RESTRICTIONS OF BANACH FUNCTION SPACES

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Let $X$ be a compact Hausdorff space. Let $C(X)$ be the space of continuous complex-valued functions on $X$ and $A$ be a function algebra on $X$, that is a uniformly closed separating subalgebra of $C(X)$ containing the constants. If $F$ is a closed subset of $X$ we say that $A$ interpolates on $F$ if $A|F = C(F)$. By a positive measure $\mu$ we shall always mean a positive regular bounded Borel measure on $X$. Let $F$ be a measurable subset of $X$. We say a subspace $S$ of $L^p(\mu)$ interpolates on $F$ if $S|F = L^p(F) = L^p(\mu_F)$, where $\mu_F$ is the restriction of $\mu$ to $F$. Let $H^p(\mu)$ be the closure of $A$ in $L^p(\mu)$ where $1 \leq p < \infty$, and let $H^\infty(\mu) = H^p(\mu) \cap L^\infty(\mu)$. One question we are concerned with here is whether interpolation of the algebra is sufficient to imply interpolation of its associated $H^p$-spaces. We therefore begin by obtaining necessary and sufficient conditions for a closed subspace of $L^p(\mu)$ to have closed restriction in $L^p(F)$. These condition are analogous to some obtained by Glicksberg for function algebras. Using these results we obtain theorems about interpolation of certain invariant subspaces, and then apply them to $H^p$-spaces. In particular we show that when $A$ approximates in modulus and $\mu$ is any measure which is not a point-mass, $H^p(\mu)$ interpolates only on sets of measure zero. (One sees that $A$ interpolates only on sets of measure zero, so our original question has a trivial answer for these algebras.) For uniformly closed weak-star Dirichlet algebras again the answer to our original question is affirmative. Finally we provide an example of an algebra which interpolates such that $H^\infty(\mu)$ interpolates and the $H^p(\mu)$ do not interpolate for $1 \leq p < \infty$. I am indebted to a paper of Glicksberg for those techniques which inspired the present effort. Below we show that these techniques apply to the $L^p$ situation and to other "similar" situations.

Glicksberg [3] has given necessary and sufficient conditions for interpolation of a closed subspace of $C(X)$. We show here that analogous theorems hold for subspaces of $L^p(X)$. Let $A \subseteq B$ be Banach spaces. $A^\perp$ will denote all bounded linear functions functionals on $B$ which annihilate $A$.

**Theorem 1.1.** Let $A, A_1, B$ all be Banach spaces with $A \subseteq A_1$ and $R: A_1 \to B$ a nonzero bounded linear transformation. Then $R(A)$ is closed in $B$ if and only if $\exists c \geq 0: \| h - R(A)^\perp \| \leq c \| h - A^\perp \| \forall h \in B^*$, where $h^* = R^* h$. It follows that $c \geq 1/\| R \|$.
Proof. The map $R_\lambda = R|_A : A \to R(A)$ induces a map
$$T = \psi \circ R_* \circ \phi : B^*/R(A)^\perp \to A^*/A^\perp$$
where $\psi : A^* \to A^*/A^\perp$ and $\phi : B^*/R(A)^\perp \to R(A)^*$ are the natural isometric isomorphisms. Further for $g \in B^*$, $g - R(A)^\perp$ is taken to $g^* - A^\perp$ by $T$, so $T$ is $1 - 1$. Now the range of $R_\lambda$ is closed if and only if the range of $R^*_\lambda$ is closed if and only if the range of $T$ is closed [1]. The latter fact is equivalent to: $\exists c \in \mathbb{R} : \| h - R(A)^\perp \| \leq c \| h^* - A^\perp \|$ for all $g \in B^*$ by the open mapping theorem. Further, $\| h^* - A^\perp \| \leq \| R \| \| h - R(A)^\perp \|$ so applying the above inequality gives $c \geq 1/\| R \|$.

The statement of the above theorem is slightly more general than those of other similar theorems appearing in the literature. The proof is virtually the same as that in [3] albeit in a more general setting. See also [2]. The next corollary follows as in [3].

**Corollary 1.2.** Let $X$ be locally compact and $A$ a uniformly closed subspace of $C_0(X)$. Let $F$ be a locally compact subset of $X$ and suppose $A|_F \subset C_0(F)$. Then
(i) $A|_F$ is uniformly closed in $C_0(F)$ if and only if $\exists c \in \mathbb{R} : \| \mu - (A|_F)^\perp \| \leq c \| \mu - A^\perp \|$ for regular bounded Borel measure $\mu$ on $F$.
(ii) $A|_F = C_0(F)$ if and only if $\exists c \in \mathbb{R} : \| \mu_F \| \leq c \| \mu_{F^\perp} \|$ for $\mu \in A^\perp$.

We now apply 1.1 to get the analogous conclusion for subspaces of $L^p$-spaces.

**Definition.** Let $\mu$ be a fixed positive measure on $X$ and $F$ a measurable subset of $X$. Set $L^p(F) = L^p(\mu_F)$, $1 \leq p \leq \infty$ where $\mu_F$ is the restriction of $\mu$ to $F$. For $f \in L^p(F)$ let $\tilde{f}$ be the function which is $f$ on $F$ and 0 on $F^\perp$. Note that if $R$ is the restriction map $L^p(X) \to L^p(F)$, then $\tilde{f} = f^\ast$. For a subspace $S$ of $L^p(X)$, $(S|F)^\perp = \{ g \in L^p(F) \mid g^\perp S \subset F \}$. Clearly $\{ \tilde{f} \mid f \in (S|F)^\perp \} \subset S^\perp$.

**Theorem 1.3.** Let $S$ be a closed subspace of $L^p(X)$, $1 \leq p < \infty$, and $F$ a measurable subset of $X$. Then:
(i) $S|F$ is closed in $L^p(F)$ if and only if
$$\exists c \in \mathbb{R} : \| g - (S|F)^\perp \| \leq c \| g - S^\perp \|$ for $g \in L^p(F)$;
(ii) $S|F = L^p(F)$ if and only if
$$\exists c \in \mathbb{R} : \| g \|_F \| \leq c \| g \|_{F^\perp}$ for $g \in S^\perp$.

If $F$ has positive measure it follows that $c \geq 1$. If $p = \infty$ then the
“only if” parts of (i) and (ii) hold for \( g \in L^1(E) \) and \( L^1(X) \cap S^\perp \) respectively.

**Proof.** (i) follows by applying 1.1 to the restriction map \( R \). As \( \| R \| \leq 1 \), we have \( c \geq 1 \). If \( S \mid F = L^p(F) \), then (1) becomes

\[
\| g \| \leq c \| g - S^\perp \| \quad \forall \ g \in L^p(F).
\]

In particular if \( g \in S^\perp \),

\[
\| g \mid F \| \leq c \| g \| \hat{F} - g \| = c \| g \mid F' \|.
\]

This shows the “only if” part of (ii). For the “if” part of (ii) we shall use a concavity property of the \( q \)-norm; namely, if \( \alpha, \beta \geq 0 \), \( \alpha + \beta \leq 1 \), then \( \| f \|_q \geq \alpha \| f \mid F \|_q + \beta \| f \mid F' \|_q \). Now taking \( g \in (S \mid F)^\perp \), and applying (2) to \( g \) shows that \( (S \mid F)^\perp = 0 \), so \( S \mid F \) is dense in \( L^p(F) \). Thus we need only show that \( S \mid F \) is closed. Here (1) reduces to \( \| g \|_q \leq c' \| \tilde{g} - S^\perp \|_q \quad \forall \ g \in L^q(F) \). But if \( g \in L^q(F) \) and \( h \in S^\perp \), then \( \| \tilde{g} - h \|_q \leq \alpha \| (g - h) \mid F \|_q + \beta \| h \mid F' \|_q \), if \( \alpha, \beta \geq 0 \) and \( \alpha + \beta \leq 1 \). Now choose \( n \) so that \( c/n + c^2/n \leq 1 \) and let \( \alpha = c/n, \beta = c^2/n \). Then

\[
\| \tilde{g} - h \|_q \geq c/n \| g \mid F \|_q - c/n \| h \mid F \|_q + c/n \| h \mid F' \|_q \geq c/n \| g \mid F \|_q
\]

after applying (2). Thus setting \( c' = n/c \) gives \( S \mid F \) is closed and thus \( S \mid F = L^p(F) \). The latter part of the conclusion is clear from the above arguments.

**Corollary 1.4.** If \( S \) is a closed subspace of \( L^p(X), 1 \leq p < \infty \) and \( S^\perp \mid F \subset (S \mid F)^\perp \) then \( S \mid F \) is closed in \( L^p(F) \).

**Proof.** \( (S \mid F)^\perp \subset S^\perp \) so in fact \( S^\perp \mid F = (S \mid F)^\perp \). Taking \( g \in L^q(F) \), and \( h \in S^\perp \) we have

\[
\| g - S^\perp \|_q \geq \| g - S^\perp \mid F \|_q = \| \tilde{g} - (S \mid F)^\perp \|_q
\]

and (1) applies.

2. Restrictions of invariant subspaces. Let \( X \) be a topological space and \( \mu \) a positive measure on \( X \). Throughout this section \( A \) will be a subalgebra of \( L^\infty(\mu) \), and \( S \) will be a closed subspace of \( L^p(X) \) for some \( 1 \leq p < \infty \). We assume that \( S \) is invariant under multiplication by elements of \( A \). A separates in modulus (SM) if \( \forall \varepsilon > 0 \), \( E, F \) disjoint closed sets in \( X \), \( \exists f \in A \) such that \( |f| < \varepsilon \) a.e., on \( E \) and \( |1 - |f|| < \varepsilon \) a.e., on \( F \). Call \( f \) a separating function. A boundedly separates in modulus (BSM) if \( \exists M > 0 : \forall \varepsilon > 0, E, F \) disjoint closed sets, \( \exists f \in A \) with \( ||f||_\infty < M \). We say that \( A \) boundedly separates in modulus by invertible func-
tions (BSMI) if \( A \) is BSM and the bounded separating functions can be chosen to be invertible. If \( A \) is a function algebra on \( X \) and the a.e., condition can be left out of the above then we say that \( A \) is BSM or BSMI on \( X \). For example, if \( A \) approximates in modulus then \( A \) is BSM on \( X \) and if \( A \) is logmodular then \( A \) is BSMI on \( X \). If \( A \) is weak-star-Dirichlet \([7]\) then \( A \) may not even be BSM, but \( H^\infty \) must be BSMI because \( \log V = L^\infty_r \) where \( V \) is the set of invertible elements in \( H^\infty \). This includes the case where \( \mu \) is a unique representing measure on \( X \), or more generally, is "minimal" in the sense of \([7, \text{pg. 238}]\). Thus BSM, etc., "localize" the separation properties to the support of the measure in question.

**Theorem 2.1.** Let \( F \) be a measurable set in \( X \). If \( A \) is BSM then \( S \setminus F = L^p(F) \) if and only if \( g \in S^\perp \Rightarrow g \mid F = 0 \). In particular, this holds if \( A \) approximates in modulus.

**Proof.** 1.4 implies the "if" part. Conversely, suppose \( S \setminus F = L^p(F) \). Then \( \exists \ c \) such that \( g \in S^\perp \Rightarrow \| g \mid F \|_q \leq c \| g \mid F' \|_q \). Choose \( \varepsilon > 0 \). Find \( K_n \) compact \( \subset F \subset V_n \) open such that \( \mu (V_n \sim K_n) < 1/n \). We can assume that the \( K_n \) are monotone. Suppose \( M \) is the bounding constant for the separating functions in \( A \). Find \( k \in A \) such that \( \| k \|_\infty \leq M \) and \( \| k \mid - 1 \| < \varepsilon \) on \( K_n \) and \( | k \mid < \varepsilon \) on \( V_n \) a.e. Then for fixed \( g \in S^\perp \),

\[
(1 - \varepsilon) \| g \mid K_n \|_q \leq \| kg \mid K_n \|_q \leq \| kg \mid F \|_q \leq c \| kg \mid F' \|_q \\
\leq c \| kg \mid F' \cap V_n \|_q + c \| kg \mid V_n' \|_q \\
\leq cM \| g \mid F' \cap V_n \|_q + c \varepsilon \| g \|_q .
\]

Letting \( \varepsilon \to 0 \), we have \( \| g \mid K_n \|_q \leq cM \| g \mid F' \cap V_n \|_q \). Letting \( n \to \infty \), we have \( g \mid F = 0 \).

**Corollary 2.2.** Let \( A \) be BSM. Suppose that \( F_i \) are measurable subsets of \( X \) and \( F'_0 = \bigcup_{i=1}^\infty F_i \). If \( S \setminus F_i = L^p(F_i) \) for \( i = 1, 2, \cdots \) then \( S \setminus F'_0 = L^p(F'_0) \).

**Proof.** Let \( g \in S^\perp \). Then \( g \mid F_i = 0 \) a.e. for \( i = 1, 2, \cdots \) and thus \( g \mid F_0 = 0 \) a.e.

**Theorem 2.3.** Let \( F \) be a closed subset of \( X \). If \( A \) is BSM then \( S \setminus F \) is closed in \( L^p(F) \) if and only if \( g \in S^\perp \Rightarrow g \mid F \in (S \setminus F)^\perp \).

**Proof.** "If." Apply Corollary 1.4. Here it is not necessary that \( F \) be closed.

"Only if." Find \( V_n \) open \( \supseteq F \) such that \( \mu (V_n \sim F) < 1/n \). Then
$\exists M > 0$ and $k_n$ invertible in $A$ such that $\|k_n\|_{\infty} \leq M$, $|1 - |k_n|| < \varepsilon$ a.e. on $F$ and $|k_n| < \varepsilon$ a.e. on $V_n$. Now $\exists c$ such that 1.3 (1) holds so $g \in S^{\perp} \Rightarrow \|g|F - (S|F)^{\perp}\|_q \leq c\|g|F'\|_q$. The same holds for $k_ng$. Thus

$$\|k_ng|F - (S|F)^{\perp}\|_q \leq c \|k_ng|V_n \cap F'\|_q + c\|k_ng|V_n'\|_q$$

$$\leq cM\|g|V_n \sim F\|_q + c\varepsilon\|g\|_q.$$ 

Now since $k_n$ are invertible, $k_n(S|F)^{\perp} = (S|F)^{\perp}$. Thus

$$(1 - \varepsilon)\|g|F - (S|F)^{\perp}\|_q \leq \|k_ng|F - (S|F)^{\perp}\|_q$$

$$\leq cM\|g|V_n \sim F\|_q + c\varepsilon\|g\|_q.$$ 

Letting $\varepsilon \to 0$ and $n \to \infty$ gives $g|F' \in (S|F)^{\perp}$.

**Corollary 2.4.** Let $A$ be BSMI. Suppose $F_i$ are closed subsets of $X$ and $F = \bigcup_{i=1}^m F_i$. If $S|F_i$ is closed in $L^p(F_i)$ for each $i$, then $S|F$ is closed in $L^p(F)$.

**Proof.** Take $g \in S^{\perp}$. Then $g|F_i \in (S|F_i)^{\perp}$, and by the dominated convergence theorem, it follows that $g|F \in (S|F)^{\perp}$.

Using the above theorem we also encounter the following phenomenon which is different from that which usually occurs in the function algebra setting.

**Corollary 2.5.** Let $F$ be a closed subset of $X$, and let $A$ be BSMI. Then $S|F$ is closed in $L^p(F) \Rightarrow S|F'$ is closed in $L^p(F')$. In particular this happens if $A$ is logmodular.

**Proof.** Let $g \in S^{\perp}$. Then $g|F' \in (S|F')^{\perp}$. Hence

$$g | F' = g - (g | F) \in S^{\perp}$$

and thus $g|F' \in (S|F')^{\perp}$, so $S|F'$ is closed.

The above is explained by the following "splitting lemma" which was pointed out to me by K.B. Laursen.

**Lemma 2.6.** Let $S$ be a closed subspace of $L^p(\mu)$, $1 \leq p < \infty$, and let $F$ be a measurable subset of $X$. Then $S = \widehat{S|F} \oplus \widehat{S|F'}$ if and only if $g \in S^{\perp} \Rightarrow g | F \in S^{\perp}$.

**Remarks.** The following illustrates 2.5. Let $X$ be the union of two disjoint disks, $\mu = m_1 + m_2$ where $m_1$ and $m_2$ are the Lebesgue measures on the two circles comprising the boundary of $X$, and let $A$ be the algebra of functions continuous on $X$ and analytic on the
interior of $X$. Then $H'(m_1) + L'(m_2)$ splits and neither $F$ nor $F'$ have measure 0.

Also it is easy to find examples of closed subspaces of $L^i(-1, 1)$ which are proper and interpolate on $(-1, 0]$ and $(0, 1)$. For example, let $S$ be the set of functions $f$ in $L^i(-1, 1)$ such that $f(x) = f(-x)$ a.e.

3. Interpolation of $H^p$-spaces and function algebras. Throughout this section unless it is otherwise stated, we assume that $A$ is a function algebra on a compact space $X$, $\mu$ is a representing measure for $A$ which is not a point-mass and $I$ is the corresponding maximal ideal.

**Proposition 3.1.** If $I$ is SM in $L^\infty(\mu)$ then the only open sets on which $H^p(\mu)$ interpolates for some $1 \leq p \leq \infty$ are those of measure 0.

**Proof.** If $H^p$ interpolates on $V$ open and $\mu(V) > 0$ then find $K$ compact in $V$ of positive measure. Find a sequence in $I$ whose moduli converge to 1 on $K$ and 0 on $V'$. This contradicts 1.3 (ii).

**Proposition 3.2.** If $I$ is BSM in $L^\infty(\mu)$ then the only measurable sets on which $H^p(\mu)$ interpolates for some $1 \leq p \leq \infty$ are those of measure 0.

**Proof.** Suppose $H^p$ interpolates on a set $F$ of positive measure. We may assume that $F$ is closed. Since $\mu$ is assumed to not be a point-mass $F'$ has positive measure. We can therefore choose $K_n$ compact and monotone in $F'$ so that $\mu(K_n) \to \mu(F')$. Find $f_n$ in $I$ which are uniformly bounded such that $||f_n| - 1| < 1/n$ on $F$ and $|f_n| < 1/n$ on $K_n$. This contradicts 1.3 (ii).

We wish to study the relation between interpolation of the algebra $A$ and its associated $H^p$-spaces. As was pointed out in the introduction, if $A$ approximates in modulus then the situation is trivial. For if $F$ is a closed set on which $A$ interpolates then because $F$ is an intersect of peak sets, we must have that $\mu(F) = 0$ by the dominated convergence theorem. So interpolation of the $H^p$-space follows vacuously. More generally we have the following.

**Proposition 3.3.** Let $A$ be BSM on $X$, and $F$ a closed subset of $X$. If $A$ interpolates on $F$ then $H^p(\mu)$ interpolate on $F$ for any measure $\mu$, and any $1 \leq p < \infty$.

**Proof.** $g \perp H^p \Rightarrow g \, d\mu \perp A \Rightarrow g \, d\mu_F = 0 \Rightarrow g \mid F = 0$ a.e., $\mu \Rightarrow H^p$ interpolates on $F$. 
PROPOSITION 3.4. If μ is a representing measure for A, and A is BSM in $L^\infty(\mu)$, then $H^p(\mu)$ interpolates only on sets of measure 0 if $1 \leq p \leq \infty$.

Proof. Suppose for some $p$, $H^p = L^p(F)$. Let $A_0$ be the ideal determined by μ. Then $A_0$ is a subset of $H^p$ so by 2.1., $g \in A_0$ if $g | F = 0$ a.e. But if $f \in H^p$, then $f - \int f d\mu$ is a pointwise a.e. limit of a sequence of elements of $A_0$ and thus $f = \int f d\mu$ a.e. on $F$, so that all $H^p$ functions are constant a.e. on $F$. Thus $L^p(F) = \text{constants}$ and thus $\mu_F$ is a point-mass at some point $x$. But $\mu$ must be continuous, for $g \in I$ such that $g(x) \neq 0$ and applying 2.1 gives $\mu\{x\} = 0$.

PROPOSITION 3.5. Let A be BSM on X, and F a closed subset of X. If $A \mid F$ is closed then $H^p(\mu)$ restricted to $F$ is closed for any measure $\mu$, and any $1 \leq p < \infty$.

Proof. $g \perp H^p \Rightarrow g d\mu \perp A \Rightarrow g d\mu_F \in (A \mid F)^\perp \Rightarrow g d\mu_F \in (H^p)^\perp \Rightarrow H^p$ restricted to $F$ is closed by 2.3.

REMARKS. Both 3.3 and 3.5 hold because $F$ is an intersect of peak sets. By the above it is easy to construct examples in which the $H^p$ spaces interpolate on sets of positive measure (where $\mu$ is not a representing measure). For another example, let $A$ be the disk algebra on the unit disk, and let $\mu = \frac{1}{2} d\theta + \frac{1}{2} \delta_0$ where $\delta_0$ is the point-mass at 0. As yet we have been unable to construct examples which are not of this discrete type when $\mu$ is a representing measure.

We now construct examples in which the algebra and $H^\infty$ interpolate but in which none of the $H^p$ spaces, $1 \leq p < \infty$, interpolate. Let $\{r_n\}$ be a nonnegative interpolating sequence in the open unit disk converging to 1. Then $F = \{r_n\} \cup \{1\}$ is an interpolating sequence for the disk algebra on the unit disk. Let $\mu_n$ be the Poisson measures for $r_n$ on the unit circle. Choose a sequence $\alpha_n \geq 0$ such that $\sum_{n=1}^{\infty} \alpha_n/\mu_n < 1/2 d\theta$. Consider the positive measure $\mu = \sum_{n=1}^{\infty} \alpha_n(\delta_n - \mu_n) + d\theta$ where $\delta_n$ is the point-mass at $r_n$. Then $\mu$ represents 0 for the disk algebra and we claim that $H^\infty(\mu)$ interpolates on $F$ while $H^p(\mu)$ $1 \leq p < \infty$ do not interpolate on $F$. To see this we need the following.

LEMMA 3.6. $H^p(\mu) = H^p \mid F \cup T$ where $H^p$ is the usual $H^p$-space for the disk algebra ($1 \leq p \leq \infty$) on the closed unit disk.

Proof. If $f \in H^p(d\theta)$ then $\exists f_n \in A \ni f_n \to f$ in $L^p(d\theta)$. If $\hat{f}$ de-
notes the harmonic extension of $f$ to $H^p$, then
\[ \int |\hat{f}_n - \hat{f}|^p \, d\mu \leq (1 + \sum 2\alpha_j(1 + r_j)/(1 - r_j)) \int |f_n - f| \, d\theta \longrightarrow 0. \]

So $H^p | F \cup T \subset H^p(d\mu)$. Conversely, if $f_n \in A$ and $f_n \to f$ in $L^p(\mu)$, then $f_n \to f$ in $L^p(d\theta)$, so $f | T \in H^p(d\theta)$ and therefore extends to $g = \hat{f} | T$ in $H^p$. So $g | F \cup T \in H^p(\mu)$ and $g | T = f | T$. But since the functions in $H^p(\mu)$ are determined by their values on $T$, we have $f = g \in H^p | F \cup T$, and we are done for $1 \leq p < \infty$. Now
\[
H^\infty(\mu) = [H^2 | F \cup T] \cap L^\infty(\mu)
\]
and this completes the proof.

Now observe that if $f \in H^p(d\mu)$, then
\[
|f(r_n)|^p \leq [(1 + r_n)/(1 - r_n)] \int |f|^p \, d\theta
\]
so that $\exists c \in \mathbb{R}$: the growth condition $|f(r_n)|^p \leq c(1 + r_n)/(1 - r_n)$ is satisfied. Thus if we choose a (nonnegative) sequence $\{x_n\}$ such that $x_n(1 - r_n)/(1 + r_n) \to \infty$ and such that $\sum x_n(1 + r_n)\alpha_n/(1 - r_n) < \infty$, we obtain an element of $L^p(\mu, F)$ which is not the restriction of a function from $H^p(d\mu)$. Such a sequence can be found for example by finding $\beta_n \geq 0$ to satisfy (*) and setting $\alpha_n = \beta_n^2$ and $x_n = (\beta_n)^{-1}\beta$.

Since $H^\infty$ interpolates on $F$, we see that $H^\infty(d\mu)$ interpolates on $F$ by 3.6.

Thus one may ask for conditions that will force interpolation of $H^p$-spaces to follow from interpolation of the algebra. The following is one such condition.

**Theorem 3.7** Let $A$ be a function algebra on $X$, $\mu$ a representing measure for $A$, and $A_0$ the corresponding maximal ideal. Suppose that $H^p(\mu) = H^\infty(\mu) \cap L^p(\mu)$, $\alpha \leq p$. If $A_0$ is weak-star dense in $H^\infty(\mu)$, then interpolation of $A$ on a closed set $F$ implies interpolation of $H^p(\mu)$ on $F$ for all $\alpha \leq p < \infty$ with integer conjugates $q$.

**Proof.** The conclusion deals only with $1 \leq \alpha \leq p \leq 2$. Suppose $1 < \alpha$ and $A | F = C(F)$. Then $\exists c \in \mathbb{R}$: $|| \mu_{F'} || \leq c || \mu_{F'} ||$ for every $\mu \in A^\perp$. Now choose $g \in A_0$. Then $g^\circ d\mu \in A^\perp$ so $\int_{F'} |g|^q d\mu \leq c^{1/q} \int_{F'} |f|^q d\mu$ or (*) $|| g | F' ||_q \leq c^{1/q} || g | F' ||_q$. Since $A_0$ is dense in $H^p(\mu)$ also, we have (*) holds for every $g \in H^p(\mu)$ and thus $H^p(\mu)$ interpolates on $F$.

Suppose $\alpha = 1$. For $g \perp H^1(\mu)$ we have $|| g | F' ||_q \leq c^{1/q} || g | F' ||_q$ for $q = 2, 3, \ldots$, and thus letting $q \to \infty$ we have $|| g | F' ||_\infty \leq || g | F' ||_\infty$ so that $H^1(\mu)$ also interpolates on $F$.
COROLLARY 3.8. If \( A \) is a function algebra which is weak-star-Dirichlet in \( L^\infty(\mu) \) then \( A \) interpolates only on sets of \( \mu \) measure 0.

Proof. \( A \) satisfies the hypotheses of 3.7 [7] and thus \( H^1 \) interpolates on \( F \). But \( H^1 \) is invariant under \( H^\infty \) which is BSMI so that \( F' \) has \( \mu \) measure 0 by 3.4.

It is also clear from 3.4 that when \( A \) is weak-star-Dirichlet, \( H^p \) interpolate only on sets of measure 0 for \( 1 \leq p \leq \infty \). Using the invariant subspace theorem we have the following.

THEOREM 3.9. Let \( A \) be weak-star-Dirichlet. If \( F \) is closed and \( H^p(\mu) \) restricted to \( F \) is closed for some \( 1 \leq p < \infty \), then \( \mu(F) = 0 \), or \( \mu(F') = 0 \).

Proof. Since \( H^p \) is invariant under \( H^\infty \) which is BSMI, applying 2.3 and 2.6 we have \( H^p = H^p|F \oplus H^p|F' \). Now if \( F \) has positive measure, then \( H^p|F \) is a simply invariant subspace of \( L^p \) and by the invariant subspace theorem [7, 4.16], \( H^p|F = qH^p \) where \( |q| = 1 \) a.e. But \( q \in H^p|F \) so we have \( \mu(F') = 0 \).

The example preceding 3.7 is clearly not weak-star-Dirichlet because the measure \( \mu \) is not minimal. In addition we have the following.

COROLLARY 3.10. In the example preceding 3.7, \( A_0 \) is not weak-star dense in \( H^1(\mu) \).

Proof. We only need to verify that \( H^p(\mu) \supset H^1(\mu) \cap L^p(\mu) \). But if \( f \in H^1(\mu) \cap L^p(\mu) \) then \( f|T = g|T \) where

\[
g \in H^1(d\theta) \cap L^p(d\theta) = H^p(d\theta).
\]

So as \( \hat{g} \mid F \cup T \in H^p(\mu) \), and \( \hat{g} \) and \( f \) agree on \( T \), we have

\[
f = \hat{g} \mid F \cup T \in H^p(\mu).
\]

Finally we remark that 1.3 should hold for function spaces whose duals restrict in some sense and whose norm satisfies the concavity condition. We hope to consider such examples at a later date.

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