RESTRICTIONS OF BANACH FUNCTION SPACES

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Let $X$ be a compact Hausdorff space. Let $C(X)$ be the space of continuous complex-valued functions on $X$ and $A$ be a function algebra on $X$, that is a uniformly closed separating subalgebra of $C(X)$ containing the constants. If $F$ is a closed subset of $X$ we say that $A$ interpolates on $F$ if $A|F = C(F)$. By a positive measure $\mu$ we shall always mean a positive regular bounded Borel measure on $X$. Let $F$ be a measurable subset of $X$. We say a subspace $S$ of $L^p(\mu)$ interpolates on $F$ if $S|F = L^p(F) = L^p(\mu_F)$, where $\mu_F$ is the restriction of $\mu$ to $F$. Let $H^p(\mu)$ be the closure of $A$ in $L^p(\mu)$ where $1 \leq p < \infty$, and let $H^\infty(\mu) = H^p(\mu) \cap L^\infty(\mu)$. One question we are concerned with here is whether interpolation of the algebra is sufficient to imply interpolation of its associated $H^p$-spaces. We therefore begin by obtaining necessary and sufficient conditions for a closed subspace of $L^p(\mu)$ to have closed restriction in $L^p(F)$. These condition are analogous to some obtained by Glicksberg for function algebras. Using these results we obtain theorems about interpolation of certain invariant subspaces, and then apply them to $H^p$-spaces. In particular we show that when $A$ approximates in modulus and $\mu$ is any measure which is not a point-mass, $H^p(\mu)$ interpolates only on sets of measure zero. (One sees that $A$ interpolates only on sets of measure zero, so our original question has a trivial answer for these algebras.) For uniformly closed weak-star Dirichlet algebras again the answer to our original question is affirmative. Finally we provide an example of an algebra which interpolates such that $H^\infty(\mu)$ interpolates and the $H^p(\mu)$ do not interpolate for $1 \leq p < \infty$. I am indebted to a paper of Glicksberg for those techniques which inspired the present effort. Below we show that these techniques apply to the $L^p$ situation and to other “similar” situations.

Glicksberg [3] has given necessary and sufficient conditions for interpolation of a closed subspace of $C(X)$. We show here that analogous theorems hold for subspaces of $L^p(X)$. Let $A \subseteq B$ be Banach spaces. $A^\perp$ will denote all bounded linear functionals on $B$ which annihilate $A$.

**Theorem 1.1.** Let $A, A_1, B$ all be Banach spaces with $A \subseteq A_1$ and $R : A_1 \rightarrow B$ a nonzero bounded linear transformation. Then $R(A)$ is closed in $B$ if and only if there exists $c : \|h - R(A)^\perp\| \leq c\|h^* - A^\perp\| \forall h \in B^*$, where $h^* = R^*h$. It follows that $c \geq 1/\|R\|$. 

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Proof. The map \( R_i = R | A : A \to R(A) \) induces a map
\[
T = \psi \circ R_i^* \circ \phi : B^*/R(A)^\perp \to A_i^*/A_i^\perp
\]
where \( \psi : A^* \to A_i^*/A_i^\perp \) and \( \phi : B^*/R(A)^\perp \to R(A)^* \) are the natural isometric isomorphisms. Further for \( g \in B^* \), \( g - R(A)^\perp \) is taken to \( g^* - A^\perp \) by \( T \), so \( T \) is 1 - 1. Now the range of \( R_i \) is closed if and only if the range of \( R_i^* \) is closed if and only if the range of \( T \) is closed \([1]\). The latter fact is equivalent to: \( \exists c \in \mathbb{R} : \| h - R(A)^\perp \| \leq c \| h^* - A^\perp \| \) for all \( g \in B^* \) by the open mapping theorem. Further, \( \| h^* - A^\perp \| \leq \| R \| \| h - R(A)^\perp \| \) so applying the above inequality gives \( c \geq 1/\| R \| \).

The statement of the above theorem is slightly more general than those of other similar theorems appearing the literature. The proof is virtually the same as that in \([3]\) albeit in a more general setting. See also \([2]\). The next corollary follows as in \([3]\).

**Corollary 1.2.** Let \( X \) be locally compact and \( A \) a uniformly closed subspace of \( C_0(X) \). Let \( F \) be a locally compact subset of \( X \) and suppose \( A | F \subseteq C_0(F) \). Then

(i) \( A | F \) is uniformly closed in \( C_0(F) \) if and only if \( \exists c \in \mathbb{R} : \| \mu - (A | F)^\perp \| \leq c \| \mu - A^\perp \| \) \( \forall \) regular bounded Borel measure \( \mu \) on \( F \).

(ii) \( A | F = C_0(F) \) if and only if \( \exists c \in \mathbb{R} : \| \mu_F \| \leq c \| \mu_{F'} \| \) \( \forall \mu \in A^\perp \).

We now apply 1.1 to get the analogous conclusion for subspaces of \( L^p \)-spaces.

**Definition.** Let \( \mu \) be a fixed positive measure on \( X \) and \( F \) a measurable subset of \( X \). Set \( L^p(F) = L^p(\mu_F), 1 \leq p \leq \infty \) where \( \mu_F \) is the restriction of \( \mu \) to \( F \). For \( f \in L^p(F) \) let \( \tilde{f} \) be the function which is \( f \) on \( F \) and 0 on \( F' \). Note that if \( R \) is the restriction map \( L^p(X) \to L^p(F) \), then \( \tilde{f} = f^* \). For a subspace \( S \) of \( L^p(X) \), \( (S | F)^\perp = \{ g \in L^p(F) | g \upharpoonright F \subseteq S \} \). Clearly \( \{ \tilde{f} | f \in (S | F)^\perp \} \subseteq S^\perp \).

**Theorem 1.3.** Let \( S \) be a closed subspace of \( L^p(X), 1 \leq p < \infty \), and \( F \) a measurable subset of \( X \). Then:

(i) \( S | F \) is closed in \( L^p(F) \) if and only if

\[
\exists c \in \mathbb{R} : \| g - (S | F)^\perp \| \leq c \| \tilde{g} - S^\perp \| \quad \forall g \in L^p(F);
\]

(ii) \( S | F = L^p(F) \) if and only if

\[
\exists c \in \mathbb{R} : \| g | F \| \leq c \| g | F' \| \quad \forall g \in S^\perp.
\]

If \( F \) has positive measure it follows that \( c \geq 1 \). If \( p = \infty \) then the
“only if” parts of (i) and (ii) hold for \( g \in L^p(E) \) and \( L^p(X) \cap S^\perp \) respectively.

**Proof.** (i) follows by applying 1.1 to the restriction map \( R \). As \( \| R \| \leq 1 \), we have \( c \geq 1 \). If \( S/F = L^p(F) \), then (1) becomes \( \| g \| \leq c \| g - S^\perp \| \forall g \in L^p(F) \). In particular if \( g \in S^\perp \),

\[
\| g \|_p \leq c \| g - S/F \|_p = c \| g - F' \|_p.
\]

This shows the “only if” part of (ii). For the “if” part of (ii) we shall use a concavity property of the \( q \)-norm; namely, if \( \alpha, \beta \geq 0 \), \( \alpha + \beta \leq 1 \), then \( \| f \|_q \geq \alpha \| f \|_p + \beta \| F' \|_q \). Now taking \( g \in (S/F)^\perp \), and applying (2) to \( g \) shows that \( (S/F)^\perp = 0 \), so \( S/F \) is dense in \( L^p(F) \). Thus we need only show that \( S/F \) is closed. Here (1) reduces to \( \| g \|_q \leq c \| g - S/F \|_q \forall g \in L^p(F) \). But if \( g \in L^p(F) \) and \( h \in S^\perp \), then \( \| g - h \|_q \leq \| (g - h) \|_p + \beta \| h \|_p \). Now choice \( n \) so that \( c/n + c^2/n \leq 1 \) and let \( \alpha = c/n, \beta = c^2/n \). Then

\[
\| g - h \|_q \leq c/n \| g \|_p - c/n \| h \|_p + c^2/n \| h \|_p \leq c/n \| g \|_p
\]

after applying (2). Thus setting \( c' = n/c \) gives \( S/F \) is closed and thus \( S/F = L^p(F) \). The latter part of the conclusion is clear from the above arguments.

**Corollary 1.4.** If \( S \) is a closed subspace of \( L^p(X) \), \( 1 \leq p < \infty \) and \( S^\perp | F \subset (S/F)^\perp \) then \( S/F \) is closed in \( L^p(F) \).

**Proof.** \((S/F)^\perp \subset S^\perp \) so in fact \( S^\perp | F = (S/F)^\perp \). Taking \( g \in L^p(F) \), and \( h \in S^\perp \) we have

\[
\| g - S^\perp \|_q \geq \| g - S^\perp | F \|_q = \| g - (S/F)^\perp \|_q
\]

and (1) applies.

2. Restrictions of invariant subspaces. Let \( X \) be a topological space and \( \mu \) a positive measure on \( X \). Throughout this section \( A \) will be a subalgebra of \( L^\infty(\mu) \), and \( S \) will be a closed subspace of \( L^p(X) \) for some \( 1 \leq p < \infty \). We assume that \( S \) is invariant under multiplication by elements of \( A \). A **separates in modulus (SM)** if \( \forall \epsilon > 0 \), \( E, F \) disjoint closed sets in \( X \), \( \exists f \in A \) such that \( |f| < \epsilon \) a.e., on \( E \) and \( 1 - |f| < \epsilon \) a.e., on \( F \). Call \( f \) a separating function. A **boundedly separates in modulus (BSM)** if \( \exists M : \forall \epsilon > 0 \), \( E, F \) disjoint closed sets, \( \exists a \) a separating function \( f \in A \) with \( ||f||_\infty < M \). We say that \( A \) **boundedly separates in modulus by invertible func-
tions (BSMI) if $A$ is BSM and the bounded separating functions can be chosen to be invertible. If $A$ is a function algebra on $X$ and the a.e., condition can be left out of the above then we say that $A$ is BSM or BSM on $X$. For example, if $A$ approximates in modulus then $A$ is BSM on $X$ and if $A$ is logmodular then $A$ is BSM on $X$. If $A$ is weak-star-Dirichlet [7] then $A$ may not even be BSM, but $H^\infty$ must be BSM because $\log V = L^\infty_\mu$ where $V$ is the set of invertible elements in $H^\infty$. This includes the case where $\mu$ is a unique representing measure on $X$, or more generally, is “minimal” in the sense of [7, pg. 238]. Thus BSM, etc., “localize” the separation properties to the support of the measure in question.

**THEOREM 2.1.** Let $F$ be a measurable set in $X$. If $A$ is BSM then $S|F = L^p(F')$ if and only if $g \in S^\perp \Rightarrow g|F = 0$. In particular, this holds if $A$ approximates in modulus.

**Proof.** 1.4 implies the “if” part. Conversely, suppose $S|F = L^p(F')$. Then $\exists \ c$ such that $g \in S^\perp \Rightarrow \|g|F\|_q \leq c \|g|F'|\|_q$. Choose $\varepsilon > 0$. Find $K_n$ compact $\subset F \subset V_n$ open such that $\mu(V_n \sim K_n) < 1/n$. We can assume that the $K_n$ are monotone. Suppose $M$ is the bounding constant for the separating functions in $A$. Find $k \in A$ such that $\|k\|_\infty \leq M$ and $|k| - 1| < \varepsilon$ on $K_n$ and $|k| < \varepsilon$ on $V_n$ a.e. Then for fixed $g \in S^\perp$,

$$(1 - \varepsilon) \|g|K_n\|_q \leq \|g|K_n\|_q \leq \|kg|F\|_q \leq c \|kg|F'|\|_q$$

$$\leq c \|kg|F' \cap V_n\|_q + c \|kg|V_n\|_q$$

$$\leq cM \|g|F' \cap V_n\|_q + c \varepsilon \|g|V_n\|_q.$$ 

Letting $\varepsilon \to 0$, we have $\|g|K_n\|_q \leq cM \|g|F' \cap V_n\|_q$. Letting $n \to \infty$, we have $g|F' = 0$.

**COROLLARY 2.2.** Let $A$ be BSM. Suppose that $F_i$ are measurable subsets of $X$ and $F_0 = \bigcup_{i=1}^\infty F_i$. If $S|F_i = L^p(F_i)$ for $i = 1, 2, \cdots$ then $S|F_0 = L^p(F_0)$.

**Proof.** Let $g \in S^\perp$. Then $g|F_i = 0$ a.e. for $i = 1, 2, \cdots$ and thus $g|F_0 = 0$ a.e.

**THEOREM 2.3.** Let $F$ be a closed subset of $X$. If $A$ is BSM then $S|F$ is closed in $L^p(F')$ if and only if $g \in S^\perp \Rightarrow g|F' \in (S|F)^\perp$.

**Proof.** “If.” Apply Corollary 1.4. Here it is not necessary that $F$ be closed.

“Only if.” Find $V_n$ open $\supset F$ such that $\mu(V_n \sim F') < 1/n$. Then
3 M > 0 and $k_n$ invertible in $A$ such that $\| k_n \|_\infty \leq M$, $1 - |k_n| < \varepsilon$ a.e. on $F$ and $|k_n| < \varepsilon$ a.e. on $V_n$. Now $\exists c$ such that 1.3 (1) holds so $g \in S^\perp = \| g \| F - (S|F)^\perp \|_e \leq c \| g \| F' \|_e$. The same holds for $k_ng$. Thus

$$
\| k_ng \| F - (S|F)^\perp \|_e \leq c \| k_ng \| V_n \cap F' \|_e + c \| k_ng \| V_n' \|_e \\
\leq cM \| g \| V_n \sim F' \|_e + c\varepsilon \| g \|_e.
$$

Now since $k_n$ are invertible, $k_n(S|F)^\perp = (S|F)^\perp$. Thus

$$(1 - \varepsilon)\| g \| F - (S|F)^\perp \|_e \leq \| k_ng \| F - (S|F)^\perp \|_e \\
\leq cM \| g \| V_n \sim F' \|_e + c\varepsilon \| g \|_e.$$

Letting $\varepsilon \to 0$ and $n \to \infty$ gives $g \in (S|F)^\perp$.

**Corollary 2.4.** Let $A$ be BSMI. Suppose $F_i$ are closed subsets of $X$ and $F = \bigcup_{i=1}^n F_i$. If $S|F_i$ is closed in $L^p(F_i)$ for each $i$, then $S|F$ is closed in $L^p(F)$.

**Proof.** Take $g \in S^\perp$. Then $g|F_i \in (S|F_i)^\perp$, and by the dominated convergence theorem, it follows that $g|F \in (S|F)^\perp$.

Using the above theorem we also encounter the following phenomenon which is different from that which usually occurs in the function algebra setting.

**Corollary 2.5.** Let $F$ be a closed subset of $X$, and let $A$ be BSMI. Then $S|F$ is closed in $L^p(F)$ if $S|F'$ is closed in $L^p(F')$. In particular this happens if $A$ is logmodular.

**Proof.** Let $g \in S^\perp$. Then $g|F \in (S|F)^\perp$. Hence $g|F' = g - (g|F) \in S^\perp$ and thus $g|F' \in (S|F')^\perp$, so $S|F'$ is closed.

The above is explained by the following “splitting lemma” which was pointed out to me by K.B. Laursen.

**Lemma 2.6.** Let $S$ be a closed subspace of $L^p(\mu)$, $1 \leq p < \infty$, and let $F$ be a measurable subset of $X$. Then $S = S \sim F \oplus S \sim F'$ if and only if $g \in S^\perp \Rightarrow g|F \in S^\perp$.

**Remarks.** The following illustrates 2.5. Let $X$ be the union of two disjoint disks, $\mu = m_1 + m_2$ where $m_1$ and $m_2$ are the Lebesgue measures on the two circles comprising the boundary of $X$, and let $A$ be the algebra of functions continuous on $X$ and analytic on the
interior of \( X \). Then \( H^i(m_\circ) + L^i(m_\circ) \) splits and neither \( F \) nor \( F' \) have measure 0.

Also it is easy to find examples of closed subspaces of \( L^i(-1, 1) \) which are proper and interpolate on \((-1, 0)\) and \((0, 1)\). For example, let \( S \) be the set of functions \( f \) in \( L^i(-1, 1) \) such that \( f(x) = f(-x) \) a.e.

### 3. Interpolation of \( H^p \)-spaces and function algebras.

Throughout this section unless it is otherwise stated, we assume that \( A \) is a function algebra on a compact space \( X \), \( \mu \) is a representing measure for \( A \) which is not a point-mass and \( I \) is the corresponding maximal ideal.

**Proposition 3.1.** If \( I \) is SM in \( L^\infty(\mu) \) then the only open sets on which \( H^p(\mu) \) interpolates for some \( 1 \leq p \leq \infty \) are those of measure 0.

**Proof.** If \( H^p \) interpolates on \( V \) open and \( \mu(V) > 0 \) then find \( K \) compact in \( V \) of positive measure. Find a sequence in \( I \) whose moduli converge to 1 on \( K \) and 0 on \( V' \). This contradicts 1.3 (ii).

**Proposition 3.2.** If \( I \) is BSM in \( L^\infty(\mu) \) then the only measurable sets on which \( H^p(\mu) \) interpolates for some \( 1 \leq p \leq \infty \) are those of measure 0.

**Proof.** Suppose \( H^p \) interpolates on a set \( F \) of positive measure. We may assume that \( F \) is closed. Since \( \mu \) is assumed to not be a point-mass \( F' \) has positive measure. We can therefore choose \( K_\circ \) compact and monotone in \( F' \) so that \( \mu(K_\circ) \rightarrow \mu(F') \). Find \( f_n \) in \( I \) which are uniformly bounded such that \( |f_n| - 1| < 1/n \) on \( F \) and \( |f_n| < 1/n \) on \( K_\circ \). This contradicts 1.3 (ii).

We wish to study the relation between interpolation of the algebra \( A \) and its associated \( H^p \)-spaces. As was pointed out in the introduction, if \( A \) approximates in modulus then the situation is trivial. For if \( F \) is a closed set on which \( A \) interpolates then because \( F' \) is an intersect of peak sets, we must have that \( \mu(F) = 0 \) by the dominated convergence theorem. So interpolation of the \( H^p \)-space follows vacuously. More generally we have the following.

**Proposition 3.3.** Let \( A \) be BSM on \( X \), and \( F \) a closed subset of \( X \). If \( A \) interpolates on \( F \) then \( H^p(\mu) \) interpolate on \( F \) for any measure \( \mu \), and any \( 1 \leq p < \infty \).

**Proof.** \( g \perp H^p = g \, d\mu \perp A = g \, d\mu_F = 0 \Rightarrow g \, |F = 0 \) a.e., \( \mu = H^p \) interpolates on \( F \).
**Proposition 3.4.** If $\mu$ is a representing measure for $A$, and $A$ is BSM in $L^\infty(\mu)$, then $H^p(\mu)$ interpolates only on sets of measure 0 if $1 \leq p \leq \infty$.

*Proof.* Suppose for some $p$, $H^p|F = L^p(F)$. Let $A_0$ be the ideal determined by $\mu$. Then $A_0 \subset (H^p)\perp$ so by 2.1., $g \in A_0 \Rightarrow g|F = 0$ a.e. But if $f \in H^p$, then $f - \int f d\mu$ is a pointwise a.e. limit of a sequence of elements of $A_0$ and thus $f = \int f d\mu$ a.e. on $F$, so that all $H^p$ functions are constant a.e. on $F$. Thus $L^p(F) = \text{constants}$ and thus $\mu_F$ is a point-mass at some point $x$. But $\mu$ must be continuous, for $\exists g \in I$ such that $g(x) \neq 0$ and applying 2.1 gives $\mu(x) = 0$.

**Proposition 3.5.** Let $A$ be BSMI on $X$, and $F$ a closed subset of $X$. If $A|F$ is closed then $H^p(\mu)$ restricted to $F$ is closed for any measure $\mu$, and any $1 \leq p < \infty$.

*Proof.* $g \perp H^p \Rightarrow g \ d\mu \perp A \Rightarrow g \ d\mu_p \in (A|F)\perp \Rightarrow g \ d\mu_p \in (H^p)\perp \Rightarrow H^p$ restricted to $F$ is closed by 2.3.

**Remarks.** Both 3.3 and 3.5 hold because $F$ is an intersect of peak sets. By the above it is easy to construct examples in which the $H^p$ spaces interpolate on sets of positive measure (where $\mu$ is not a representing measure). For another example, let $A$ be the disk algebra on the unit disk, and let $\mu = \frac{1}{2} d\theta + \frac{1}{2} \delta_0$ where $\delta_0$ is the point-mass at 0. As yet we have been unable to construct examples which are not of this discrete type when $\mu$ is a representing measure.

We now construct examples in which the algebra and $H^\infty$ interpolate but in which none of the $H^p$-spaces, $1 \leq p < \infty$, interpolate. Let $\{r_n\}$ be a nonnegative interpolating sequence in the open unit disk converging to 1. Then $F = \{r_n\} \cup \{1\}$ is an interpolating sequence for the disk algebra on the unit disk [6]. Let $\mu_n$ be the Poisson measures for $r_n$ on the unit circle. Choose a sequence $\alpha_n \geq 0$ such that $\sum_{n=1}^{\infty} \alpha_n \mu_n < 1/2 d\theta$ (*). Consider the positive measure $\mu = \sum_{n=1}^{\infty} \alpha_n (\delta_{r_n} - \mu_n) + d\theta$ where $\delta_{r_n}$ is the point-mass at $r_n$. Then $\mu$ represents 0 for the disk algebra and we claim that $H^\infty(\mu)$ interpolates on $F$ while $H^p(\mu)$ $1 \leq p < \infty$ do not interpolate on $F$. To see this we need the following.

**Lemma 3.6.** $H^p(\mu) = H^p|F \cup T$ where $H^p$ is the usual $H^p$-space for the disk algebra ($1 \leq p \leq \infty$) on the closed unit disk.

*Proof.* If $f \in H^p(d\theta)$ then $\exists f_n \in A : f_n \rightarrow f$ in $L^p(d\theta)$. If $\hat{f}$ de-
notes the harmonic extension of \( f \) to \( H^p \), then

\[
\int |\hat{f}_n - \hat{f}|^p \, d\mu \leq (1 + \sum 2\alpha_j(1 + r_j)/(1 - r_j)) \int |f_n - f| \, d\theta \to 0.
\]

So \( H^p \cap F \subset H^p(d\mu) \). Conversely, if \( f_n \in A \) and \( f_n \to f \) in \( L^p(d\theta) \), then \( f_n \to f \) in \( L^p(d\theta) \), so \( f \mid T \in H^p(d\theta) \) and therefore extends to \( g = \hat{f} \mid T \) in \( H^p \). So \( g \mid F \cup T \in H^p(\mu) \) and \( g \mid T = f \mid T \). But since the functions in \( H^p(\mu) \) are determined by their values on \( T \), we have \( f = g \in H^p \cap F \cup T \), and we are done for \( 1 \leq p < \infty \). Now

\[
H^\infty(d\mu) = H^2(d\mu) \cap L^\infty(d\mu) = [H^1 \mid F \cup T] \cap L^\infty(\mu)
\]

\[
= [H^2(d\theta) \cap L^0(d\theta)] \mid F \cup T = H^\infty \mid F \cup T,
\]

and this completes the proof.

Now observe that if \( f \in H^p(d\mu) \), then

\[
|f(r_n)|^p \leq [(1 + r_n)/(1 - r_n)] \int |f|^p \, d\theta
\]

so that \( \exists \ c \ \exists \ |\ f(r_n)|^p \leq c(1 + r_n)/(1 - r_n) \) is satisfied. Thus if we choose a (nonnegative) sequence \( \{x_n\} \) such that \( x_n(1 - r_n)/(1 + r_n) \to \infty \) and such that \( \sum x_n(1 + r_n) \alpha_n/(1 - r_n) < \infty \), we obtain an element of \( L^p(\mu_F) \) which is not the restriction of a function from \( H^p(d\mu) \). Such a sequence can be found for example by finding \( \beta_n \geq 0 \) to satisfy \(^*\) and setting \( \alpha_n = \beta_n^2 \) and \( x_n = (\beta_n)^{-1/p} \).

Since \( H^\infty \) interpolates on \( F \), we see that \( H^\infty(d\mu) \) interpolates on \( F \) by 3.6.

Thus one may ask for conditions that will force interpolation of \( H^\infty \)-spaces to follow from interpolation of the algebra. The following is one such condition.

**Theorem 3.7** Let \( A \) be a function algebra on \( X \), \( \mu \) a representing measure for \( A \), and \( A_0 \) the corresponding maximal ideal. Suppose that \( H^\infty(\mu) = H^\alpha(\mu) \cap L^\alpha(\mu) \), \( \alpha \leq p \). If \( A_0 \) is weak-star dense in \( H^\alpha(\mu)^1 \), then interpolation of \( A \) on a closed set \( F \) implies interpolation of \( H^\alpha(\mu) \) on \( F \) for all \( \alpha \leq p < \infty \) with integer conjugates \( q \).

**Proof.** The conclusion deals only with \( 1 \leq \alpha \leq p \leq 2 \). Suppose \( 1 < \alpha \) and \( A \mid F = C(F) \). Then \( \exists \ c \ \exists \ |\ \mu_F\mid \leq c |\ \mu_P\mid \) for every \( \mu \in A^1 \). Now choose \( g \in A_\alpha \). Then \( g^*d\mu \in A^1 \) so \( \int |g|^p d\mu \leq c \int |g|^q d\mu \) or \(^*\) \( \|g\mid F\|_q \leq c^{1/p} \|g\mid F'\|_q \). Since \( A_0 \) is dense in \( H^\infty(\mu)^1 \) also, we have \(^*\) holds for every \( g \in H^p(\mu)^1 \) and thus \( H^p(\mu) \) interpolates on \( F \). Suppose \( \alpha = 1 \). For \( g \in H^1(\mu) \) we have \( \|g\mid F\|_q \leq c^{1/p} \|g\mid F'\|_q \) for \( q = 2, 3, \ldots \), and thus letting \( q \to \infty \) we have \( \|g\mid F\|_\infty \leq \|g\mid F'\|_\infty \) so that \( H^1(\mu) \) also interpolates on \( F \).
COROLLARY 3.8. If \( A \) is a function algebra which is weak-star-Dirichlet in \( L^\infty(\mu) \) then \( A \) interpolates only on sets of \( \mu \) measure 0.

**Proof.** \( A \) satisfies the hypotheses of 3.7 [7] and thus \( H^1 \) interpolates on \( F \). But \( H^1 \) is invariant under \( H^\infty \) which is BSMI so that \( F \) has \( \mu \) measure 0 by 3.4.

It is also clear from 3.4 that when \( A \) is weak-star-Dirichlet, \( H^p \) interpolate only on sets of measure 0 for \( 1 \leq p \leq \infty \). Using the invariant subspace theorem we have the following.

**Theorem 3.9.** Let \( A \) be weak-star-Dirichlet. If \( F \) is closed and \( H^p(\mu) \) restricted to \( F \) is closed for some \( 1 \leq p < \infty \), then \( \mu(F) = 0 \), or \( \mu(F') = 0 \).

**Proof.** Since \( H^p \) is invariant under \( H^\infty \) which is BSMI, applying 2.3 and 2.6 we have \( H^p = \widetilde{H^p(F \oplus H^pF')} \). Now if \( F \) has positive measure, then \( \widetilde{H^pF} \) is a simply invariant subspace of \( L^p \) and by the invariant subspace theorem [7, 4.16], \( \widetilde{H^pF} = qH^p \) where \( |q| = 1 \) a.e. But \( q \in \widetilde{H^pF'} \) so we have \( \mu(F') = 0 \).

The example preceding 3.7 is clearly not weak-star-Dirichlet because the measure \( \mu \) is not minimal. In addition we have the following.

**Corollary 3.10.** In the example preceding 3.7, \( A_0 \) is not weak-star dense in \( H^1(\mu)^\perp \).

**Proof.** We only need to verify that \( H^p(\mu) \supset H^1(\mu) \cap L^p(\mu) \). But if \( f \in H^1(\mu) \cap L^p(\mu) \) then \( f|T = g|T \) where

\[
g \in H^1(d\vartheta) \cap L^p(d\vartheta) = H^p(d\vartheta) .
\]

So as \( \hat{g} \big| F \cup T \in H^p(\mu) \), and \( \hat{g} \) and \( f \) agree on \( T \), we have

\[
f = \hat{g} \big| F \cup T \in H^p(\mu) .
\]

Finally we remark that 1.3 should hold for function spaces whose duals restrict in some sense and whose norm satisfies the concavity condition. We hope to consider such examples at a later date.

**References**


Received April 29, 1970. Supported by Western Washington State College Bureau for Faculty Research.

Western Washington State College
Pacific Journal of Mathematics
Vol. 39, No. 3 July, 1971

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