

Pacific Journal of Mathematics

REAL-VALUED CHARACTERS OF METACYCLIC GROUPS

B. G. BASMAJI

REAL-VALUED CHARACTERS OF METACYCLIC GROUPS

B. G. BASMAJI

The nonlinear real-valued irreducible characters of metacyclic groups are determined, and the defining relations are given for the metacyclic groups with every nonlinear irreducible character real-valued.

Consider the metacyclic group

$$G = \langle a, b \mid a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r \rangle$$

where $r^t - 1 \equiv kr - k \equiv 0 \pmod{n}$ and $t \mid m$. Let s be a positive divisor of n and let t_s be the smallest positive integer such that $r^{t_s} \equiv 1 \pmod{s}$. Let χ_s be a linear character of $\langle a \rangle$ with kernel $\langle a^s \rangle$ and $\bar{\chi}_s$ be an extension of χ_s to $K_s = \langle a, b^{t_s} \rangle$, see [1]. From [1] the induced character $\bar{\chi}_s^G$ is irreducible of degree t_s and every irreducible character of G is some $\bar{\chi}_s^G$.

Assume $\bar{\chi}_s^G$ is nonlinear. Then $K_s \subset G \neq K_s$. From Lemma 1 of [2], $\bar{\chi}_s^G$ is real-valued if and only if there is $y \in G$ such that $\langle K_s, y \rangle / D_s$ is dihedral or quaternion, where D_s is the kernel of $\bar{\chi}_s$. Assume such a y exists. Since G/K_s is cyclic, t_s is even and we may let $y = b^{t_s/2}$. Hence $r^{t_s/2} \equiv -1 \pmod{s}$ and $\bar{\chi}_s(b^{t_s}) = \pm 1$. Since $\bar{\chi}_s(b^{t_s})^{t/t_s} = \chi_s(a^k)$, $\bar{\chi}_s(b^{t_s}) = \pm 1$ implies either (i) $s \mid (k, n)$, or (ii) $s \mid 2(k, n)$, $s \nmid (k, n)$, and $t = t_s$. When (i) occurs, $\bar{\chi}_s(b^{t_s}) = 1$ if t/t_s is odd and $\bar{\chi}_s(b^{t_s}) = \pm 1$ if t/t_s is even. When (ii) occurs $\bar{\chi}_s(b^{t_s}) = -1$. Note that if $\bar{\chi}_s(b^{t_s}) = -1$ then $\bar{\chi}_s^G$ is not realizable in the real field. Using [1] the number of the nonlinear irreducible real-valued characters not realizable in the real field is $\Sigma''\phi(s)/t_s$ where Σ'' is over all positive divisors s of n such that t_s is even, $r^{t_s/2} \equiv -1 \pmod{s}$ and either (i) $s \mid (k, n)$ and t/t_s even or (ii) $s \mid 2(k, n)$, $s \nmid (k, n)$, and $t = t_s$. The number of the nonlinear irreducible real-valued characters is $\Sigma'\phi(s)/t_s + \Sigma''\phi(s)/t_s$ where Σ' is defined in [2, § 2].

Let $\pi = \{p \mid p \text{ an odd prime dividing } n\}$.

THEOREM. *Assume G , given as above, is non-abelian. Then every nonlinear irreducible character of G is real-valued if and only if n, m, k, t, t_p and r satisfy one of the conditions below.*

(a) $m = t$ with either (i) $4 \nmid n$, $2 \mid t$, and $t_p = t$ for all $p \in \pi$,
 (ii) $4 \nmid n$, $4 \mid t$, and $t_p = t/2$ for all $p \in \pi$, or (iii) $4 \mid n$, $r = -1$, $t = 2$ or $t = 4$.

(b) $m = 2t$ with either (i) $2 \parallel n$, $2 \mid t$, and $t_p = t$ for all $p \in \pi$, or

(ii) $4 \mid n$, $r = -1$, and $t = 2$.

REMARK. Since (b)—(i) is the semi-direct product $\langle a^2 \rangle \circ \langle b \rangle$, and thus a special case of (a)—(ii), we have (b)—(ii) (where G is quaternion) the only nonsplitting case.

Proof. Consider χ_n with kernel $\langle 1 \rangle$. Assume $m/t > 2$. Since $b^t \in \langle a \rangle \cong K_n = \langle a, b^{t_n} \rangle$ we have $\chi_n(b^t)$, and hence $\bar{\chi}_n^G(b^t) = t_n \chi_n(b^t)$, complex. Thus $m/t \leq 2$, i.e., $m = t$ or $m = 2t$. Assume $t_s = 1$ for some $s \mid n$, $s > 2$. Since $b^{-1} a^{n/s} b = a^{n/s}$ we have $\bar{\chi}_n^G(a^{n/s}) = t_n \chi_n(a^{n/s})$ complex. Thus $t_s > 1$ for all $s \mid n$, $s > 2$.

Consider χ_s with kernel $\langle a^s \rangle$, $s > 2$. Assume $t/t_s > 2$. Then the extension $\bar{\chi}_s$ to $K_s = \langle a, b^{t_s} \rangle$ can be chosen such that $\bar{\chi}_s(b^{t_s})$ is complex. Thus $\bar{\chi}_s^G(b^{t_s}) = t_s \bar{\chi}_s(b^{t_s})$ is complex and hence $t = t_s$ or $t = 2t_s$. Also, since G/K_s is cyclic of even order, $2 \mid t_s$.

Let p and q be in π , $p \neq q$ and assume $t_p = t$ and $t_q = t/2$. Then $t_{pq} = t$. Thus $K_{pq} = \langle a \rangle$, and since $b^{-t/2} a b^{t/2} \notin \alpha^{-1} \langle a^{pq} \rangle$, it follows that $\langle a, b^{t/2} \rangle / \langle a^{pq} \rangle$ is neither dihedral nor quaternion, a contradiction. Thus either $t_p = t$ for all $p \in \pi$ or $t_p = t/2$ for all $p \in \pi$.

Now assume $\lambda \mid n$, $\lambda = 2^e$, $e > 1$. Then t_λ is a power of 2. Consider χ_λ . Then since $\langle a, b^{t_\lambda/2} \rangle / \langle a^\lambda \rangle$ is dihedral or quaternion, it follows that $r^{t_\lambda/2} \equiv -1 \pmod{\lambda}$. The only solution is $r \equiv -1 \pmod{\lambda}$. Thus $t_\lambda = 2$. As above, if $e > 1$ then $2 = t_\lambda = t_p$ for all $p \in \pi$, so $r \equiv -1 \pmod{n}$.

Assume $t_p = t$, $2 \mid t$, for all $p \in \pi$. If n is odd then $t = m$, and if $n = 2v$, v odd, then $t = m$ or $2t = m$. If $4 \mid n$ then $r = -1$ and $2 = t_4 = t_p = t = m$ or $2t = m = 4$.

Assume $t_p = t/2$, $4 \mid t$, for all $p \in \pi$. If $4 \nmid n$ then $t = m$. [The case $m = 2t$, $n = 2v$, v odd can not occur, since then we have $b^t = a^v \neq 1$, $K_p = \langle a, b^{t/2} \rangle$, which implies $\bar{\chi}_n(b^{t/2}) = \pm \sqrt{-1}$ and thus $\bar{\chi}_n^G$ is complex.] If $4 \mid n$ then $r = -1$ and $2 = t_4 = t_p = t/2$. Thus $4 = t = m$. [The case $m = 2t = 8$ cannot occur for a similar reason as above.]

The above give all the cases in the Theorem. Conversely if G is as in the Theorem and $s \mid n$, $s > 2$, it is easy to show that $\langle a, b^{t_s/2} \rangle / D_s$ is dihedral or quaternion, where D_s is the kernel of $\bar{\chi}_s$, and thus $\bar{\chi}_s^G$ is real-valued. This completes the proof.

We remark that a similar result to Theorem 1 of [2], with a parallel proof, could be given for the real-valued characters of metabelian groups.

REFERENCES

1. B. Basmaji, *Monomial representations and metabelian groups*, Nagoya Math. J., **35** (1969), 99-107.

2. B. Basmaji, *Representations of metabelian groups realizable in the real field*, Trans. Amer. Math. Soc., **156** (1971), 109-118.

Received October 12, 1970 and in revised form June 6, 1971.

CALIFORNIA STATE COLLEGE AT LOS ANGELES

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 3

July, 1971

William O'Bannon Alltop, <i>5-designs in affine spaces</i>	547
B. G. Basmaji, <i>Real-valued characters of metacyclic groups</i>	553
Miroslav Benda, <i>On saturated reduced products</i>	557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups. II</i>	573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, <i>Fixed points and stability for a sum of two operators in locally convex spaces</i>	581
Donald Richard Chalice, <i>Restrictions of Banach function spaces</i>	593
Eugene Frank Cornelius, Jr., <i>A generalization of separable groups</i>	603
Joel L. Cunningham, <i>Primes in products of rings</i>	615
Robert Alan Morris, <i>On the Brauer group of \mathbb{Z}</i>	619
David Earl Dobbs, <i>Amitsur cohomology of algebraic number rings</i>	631
Charles F. Dunkl and Donald Edward Ramirez, <i>Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups</i>	637
Hicham Fakhoury, <i>Structures uniformes faibles sur une classe de cônes et d'ensembles convexes</i>	641
Leslie R. Fletcher, <i>A note on $C\theta\theta$-groups</i>	655
Humphrey Sek-Ching Fong and Louis Sucheston, <i>On the ratio ergodic theorem for semi-groups</i>	659
James Arthur Gerhard, <i>Subdirectly irreducible idempotent semigroups</i>	669
Thomas Eric Hall, <i>Orthodox semigroups</i>	677
Marcel Herzog, <i>$C\theta\theta$-groups involving no Suzuki groups</i>	687
John Walter Hinrichsen, <i>Concerning web-like continua</i>	691
Frank Norris Huggins, <i>A generalization of a theorem of F. Riesz</i>	695
Carlos Johnson, Jr., <i>On certain poset and semilattice homomorphisms</i>	703
Alan Leslie Lambert, <i>Strictly cyclic operator algebras</i>	717
Howard Wilson Lambert, <i>Planar surfaces in knot manifolds</i>	727
Robert Allen McCoy, <i>Groups of homeomorphisms of normed linear spaces</i>	735
T. S. Nanjundiah, <i>Refinements of Wallis's estimate and their generalizations</i>	745
Roger David Nussbaum, <i>A geometric approach to the fixed point index</i>	751
John Emanuel de Pillis, <i>Convexity properties of a generalized numerical range</i>	767
Donald C. Ramsey, <i>Generating monomials for finite semigroups</i>	783
William T. Reid, <i>A disconjugacy criterion for higher order linear vector differential equations</i>	795
Roger Allen Wiegand, <i>Modules over universal regular rings</i>	807
Kung-Wei Yang, <i>Compact functors in categories of non-archimedean Banach spaces</i>	821
R. Grant Woods, <i>Correction to: "Co-absolutes of remainders of Stone-Čech compactifications"</i>	827
Ronald Owen Fulp, <i>Correction to: "Tensor and torsion products of semigroups"</i>	827
Bruce Alan Barnes, <i>Correction to: "Banach algebras which are ideals in a Banach algebra"</i>	828