

Pacific Journal of Mathematics

UNIQUELY REPRESENTABLE SEMIGROUPS. II

J. T. BORREGO, HASKELL COHEN AND ESMOND ERNEST DEVUN

UNIQUELY REPRESENTABLE SEMIGROUPS II

J. T. BORREGO, H. COHEN, AND E. E. DEVUN

A semigroup S is said to be uniquely representable in terms of two subsets X and Y if $X \cdot Y = Y \cdot X = S$, $x_1 y_1 = x_2 y_2$ is a nonzero element of S implies $x_1 = x_2$ and $y_1 = y_2$ and $y_1 x_1 = y_2 x_2$ is a nonzero element of S implies $y_1 = y_2$ and $x_1 = x_2$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

In this paper we are concerned with semigroups S with no zero divisors, $E(S) = \{0, 1\}$, and which are uniquely representable in terms of two subsets X and Y which are isomorphic copies of the unusual unit interval. Here we show the nonzero elements of the semigroup S can be embedded in a Lie group.

The authors would like to express their appreciation to Professor G. D. Crown for taking part in discussions concerning this paper.

NOTATION. S will represent a semigroup without zero-divisors, $E(S) = \{0, 1\}$ ($E(S)$ is the set of idempotents of S), and which is uniquely representable in terms of X and Y which are isomorphic copies of the usual unit interval. We will let $T = S - \{0\}$ where 0 is the zero of S . Also $X^0 = X - \{0\}$ and $X^{01} = X - \{0, 1\}$ where 1 is the identity for S . Similarly, $Y^0 = Y - \{0\}$, and $Y^{01} = Y - \{0, 1\}$.

Define $\phi: X^0 \times Y^0 \rightarrow X^0 \times Y^0$ by $\phi(x, y) = (x', y')$ where x' and y' are the unique elements of X^0 and Y^0 respectively such that $xy = y'x'$. Also define $\psi: X^0 \times Y^0 \rightarrow X^0 \times Y^0$ by $\psi(x, y) = (x', y')$ where x' and y' are the unique elements of X^0 and Y^0 respectively such that $yx = x'y'$. It is easy to show ϕ and ψ are homeomorphisms. Also for fixed y , $\pi_1 \phi | : X^0 \times \{y\} \rightarrow X^0$, $\pi_1 \psi | : X^0 \times \{y\} \rightarrow X^0$ are strictly increasing functions, and for fixed x , $\pi_2 \phi | : \{x\} \times Y^0 \rightarrow Y^0$ and

$$\pi_2 \psi | : \{x\} \times Y^0 \rightarrow Y^0$$

are strictly increasing functions.

LEMMA 1. Let $x_1 \in X^{01}$. If $\pi_2 \phi | : \{x_1\} \times Y^0 \rightarrow Y^0$ is a homeomorphism, then $\pi_2 \phi | : \{x\} \times Y^0 \rightarrow Y^0$ is a homeomorphism for all $x \in X^0$.

Proof. Fix $\{y_n\}$ a decreasing sequence in Y^0 with $y_n \rightarrow 0$, and $x \in X^0$ with $x \geq x_1$. To show $\pi_2 \phi | : \{x\} \times Y^0 \rightarrow Y^0$ is a homeomorphism we need only show $\pi_2 \phi | (x, y_n) \rightarrow 0$. Let $x_2 \in X^0$ with $x_2 x = x_1$. Then there exist sequences $\{q_n\}$, $\{r_n\}$ contained in X^0 , $\{s_n\}$, $\{t_n\}$ contained in Y^0 such that $x_1 y_n = x_2 x y_n = x_2 s_n q_n = t_n r_n$. Since

$$t_n = \pi_2 \phi | (x_1, y_n) \longrightarrow 0, s_n = \pi_2 \phi | (x, y_n) \longrightarrow 0.$$

Since $x_1^{2^n} \rightarrow 0$, to finish the proof we need only show

$$\pi_2\phi | : \{x_1^2\} \times Y^0 \longrightarrow Y^0$$

is a homeomorphism. Select sequence $\{q_n\}, \{r_n\}$ in X^0 , $\{s_n\}, \{t_n\}$ in Y^0 such that $x_1^2 y_n = x_1 s_n q_n = t_n r_n$. Since $s_n = \pi_2\phi | (x_1, y_n), s_n \rightarrow 0$. Thus $t_n = \pi_2\phi | (x_1, q_n) \rightarrow 0$. Thus $\pi_2\phi | : \{x_1^2\} \times Y^0 \rightarrow Y^0$ is a homeomorphism. A similar statement for $\pi_1\phi | : X^0 \times \{y\} \rightarrow X^0$ can be made.

LEMMA 2. $\pi_2\phi | : \{x\} \times Y^0 \rightarrow Y^0$ is a homeomorphism for all $x \in X^0$ or $\pi_1\phi | : X^0 \times \{y\} \rightarrow X^0$ is a homeomorphism for all $y \in Y^0$.

Proof. Let $x \in X^{01}$ with $\pi_2\phi | : \{x\} \times Y^0 \rightarrow Y^0$ not a homeomorphism, and let $y \in Y^{01}$. Fix $\{y_n\}$ a decreasing sequence in Y^0 with $y_n \rightarrow 0$. There exist sequences $\{q_n\}$ in X^0 , $\{s_n\}$ in Y^0 such that

$$xy_n = s_n q_n, s_n \not\rightarrow 0$$

and $q_n \rightarrow 0$. Also there exist sequences $\{r_n\}$ in $X^0, \{t_n\}$ in Y^0 with $q_n y = t_n r_n$. We claim $r_n \rightarrow 0$. For if not $t_n \rightarrow 0$ and thus

$$xy_n y = s_n t_n r_n,$$

with $s_n t_n \rightarrow 0$. However this implies $\pi_2\phi | : \{x\} \times Y^0 \rightarrow Y^0$ is a homeomorphism. This is a contradiction. So $r_n \rightarrow 0$, and thus

$$\pi_1\phi | : X^0 \times \{y\} \longrightarrow X^0$$

is a homeomorphism.

LEMMA 3. T is right reversible or T is left reversible.

Proof. We will assume $\pi_1\phi | : X^0 \times \{y\} \rightarrow X^0$ is a homeomorphism for all $y \in Y^0$. We will show T is right reversible. Let $s_1, s_2 \in T$ with $s_1 = x_1 y_1, s_2 = x_2 y_2$ and $y_1 \leq y_2$. Thus $Ts_1 \cap Ts_2 = Tx_1 y_1 \cap Tx_2 y_2 \neq \phi$ if $Tx_1 y_1 y_2^{-1} \cap Tx_2 \neq \phi$. Let $y_3 x_3 = x_1 y_1 y_2^{-1}$. If $x_3 \leq x_2$, then

$$Ty_3 x_3 x_2^{-1} \cap T \neq \phi$$

and hence $Ty_3 x_3 \cap Tx_2 \neq \phi$. If $x_2 < x_3$, then $Ty_3 x_3 \cap Tx_2 \neq \phi$ if

$$Ty_3 \cap Tx_2 x_3^{-1} \neq \phi.$$

Thus to show T is right reversible we need only show $Tx_i \cap Ty_i \neq \phi$ for all $x_i \in X^{01}, y_i \in Y^{01}$. Now $\pi_1\phi | : X^0 \times \{y_i\} \rightarrow X^0$ is onto and thus there exists $x_5 \in X^{01}$ such that $\pi_1\phi | (x_5, y_i) = x_i$ and thus $x_5 y_i = y_5 x_i$ for some $y_5 \in Y^0$. Hence $Tx_i \cap Ty_i \neq \phi$. If $\pi_2\phi | : \{x\} \times Y^0 \rightarrow Y^0$ is a homeomorphism for all $x \in X^0$, T is left reversible.

Now T is a right (left) reversible cancellative semigroup [2]. Hence

[4] T is algebraically embedded in a group G of left (right) quotents of T . Note that for every element $g \in G$ we have $g = st$ where

$$s, t \in X^0 \cup X^{0^{-1}} \cup Y^0 \cup Y^{0^{-1}}.$$

Also it is easy to see that there exist $x \in X^{0^1}, y \in Y^{0^1}$ such that $xT \cap yT \neq \phi$ and $Tx \cap Ty \neq \phi$.

LEMMA 4. *If $x_1 \in X^{0^1}, y_1 \in Y^{0^1}$ with $x_1T \cap y_1T \neq \phi$ and $Tx_1 \cap Ty_1 \neq \phi$, then for $x_2 \in X^{0^1}, x_2 \geq x_1, y_2 \in Y^{0^1}, y_2 \geq y_1$ there exist $x \in X^{0^1}, y \in Y^{0^1}$ such that $x_2y_2^{-1} = y^{-1}x$.*

Proof. Now $Tx_2 \cap Ty_2 \neq \phi$ for $x_2 \geq x_1$ and $y_2 \geq y_1$. Thus there exist $s, t \in T$ $sx_2 = ty_2$. Let $s = x_3y_3$ and $t = x_4y_4$. Thus $x_3y_3x_2 = x_4y_4y_2$. If $x_3 < x_4$, then $x_4^{-1}x_3 \in X^{0^1}$. Thus $x_4^{-1}x_3y_3x_2 = y_4y_2$ or letting $y_3x_2 = x_5y_5$ with $x_5 \in X^0, y_5 \in Y^0$ we have $x_4^{-1}x_3x_5y_5 = y_4y_2$. This contradicts S being uniquely representable, so $x_3 \geq x_4$. Hence $x_3^{-1}x_4 \in X^0$ and thus $y_3x_2 = x_3^{-1}x_4y_4y_2$ or $x_2y_2^{-1} = y_3^{-1}x_3^{-1}x_4y_4$. But $x_3^{-1}x_4 \in X^0$, so there exist $x_6 \in X^0, y_6 \in Y^0$ such that $x_3^{-1}x_4y_4 = y_6x_6$. Hence $x_2y_2^{-1} = y_3^{-1}y_6x_6$. Now $y_3^{-1}y_6 \in Y^{0^-}$. For if $y_3^{-1}y_6 \in Y^{0^1}$ we would have $x_2 = y_3^{-1}y_6x_6y_2$ and letting $x_6y_2 = y_7x_7$ with $x_7 \in X^0, y_7 \in Y^0$ we would have $x_2 = y_3^{-1}y_6y_7x_7$ with $y_3^{-1}y_6y_7 \in Y^{0^1}, x_7 \in X^0$. But this contradicts S being uniquely representable. Note that a similar argument yields that there exist $x \in X^0, y \in Y^{0^1}$ such that $y_2x_2^{-1} = x^{-1}y$.

LEMMA 5. *If there exist $x_1 \in X^{0^1}, y_1, y_2 \in Y^{0^1}$ with $y_1x_1 = x_1y_2$, then for each $x \in X^0, y \in Y^0$, there exist $y' \in Y^0$ such that $yx = xy'$.*

Proof. Let $x_1 \in X^{0^1}, y_1, y_2 \in Y^{0^1}$ with $y_1x_1 = x_1y_2$. We will divide the proof into two parts.

Part 1. We will show that for each $y \in Y^{0^1}$ there exist $y' \in Y^{0^1}$ such that $yx_1 = x_1y'$. To prove the above we need only show that there exist $y_3 \in Y^{0^1}$ such that $\sqrt{y_1}x_1 = x_1y_3$. Now $\sqrt{y_1}x_1 \in T$ so there exist $x_4 \in X^0, y_4 \in Y^0$ such that $\sqrt{y_1}x_1 = x_4y_4$. Also let $x_5 \in X^0, y_5 \in Y^0$ with $\sqrt{y_1}x_4 = x_5y_5$. Now $y_1x_1 = \sqrt{y_1}\sqrt{y_1}x_1 = \sqrt{y_1}x_4y_4 = x_5y_5y_4$. Thus $x_5 = x_1$ and $y_2 = y_5y_4$. The map $\pi_1\psi | : X^0 \times \{\sqrt{y_1}\} \rightarrow X^0$ is strictly increasing and $\pi_1\psi | (\pi_1\psi | (x_1, \sqrt{y_1}), \sqrt{y_1}) = \pi_1\psi | (x_4, \sqrt{y_1}) = x_5 = x_1$, thus $\pi_1\psi | (x_1, \sqrt{y_1}) = x_1$. Hence $\sqrt{y_1}x_1 = x_1y_4$.

Part 2. To finish the theorem we need only show that there exist $x_2 \in X^{0^1}$ with $x_2 > x_1$ and $y, y' \in Y^{0^1}$ such that $yx_2 = x_2y'$. Since the map $s \rightarrow s^2$ is onto we can pick $x_3, x_4 \in X^{0^1}, y_3, y_4 \in Y^{0^1}$ with $y_1x_1 = (x_3y_3)^2$ and $x_3y_3 = y_4x_4$. Now $y_4x_4x_3y_3 = y_1x_1 = x_1y_2$. Pick $x_5 \in X^{0^1}$,

$y_5 \in Y^{01}$ such that $x_5 y_5 = y_4 x_4 x_3$. Then $x_1 y_2 = y_1 x_1 = y_4 x_4 x_3 y_3 = x_5 y_5 y_3$. Thus $x_5 = x_1$. Select $y_6 \in Y^{01}$ such that $x_1 y_5 = y_6 x_1$. So

$$y_4 x_4 x_3 = x_5 y_5 = y_6 x_1 .$$

Thus $x_4 x_3 = x_1$. Hence $x_3 > x_1$. Now there exist $x_0 \in X^{01}$, $y_0 \in Y^{01}$ such that $x_3 y_3 y_4 = y_0 x_0$. So $y_1 x_1 = x_3 y_3 y_4 x_4 = y_0 x_0 x_4$. Hence $x_1 = x_0 x_4$. But $x_1 = x_4 x_3 = x_3 x_4$. Thus $x_0 = x_3$ and $y_0 x_3 = y_0 x_0 = x_3 (y_3 y_4)$. This completes the proof.

Let R and R' be the relation \geq or \leq .

LEMMA 6. *If $x_1, x_2 \in X^{01}$, $y_1, y_2 \in Y^{01}$ with $x_1 y_1 = y_2 x_2$, $x_1 R x_2$, and $y_1 R' y_2$, then for $x_3, x_4 \in X^{01}$, $y_3, y_4 \in Y^{01}$ with $x_3 y_3 = y_4 x_4$ we have $x_3 R x_4$ and $y_3 R' y_4$.*

Proof. Consider the map $\pi_1 \phi | : X^0 \times \{y_1\} \rightarrow X^0$, and let

$$\pi_1 \phi(x_1, y_1) = x_2$$

and $\pi_1 \phi(x_3, y_1) = x_5$. Suppose $x_5 R x_3$. Now $x_1 R x_2$ and thus there exist $x \in X^{01}$ such that $\pi_1 \phi(x, y_1) = x$. Hence there exist $y \in Y^{01}$ such that $x y_1 = y x$. By Lemma 5 we see $x_3 y_1 = y' x_3$ for some $y' \in Y^{01}$. Thus $x_3 = x_5$ and $x_3 R x_5$. The same type of argument yields $y_1 R' y_5$ where $x_3 y_1 = y_5 x_5$. Applying them again we get $x_3 R x_4$ and $y_3 R' y_4$. This completes the proof.

For $s \in T$, let $s^0 = 1$. Fix $x \in X^{01}$, $y \in Y^{01}$ with $xT \cap yT \neq \phi$ and $Tx \cap Ty \neq \phi$. Now consider G with the topology generated by the following neighborhoods. For t real $t \in (0, 1)$ define

$$N(1, t) = \{x^\alpha y^\beta : \alpha, \beta \in (-t, t)\} .$$

For $g \in G$, $g = sr$ with $s, r \in X^0 \cup X^{0-1} \cup Y^0 \cup Y^{0-1}$. The neighborhoods for g will consist of $sN(1, t)r$ where $N(1, t)$ is a neighborhood of the identity.

LEMMA 7. *If $N(1, t)$ is a neighborhood of the identity, then there exist $N(1, q)$ a neighborhood of the identity such that*

$$N(1, q) \cdot N(1, q) \subset N(1, t) .$$

Proof. From Lemma 6 and from the fact $y^{1/n} \rightarrow 1, x^{1/n} \rightarrow 1$ we can pick N such that for $n > N$ the following hold: (1) $y^{1/n} x^q = x_n y_n$ and $x_n \in (x^{t/4}, 1]$ implies $x^q \in (x^{t/2}, 1]$, (2) $x^{1/n} \in (x^{t/4}, 1]$, and (3)

$$y^{1/n} x^q = x'_n y'_n$$

with $x^q \in (x^{t/2}, 1]$ implies $y'_n \in (y^{t/2}, 1]$.

From Lemma 4 there exist $\bar{x}_n \in X^{01}$, $\bar{y}_n \in Y^{01}$ such that

$$y^{-1/n}x^{1/n} = \bar{x}_n\bar{y}_n^{-1} .$$

Since $x^{1/n} = y^{1/n}y^{-1/n}x^{1/n} = y^{1/n}\bar{x}_n\bar{y}_n^{-1}$ we see that $y^{1/n}\bar{x}_n = x^{1/n}\bar{y}_n$. Thus from the above $\bar{x}_n \in (x^{t/2}, 1]$ and $\bar{y}_n \in (y^{t/2}, 1]$ or $\bar{y}_n^{-1} \in [1, y^{-t/2})$. That is there exist N such that for

$$n > N, \psi(x^{1/n}, y^{-1/n}) \subset \{(x^\alpha, y^\beta) : \alpha, \beta \in (-t/2, 0) \cup (0, t/2)\} .$$

Using the same procedure we can find M large enough such that $\{\psi(x^{1/M}, y^{1/M}), \psi(x^{-1/M}, y^{1/M}), \psi(x^{1/M}, y^{-1/M}), \psi(x^{-1/M}, y^{-1/M}), (x^{1/M}, y^{1/M}), (x^{-1/M}, y^{-1/M})\} \subset \{(x^\alpha, y^\beta) : \alpha, \beta \in (-t/2, 0) \cup (0, t/2)\} .$

Now by Lemma 4 and Lemma 6

$$\{y^\alpha x^\beta : \alpha, \beta \in (-1/M, 1/M)\} \subset \{x^\alpha y^\beta : \alpha, \beta \in (-t/2, t/2)\} .$$

Hence $N(1, 1/M) N(1, 1/M) \subset N(1, t)$.

LEMMA 8. G is a topological semigroup.

Proof. To prove this we need only show that for each

$$s \in X^0 \cup X^{0-1} \cup Y^0 \cup Y^{0-1}$$

and $N(1, t)$ a neighborhood of the identity there exist $N(1, q)$ a neighborhood of the identity such that $sN(1, q) \subset N(1, t)s$. We will assume $s \in Y^0 \cup Y^{0-1}$. Now $N(1, t)s = \{x^\alpha s y^\beta : \alpha, \beta \in (-t, t)\}$. Now pick r such that $\{sx^\alpha : \alpha \in (-r, r)\} \subset \{x^\alpha y^\beta s : \alpha, \beta \in (-t/2, t/2)\}$. Set $q = \min \{r, t/2\}$. Then $sN(1, q) \subset N(1, t)s$.

Now G is a locally compact topological semigroup which is algebraically a group. By [9] G is a topological group. Moreover since G is locally euclidean [8] G is a two-dimensional Lie group.

THEOREM 9. T is embedded in G .

Proof. The inclusion map $i: T \rightarrow G$ is an isomorphism into.

It should be pointed out here that an alternate and more general method for embedding semigroups in groups has been constructed by D. R. Brown and Michael Friedberg [4].

COROLLARY 10. If D is a uniquely divisible semigroup on the two-cell with $E(D) = \{0, 1\}$ ($E(D)$ is the set of idempotents for D), then $D - \{0\}$ is embedded in a Lie group.

Proof. In [2] it was shown that $D - \{0\}$ is uniquely representable in terms of two usual unit intervals. Thus $D - \{0\}$ is embedded in

a Lie group.

Examples and characterization. The authors would like to extend their appreciation to J. Lawson for supplying us with the information for the characterization of the uniquely representable semigroups.

(1) Let (I, \cdot) denote the closed unit interval with the usual multiplication. Then $(I, \cdot) \times (I, \cdot) / [(\{0\} \times I) \cup (I \times \{0\})]$ is the only commutative which is uniquely representable in terms of two usual unit intervals [6], [7].

If S is non-abelian, then G is a non-abelian Lie group and G can be represented by the real matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0$ [1].

In the examples below we will take S to be the semigroup induced by one point compactification of the subsemigroups of G . The point added will always be the zero for S .

It is to be noted that Example 4 is anti-isomorphic to Example 2 and Example 5 is anti-isomorphic to Example 3.

(2) Let S be the topological semigroup generated by taking the one point compactification of the semigroup of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0, y \geq 0, x + y \leq 1$. Note S is uniquely divisible and thus S is uniquely representable in terms of two usual unit intervals [2]. Also S is not left reversible. It is easy to see that if W is the semigroup induced by the one point compactification of any collection of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $0 < x \leq 1$ and $y \geq \alpha(x - 1), y \leq \beta(x - 1)$ for two real numbers α and β , W is isomorphic to S .

(3) The one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $0 < x \leq 1, y \geq 0$ is a uniquely divisible semigroup on the two-cell. S is uniquely representable in terms of $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \cup \{0\}$ and $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cup \{0\}$. This semigroup is both left and right reversible. Furthermore,

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & xy \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Also if W is the one point compactification of any semigroup of matrices $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $y \geq \alpha(x - 1), 0 < x \leq 1$ or $y \leq \alpha(x - 1), 0 < x \leq 1$ for some real number α , then S is isomorphic to W . We will say S is half commutative if for each $x \in X^0, y \in Y^0$ there exists $y' \in Y^0$ such that $xy = y'x$.

(4) Let S be the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x \geq 1, y \geq 0, y \leq x - 1$. Then S is uniquely divisible,

left but not right reversible, it is not half commutative. Also if W is the semigroup formed by the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x \geq 1, y \geq \alpha(x-1), y \leq \beta(x-1), \beta > \alpha$, W is isomorphic to S .

(5) Consider the semigroup S formed by the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} x \geq 1, y \geq 0$. S is uniquely divisible, half commutative, right and left reversible. S differs from Example 3, since S has no copy of Example 2 contained in it, but Example 3 has a copy of Example 2 in it. Also if W is the one point compactification of the semigroup $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} x \geq 1, y \geq \alpha(x-1)$, α real, or $x \geq 1, y \leq \alpha(x-1)$, then W is isomorphic to S .

These are all of the semigroups which are uniquely representable in terms of two usual unit intervals. Note that they are all uniquely divisible.

COROLLARY 11. *If S is uniquely representable in terms of two usual unit intervals and without zero divisor and $E(S) = \{0, 1\}$, then S is uniquely divisible.*

REFERENCES

1. J. G. Belinfante, B. Kolman, H. A. Smith, *An introduction to Lie groups and Lie algebras with applications*, SIAM Review. **8**, No. 1, (Jan., 1966), 11-46.
2. J. T. Borrego, H. Cohen, and E. E. DeVun, *Uniquely representable semigroups on the two-cell*, Pacific J. Math., **38** (1971), 565-569.
3. D. R. Brown, *Topological Semilattices on the two-cell*, Pacific J. Math., **15** (1965), 35-46.
4. D. R. Brown and Michael Friedberg. *Linear representations of certain compact semigroups*, To appear.
5. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I, Mathematical Surveys, 7. Amer. Math. Soc., 1961.
6. E. E. DeVun, *Special semigroups on the two-cell*, Pacific J. Math., **34** (1970), 639-645.
7. J. A. Hildebrandt, *On uniquely divisible semigroups on the two-cell*, Pacific J. Math., **23** (1967), 91-95.
8. D. Montgomery and L. Zippen, *Topological Transformation Groups*, (Intersc. Tracts 1, Intersc. Publ.) John Wiley and Sons, New York, 1955.
9. Paalman-DeMiranda, A. B., *Topological Semigroups*, Amsterdam; Mathematical Centre Tracts, 1964.

Received May 4, 1970 and in revised form October 16, 1970,

WICHITA STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 3

July, 1971

William O'Bannon Alltop, <i>5-designs in affine spaces</i>	547
B. G. Basmaji, <i>Real-valued characters of metacyclic groups</i>	553
Miroslav Benda, <i>On saturated reduced products</i>	557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups. II</i>	573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, <i>Fixed points and stability for a sum of two operators in locally convex spaces</i>	581
Donald Richard Chalice, <i>Restrictions of Banach function spaces</i>	593
Eugene Frank Cornelius, Jr., <i>A generalization of separable groups</i>	603
Joel L. Cunningham, <i>Primes in products of rings</i>	615
Robert Alan Morris, <i>On the Brauer group of Z</i>	619
David Earl Dobbs, <i>Amitsur cohomology of algebraic number rings</i>	631
Charles F. Dunkl and Donald Edward Ramirez, <i>Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups</i>	637
Hicham Fakhoury, <i>Structures uniformes faibles sur une classe de cônes et d'ensembles convexes</i>	641
Leslie R. Fletcher, <i>A note on $C\theta\theta$-groups</i>	655
Humphrey Sek-Ching Fong and Louis Sucheston, <i>On the ratio ergodic theorem for semi-groups</i>	659
James Arthur Gerhard, <i>Subdirectly irreducible idempotent semigroups</i>	669
Thomas Eric Hall, <i>Orthodox semigroups</i>	677
Marcel Herzog, <i>$C\theta\theta$-groups involving no Suzuki groups</i>	687
John Walter Hinrichsen, <i>Concerning web-like continua</i>	691
Frank Norris Huggins, <i>A generalization of a theorem of F. Riesz</i>	695
Carlos Johnson, Jr., <i>On certain poset and semilattice homomorphisms</i>	703
Alan Leslie Lambert, <i>Strictly cyclic operator algebras</i>	717
Howard Wilson Lambert, <i>Planar surfaces in knot manifolds</i>	727
Robert Allen McCoy, <i>Groups of homeomorphisms of normed linear spaces</i>	735
T. S. Nanjundiah, <i>Refinements of Wallis's estimate and their generalizations</i>	745
Roger David Nussbaum, <i>A geometric approach to the fixed point index</i>	751
John Emanuel de Pillis, <i>Convexity properties of a generalized numerical range</i>	767
Donald C. Ramsey, <i>Generating monomials for finite semigroups</i>	783
William T. Reid, <i>A disconjugacy criterion for higher order linear vector differential equations</i>	795
Roger Allen Wiegand, <i>Modules over universal regular rings</i>	807
Kung-Wei Yang, <i>Compact functors in categories of non-archimedean Banach spaces</i>	821
R. Grant Woods, <i>Correction to: "Co-absolutes of remainders of Stone-Čech compactifications"</i>	827
Ronald Owen Fulp, <i>Correction to: "Tensor and torsion products of semigroups"</i>	827
Bruce Alan Barnes, <i>Correction to: "Banach algebras which are ideals in a banach algebra"</i>	828