REAL-VALUED CHARACTERS OF METACYCLIC GROUPS

B. G. Basmaji
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B. G. BASMAJI

The nonlinear real-valued irreducible characters of metacyclic groups are determined, and the defining relations are given for the metacyclic groups with every nonlinear irreducible character real-valued.

Consider the metacyclic group

\[ G = \langle a, b \mid a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r \rangle \]

where \( r^t - 1 \equiv kr - k \equiv 0 \pmod{n} \) and \( t \mid m \). Let \( s \) be a positive divisor of \( n \) and let \( t_g \) be the smallest positive integer such that \( r^{ts} \equiv 1 \pmod{s} \). Let \( \chi_s \) be a linear character of \( \langle a \rangle \) with kernel \( \langle a^s \rangle \) and \( \chi_g \) be an extension of \( \chi_s \) to \( K_g = \langle a, b^s \rangle \), see [1]. From [1] the induced character \( \chi_g \) is irreducible of degree \( t \) and every irreducible character of \( G \) is some \( \chi_g \).

Assume \( \chi_g \) is nonlinear. Then \( K_g \subset G \neq K_g \). From Lemma 1 of [2], \( \chi_g \) is real-valued if and only if there is \( y \in G \) such that \( \langle K_g, y \rangle /D_g \) is dihedral or quaternion, where \( D_g \) is the kernel of \( \chi_g \). Assume such a \( y \) exists. Since \( G/K_g \) is cyclic, \( t_g \) is even and we may let \( y = b^{s/t_g} \). Hence \( r^{ts/t_g} \equiv -1 \pmod{s} \) and \( \chi_g(b^{s/t_g}) = \pm 1 \). Since \( \chi_g(b^{ts}) = \chi_g(a^k) \), \( \chi_g(b^{ts}) = \pm 1 \) implies either (i) \( s \mid (k, n) \), or (ii) \( s \mid 2(k, n), s \not\mid (k, n) \), and \( t = t_g \). When (i) occurs, \( \chi_g(b^{ts}) = 1 \) if \( t/t_g \) is odd and \( \chi_g(b^{ts}) = \pm 1 \) if \( t/t_g \) is even. When (ii) occurs \( \chi_g(b^{ts}) = -1 \). Note that if \( \chi_g(b^{ts}) = -1 \) then \( \chi_g \) is not realizable in the real field. Using [1] the number of the nonlinear irreducible real-valued characters not realizable in the real field is \( \Sigma''(s/t_g) \phi(s) \) where \( \Sigma'' \) is over all positive divisors \( s \) of \( n \) such that \( t_g \) is even, \( r^{ts/t_g} \equiv -1 \pmod{s} \) and either (i) \( s \mid (k, n) \) and \( t/t_g \) even or (ii) \( s \mid 2(k, n), s \not\mid (k, n) \), and \( t = t_g \). The number of the nonlinear irreducible real-valued characters is \( \Sigma'(s/t_g) + \Sigma''(s/t_g) = n' \phi(s) \) where \( n' \) is defined in [2, § 2].

Let \( \pi = \{ p \mid p \text{ an odd prime dividing } n \} \).

**Theorem.** Assume \( G \), given as above, is non-abelian. Then every nonlinear irreducible character of \( G \) is real-valued if and only if \( n, m, k, t, t_g \), and \( r \) satisfy one of the conditions below.

(a) \( m = t \) with either (i) \( 4 \mid n, 2 \mid t \), and \( t_g = t \) for all \( p \in \pi \),
(ii) \( 4 \mid n, 4 \mid t \), and \( t_g = t/2 \) for all \( p \in \pi \), or (iii) \( 4 \mid n, r = -1, t = 2 \) or \( t = 4 \).

(b) \( m = 2t \) with either (i) \( 2 \parallel n, 2 \parallel t \), and \( t_g = t \) for all \( p \in \pi \), or
(ii) \(4 \mid n, \ r = -1, \text{ and } t = 2\).

**Remark.** Since (b)—(i) is the semi-direct product \(\langle a^s \rangle \circ \langle b \rangle\), and thus a special case of (a)—(ii), we have (b)—(ii) (where \(G\) is quaternion) the only nonsplitting case.

**Proof.** Consider \(\chi_n\) with kernel \(\langle 1 \rangle\). Assume \(m/t > 2\). Since \(b' \in \langle a \rangle \subseteq K_s = \langle a, b^{is} \rangle\) we have \(\chi_n(b')\), and hence \(\overline{\chi}_n(b') = t_s \chi_n(b')\), complex. Thus \(m/t \leq 2\), i.e., \(m = t\) or \(m = 2t\). Assume \(t_s = 1\) for some \(s \mid n, \ s > 2\). Since \(b^{-1}a^{ns}b = a^{ns}\) we have \(\overline{\chi}_n(a^{ns}) = t_s \chi_n(a^{ns})\) complex. Thus \(t_s > 1\) for all \(s \mid n, \ s > 2\).

Consider \(\chi_s\) with kernel \(\langle 1 \rangle\), \(s > 2\). Assume \(t/s > 2\). Then the extension \(\overline{\chi}_s\) to \(K_s = \langle a, b^{is} \rangle\) can be chosen such that \(\overline{\chi}_s(b^{is})\) is complex. Thus \(\overline{\chi}_s(b^{is}) = t_s \overline{\chi}_s(b^{is})\) is complex and hence \(t = t_s\) or \(t = 2t_s\). Also, since \(G/K_s\) is cyclic of even order, \(2 \mid t_s\).

Let \(p\) and \(q\) be in \(\pi, \ p \neq q\) and assume \(t_p = t\) and \(t_q = t/2\). Then \(t_{pq} = t\). Thus \(K_{pq} = \langle a \rangle\), and since \(b^{-t/2}a b^{t/2} \in a^{-1} \langle a^{pq} \rangle\), it follows that \(\langle a, b^{is} \rangle \langle a^{pq} \rangle\) is neither dihedral nor quaternion, a contradiction. Thus either \(t_p = t\) for all \(p \in \pi\) or \(t_p = t/2\) for all \(p \in \pi\).

Now assume \(\lambda \mid n, \ \lambda = 2^e, \ e > 1\). Then \(t_2\) is a power of 2. Consider \(\chi_2\). Then since \(\langle a, b^{is} \rangle \langle a^s \rangle\) is dihedral or quaternion, it follows that \(a^{t/2} \equiv -1 \pmod{\lambda}\). The only solution is \(r = -1 \pmod{\lambda}\). Thus \(t_2 = 2\). As above, if \(e > 1\) then \(2 = t_1 = t_p\) for all \(p \in \pi\), so \(r = -1 \pmod{\lambda}\).

Assume \(t_p = t, \ 2 \mid t, \) for all \(p \in \pi\). If \(n\) is odd then \(t = m\), and if \(n = 2v, \ v\) odd, then \(t = m\) or \(2t = m\). If \(4 \mid n\) then \(r = -1\) and \(2 = t_4 = t_p = t = m\) or \(2t = m = 4\).

Assume \(t_p = t/2, \ 4 \mid t, \) for all \(p \in \pi\). If \(4 \nmid n\) then \(t = m\). [The case \(m = 2t, \ n = 2v, \ v\) odd can not occur, since then we have \(b' = a^s \neq 1, \ K_s = \langle a, b^{is} \rangle\), which implies \(\overline{\chi}_s(b^{is}) = \pm \sqrt{-1}\) and thus \(\overline{\chi}_s\) is complex.] If \(4 \mid n\) then \(r = -1\) and \(2 = t_4 = t_p = t/2\). Thus \(4 = t = m\). [The case \(m = 2t = 8\) cannot occur for a similar reason as above.]

The above give all the cases in the Theorem. Conversely if \(G\) is as in the Theorem and \(s \mid n, \ s > 2\), it is easy to show that \(\langle a, b^{is} \rangle / D_s\) is dihedral or quaternion, where \(D_s\) is the kernel of \(\overline{\chi}_s\), and thus \(\overline{\chi}_s\) is real-valued. This completes the proof.

We remark that a similar result to Theorem 1 of [2], with a parallel proof, could be given for the real-valued characters of metabelian groups.

**References**


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