$ 

Since $R_0$ contains no member of $\mathcal{I}$, the $\omega_i$'s must be linearly independent in $\Omega$ considered as a vector space. Since $L$ is transitive on linearly independent quadruples in $\Omega - \{0\}$, it follows that $A$ must be transitive on the family $\mathcal{I}_0$, where $\mathcal{I}_0 = \Sigma - \mathcal{I}_0$. Therefore, $A$ also decomposes $\Sigma_5$ into only two orbits. From our knowledge of $|\mathcal{I}_0|$ we can deduce that

$$|\mathcal{I}_0| = (2^n - 4) |\mathcal{I}_0|,$$
$$|\mathcal{I}_1| = \frac{1}{5} (2^n - 4) (2^n - 8) |\mathcal{I}_0|.$$ 

Geometrically $\mathcal{I}_0$ consists of the 5-subsets which generate 3-dimensional affine subspaces of $\Omega$, while the members of $\mathcal{I}_1$ generate 4-dimensional subspaces. This classification of orbits in $\Sigma_4$ and $\Sigma_5$ will provide the information needed to investigate 4- and 5-designs which arise from orbits under $A$.

Suppose $\Delta$ is a $k$-subset of $\Omega$ and let $\mathcal{B}$ denote the orbit of $\Delta$ under $A$. Let $\sigma_i$, $\tau_i$ denote the number of members of $\mathcal{I}$, $\mathcal{I}_i$ contained in $\Delta$ respectively, $i = 0, 1$. Let $\lambda_i$, $\mu_i$ denote the number of members of $\mathcal{B}$ containing a fixed member of $\mathcal{I}$, $\mathcal{I}_i$ respectively, $i = 0, 1$. If $\lambda_0 = \lambda_1$ ($\mu_0 = \mu_1$), then $(\Omega, \mathcal{B})$ is a 4-design (5-design).

The following equations relating the $\sigma_i$, $\tau_i$, $\lambda_i$, $\mu_i$ are the result of straightforward counting arguments:

(1) $\sigma_i |\mathcal{B}| = \lambda_i |\mathcal{I}_i|$

(2) $\tau_i |\mathcal{B}| = \mu_i |\mathcal{I}_i|$

(3) $\tau_0 = \sigma_0 (k - 4)$.

From (1) and the fact that

$$|\mathcal{I}_0| / |\mathcal{I}| = 1/(2^n - 4)$$

we see that $(\Omega, \mathcal{B})$ is a 4-design if and only if

(4) $\sigma_1 = \sigma_0 (2^n - 4)$.

Likewise from (2) and the fact that

$$|\mathcal{I}_0| / |\mathcal{I}_1| = 5/(2^n - 8)$$

we see that $(\Omega, \mathcal{B})$ is a 5-design if and only if

(5) $\tau_1 = \tau_0 (2^n - 8)/5$.
Since \( \sigma_1 = \left( \frac{k}{4} \right) - \sigma_0 \) and \( \tau_1 = \left( \frac{k}{5} \right) - \tau_0 \), we can use (3) to express \( \sigma_1, \tau_0, \tau_1 \) in terms of \( \sigma_0 \) and \( k \). Substituting accordingly for \( \sigma_1, \tau_0, \tau_1 \) in (4) and (5) we obtain

\[
\left( 4' \right) \quad \left( \frac{k}{4} \right) - \sigma_0 = \sigma_0(2^n - 4)
\]

\[
\left( 5' \right) \quad \left( \frac{k}{5} \right) - \sigma_0(k - 4) = \sigma_0(k - 4)(2^n - 8)/5 .
\]

After simplifying the preceding equations we see that both (4') and (5') are equivalent to

\[
\left( 6 \right) \quad \sigma_0 = \left( \frac{k}{4} \right)/(2^n - 3) .
\]

We have in effect proved the following

**Theorem.** \((\Omega, \mathcal{D})\) is a 5-design whenever \((\Omega, \mathcal{D})\) is a 4-design. A necessary and sufficient condition for this to take place is that \( \sigma_0 = \left( \frac{k}{4} \right)/(2^n - 3) .
\]

The first thing to note is that \( 2^n - 3 \) must divide \( \left( \frac{k}{4} \right) \) for such a 5-design to exist. This is not possible for \( 6 \leq k \leq 2^{n-1} \) if \( 2^n - 3 \) is a prime power. Therefore, the first feasible value of \( n \) is eight. For \( n = 8 \), the values of \( k \leq 2^7 \) for which \( 2^n - 3 \) divides \( \left( \frac{k}{4} \right) \) are 23, 24, 25, 46, 47 and 69. We pursue the case \( n = 8, k = 24 \).

Our theorem tells us that for \( |\Delta| = 24 \), \((\Omega, \mathcal{D})\) is a 5-design provided \( \sigma_0 = 42 \). We must select a 24-subset \( \Delta \) which contains exactly 42 members of \( \mathcal{S} \). One example of such a \( \Delta \) is the following. Let \( (u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2) \) be a basis for the vector space \( \Omega \). We define 3-dimensional vector subspaces of \( \Omega \):

\[
U_0 = (u_1, u_2, u_3)
\]

\[
V_0 = (v_1, v_2, v_3)
\]

\[
W_0 = (u_1 + v_1, u_2 + v_2, u_3 + v_3) .
\]

Now let \( \Delta = U \cup V \cup W \), where

\[
U = U_0 + w_1
\]

\[
V = V_0 + w_2
\]

\[
W = W_0 + (w_1 + w_2) .
\]

For this \( \Delta \) it is clear that \( \sigma_0 \geq 42 \) since each of the 3-dimensional
affine subspaces $U$, $V$, $W$ contains 14 members of $\mathcal{S}$. Suppose $\mathcal{A}$ contains additional members of $\mathcal{S}$. There exists $Q \in \mathcal{S}$ such that $Q$ meets at least two members of \{U, V, W\}. In order to decrease the number of cases to be considered we investigate the action of the stabilizer of $\mathcal{A}$ on \{U, V, W\}. Let $x, y \in L$ be defined by

$$
\begin{align*}
  \begin{array}{l}
    x: \\
    y:
  \end{array}
\begin{align*}
  u_i \rightarrow v_i \rightarrow (u_i + v_i) \rightarrow u_i, & \quad 1 \leq i \leq 3 \\
  w_1 \rightarrow w_2 \rightarrow (w_1 + w_2) \rightarrow w_1
\end{align*}
\end{align*}
$$

Letting $x^*, y^*$ denote the action of $x$, $y$ on \{U, V, W\}, we have

$$
\begin{align*}
  x^*: & \quad U \rightarrow V \rightarrow W \rightarrow U \\
  y^*: & \quad U \rightarrow V \rightarrow U, \quad W \rightarrow W
\end{align*}
$$

Hence, $\langle x^*, y^* \rangle$ acts as the symmetric group $S_3$ on \{U, V, W\}. We must only consider the cases where the partition of $Q$ induced by $(U, V, W)$ is of the form (2, 2, 0), (3, 1, 0) or (2, 1, 1). These three cases are easily seen to be impossible, so no such $Q$ exists. It follows that $\sigma_0 = 42$, and we have a 5-design on 256 varieties with blocks of size 24.

One wonders in how many affine spaces $\Omega$ such 5-designs exist. Since $143$ divides $2^n - 3$ whenever $n = 28 \pmod{60}$, there are infinitely many values of $n$ for which $2^n - 3$ is not a prime power. For fixed $k, n$, with $6 \leq k \leq 2^{n-1}$, let us consider the problem heuristically. Suppose we select $\mathcal{A}$ from $\Sigma_k$ randomly, each member of $\Sigma_k$ having probability $1/(2^n)$ of being selected. Now $\sigma_0$ is a random variable on the probability space $\Sigma_k$. The expectation of $\sigma_0$ is

$$
E = \left( \frac{k}{4} \right) / (2^n - 3).
$$

A 5-design of the type under consideration exists if and only if $\sigma_0$ achieves its expectation in $\Sigma_k$. When $E$ is an integer, it does not seem unreasonable that $\sigma_0$ would achieve its expectation.

The author has not investigated the construction of designs in affine spaces over $GF(2)$ by using more than one orbit under $A$.

REFERENCES


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