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A GENERALIZATION OF SEPARABLE GROUPS

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This paper introduces a new class of torsion free abelian groups, the class of quasi-separable groups, which is the quasi-isomorphism analog of the class of separable groups and which properly contains the latter. Our purpose is two-fold: first, to further explore the phenomena of quasi-isomorphism, which has proved fruitful in the study of torsion free groups, and second, to shed further light on separable groups.

The term "group" herein refers to a torsion free abelian group. As is customary when dealing with quasi-isomorphism, we assume that all groups are subgroups of a fixed vector space V over the rational number field Q. L(V) denotes the algebra of linear transformations of V. L(V) is equipped with the finite topology [7] throughout and topological terms refer to this topology unless otherwise stated. G always denotes a full subgroup of V, i.e., a subgroup with torsion quotient V/G; G is full in V if and only if V is its unique minimal divisible extension. QE(G) is the quasi-endomorphism algebra of G and QF(G) is the ideal of QE(G) consisting of elements of finite rank.

Our approach is to recall that there is a one-to-one correspondence between quasi-decompositions of a group G and idempotents in QE(G)[8]. Thus a group with "many" quasi-decompositions has "many" quasi-endomorphisms of a particular type. In §1, quasi-separable groups are defined and basic properties are explored. A principal result is that every pure subgroup of finite rank in G is a quasisummand of G if and only if G is quasi-separable with linearly ordered In §2, a characterization of homogeneous quasitype set, T(G). separable groups is obtained, namely, G is homogeneous and quasiseparable if and only if QF(G) is dense in the finite topology of L(V). In §3, attention focuses on separable groups. It is shown that a countable group G is homogeneous and completely decomposable if and only if QE(G) is dense. Finally, a description of homogeneous separable groups is obtained in terms of their endomorphisms. For example, a countable group G is homogeneous and completely decomposable if and only if for any pair of independent elements a_1, a_2 in G and any arbitrary pair of elements b_1, b_2 in G, there exists an endomorphism f of G such that $fa_i = nb_i$, i = 1, 2, n some positive integer.

General abelian group theory [5] is assumed. By this date, quasi-isomorphism is a familiar concept of this theory so basic facts

are used here without comment; a complete background may be obtained from [1, 2, 8, 9]. $\stackrel{\cdot}{\subseteq}$ and $\stackrel{\cdot}{=}$ denote quasi-contained and quasi-equal, respectively. Recall that $QE(G) = \{f \in L(V): fG \subseteq G\}$. Each endomorphism of G has a unique extension to a linear transformation of V and we use the same symbol to denote both. h(a)denotes the height of the element a; if it is not clear from context in which group height is computed, a subscript is appended, e.g., $h_{\alpha}(a)$. Similarly, t(a) denotes the type of the element a; t(H) may also denote the type of a homogeneous group H. Notation is abused for the sake of conciseness; e.g., the same symbol Z is used to denote both the ring of integers and its additive group. S^* denotes the subspace spanned by the subset S of V; it is also used to denote the subalgebra generated by a subset of L(V). All sums are direct; e.g., notation such as $G \doteq A + B$ implies that A and B are disjoint subgroups of V and we call A a quasi-summand of G. Additional notation is introduced as needed.

1.0. Quasi-separable groups.

DEFINITION 1.1. Call a group G quasi-separable if and only if every finite subset of G is contained in a completely decomposable quasi-summand.

REMARK 1.2. Suppose G is quasi-separable and suppose F is a finite subset of G; by definition $G \doteq A + B$ for some groups A and B contained in V, with A completely decomposable and containing F. Clearly A may be assumed to have finite rank without any loss of generality. Now $G \doteq A \cap G + B \cap G$ and $F \subseteq A \cap G$, but $A \cap G$ need not be completely decomposable even if A has finite rank; see for example Lemma 9.3 [2]. However, if A has finite rank and T(A) is linearly ordered (especially f A is homogeneous), then $A \cap G$ is also completely decomposable by Corollary 9.6 [1]. Thus if T(G) is linearly ordered, A may be assumed to be a completely decomposable, pure subgroup [1, p. 95] of finite rank in G.

The following modular law will prove indispensable.

PROPOSITION 1.3. Suppose $H \stackrel{\cdot}{\subseteq} A + B$ and $A \stackrel{\cdot}{\subseteq} H$ for groups H, A, and B. Then $H \stackrel{\cdot}{=} A + H \cap B$.

Proof. For some positive integer $n, nA \subseteq H$ so

$$n(A + H \cap B) \subseteq H$$
.

If $mH \subseteq A + B$ for m a positive integer, then $nmH \subseteq nA + nB$;

i.e., for $c \in H$, mnc may be written mnc = na + nb with $a \in A$ and $b \in B$. Now $nb = mnc - na \in H \cap B$ so $mnH \subseteq A + H \cap B$.

REMARK 1.4. Let n be a positive integer. Consider a group having the property: (1) every pure subgroup of rank n is a quasisummand. It is easy to see that every pure subgroup of rank n is a quasi-summand of G if and only if QE(G) contains a projection onto any n-dimensional subspace of V. Consequently if G has property (1), so does any quasi-summand of G. Also, by Proposition 1.3, if Gsatisfies (1), so does any pure subgroup of G. Corresponding results hold for the property: (2) every pure subgroup is a quasi-summand.

LEMMA 1.5. If every pure subgroup of rank one is a quasisummand of G, then every pure subgroup of finite rank is a quasisummand which is quasi-equal to a completely decomposable group.

Proof. Assume the result for pure subgroups of rank $\leq n$ and let H be a pure subgroup of rank $n + 1 \geq 2$. Let $A \subset H$ be pure of rank n; by hypothesis $G \doteq A + B$ with A quasi-equal to a completely decomposable group; take B pure in G [1, p. 95]. By Proposition 1.3, $H \doteq A + H \cap B$; clearly $H \cap B$ is a pure subgroup of rank one in B. By Remark 1.4, $B \doteq H \cap B + C$ and so $G \doteq A + H \cap B + C \doteq H + C$, which completes the proof.

We shall shortly be able to strengthen the conclusion of Lemma 1.5 (see Corollary 1.7). A complete description of groups with the property that every pure subgroup of finite rank is a quasi-summand can be obtained from the following theorem, which is the quasi-isomorphism analog of Theorem 46.8 [5].

THEOREM 1.6. Every pure subgroup of G is a quasi-summand if and only if $G = D + G_1 + \cdots + G_n$ with D divisible and the G_i reduced rank-one groups satisfying $t(G_1) \leq \cdots \leq t(G_n)$.

Proof. Suppose G has the property that every pure subgroup is a quasi-summand and write G = D + H with D divisible and H reduced; by Remark 1.4, H inherits this property. To see that H has finite rank, suppose $\{a_i\}_{i=1}^{\infty}$ is an independent set in H. Let A be the pure subgroup of H generated by $\{a_i - (i+1)a_{i+1}\}_{i=1}^{\infty}; a_1 \notin A$. Now H/A contains a divisible subgroup generated by $\{a_i + A\}_{i=1}^{\infty}$, so A could not be a quasi-summand of the reduced group H [2, p. 26]. Thus H has finite rank and by Lemma 1.5, $H \doteq H_1 + \cdots + H_n$ with H_i of rank one, $i = 1, \dots, n$. It will be sufficient to show that the types of any two of the H_i are comparable, for then a suitable

relabeling of the H_i and Corollary 9.6 [1] will complete the proof. Let B and C be distinct among the H_i ; by Remark 1.4, every pure subgroup of B + C is a quasi-summand since B + C is a quasi-sum-Suppose the types of B and C are incomparable; then mand of H. B+C contains elements of three different types, t(B), t(C), and $t(B) \cap t(C)$. Pick nonzero elements b and c of B and C, respectively, and let M be the pure subgroup of B + C generated by b + c. But $B + C \doteq M + N$ is impossible because M + N cannot contain both elements of type t(B) and of type t(C), since $t(M) = t(B) \cap t(C)$ This contradiction shows that t(B) and t(C) are in fact [2, p. 26]. comparable and so completes the first half of the proof. Conversely suppose G = D + H with D divisible, $H = G_1 + \cdots + G_n$, and the G_i reduced rank-one groups satisfying $t(G_1) \leq \cdots \leq t(G_n)$. First, to see that it will be sufficient to treat the case D = 0, recall that any pure subgroup A of G decomposes into A = B + C with B divisible and C reduced and that $D \cap C = 0$ because C is pure in G. Thus the complement H of D may be chosen to contain C [5, p. 63]. Since B is a direct summand of D, it will be enough to show that C is a quasi-summand of H, so we assume D = 0. By Remark 1.4 and Lemma 1.5, it will be sufficient to show that QE(G) contains a projection onto any one-dimensional subspace of V. Let $x \in V$ be nonzero; $kx \in G$ for some positive integer k and so $kx = a_1 + \cdots + a_n$ with $a_i \in G_i$, $i = 1, \dots, n$. Let a_j be the first nonzero a_i ;

$$t_G(kx) = t_G(a_j) = t(G_j)$$
.

If S denotes the pure subgroup of G generated by kx, then G_j is isomorphic to S via some map f. Since G_j has rank one, for some non-zero integers r and s, $rf^{-1}(kx) = sa_j$. If g denotes the map from G onto $S \subseteq G$ induced by f, then $(s/r)g \in QE(G)$ projects V onto the subspace spanned by x.

COROLLARY 1.7. These properties of a group G are equivalent: (1) Every pure subgroup of rank one in G is a quasi-summand; (2) every pure subgroup of finite rank in G is a completely decomposable quasi-summand; (3) G is quasi-separable with linearly ordered type set.

Proof. Assume (1) is true and let S be a pure subgroup of finite rank in G. By Lemma 1.5, S is a quasi-summand of G and thus by Remark 1.4, every pure subgroup of S is a quasi-summand of S. Theorem 1.6 shows that S is completely decomposable with linearly ordered type set. Thus we have (1) implies (2) and (2) implies (3). Finally, suppose (3) holds and let H be a pure subgroup of rank one in G. By Remark 1.2, H is contained in a pure subgroup S of G which is a completely decomposable group of finite rank with linearly ordered type set. By Theorem 1.6, H is a quasi-summand of S and thus of G.

From the foregoing results it is perhaps clear that a quasiseparable group need not be separable; a specific example is the following. It is well known that the subgroup S of $\pi = \prod_{i=1}^{\infty} Z$ generated by 2π and $\Sigma = \sum_{i=1}^{\infty} Z \subseteq \pi$ is not separable. Since $S \doteq \pi$, both groups have the same quasi-endomorphism algebra [8]. It is also known that π is homogeneous and separable, so by Theorem 2.5, S is quasiseparable. In fact, there exist rank-two groups which are quasiseparable but not separable, i.e., not the direct sum of two rank-one groups; see for example Lemma 9.3 [2].

Just as for separable groups, the direct sum of a collection of quasi-separable groups is quasi-separable and the tensor product of two quasi-separable groups is quasi-separable.

Having proved basic results about quasi-separable groups, we turn our attention to the homogeneous case.

2.0. Homogeneous quasi-separable groups. We proceed to obtain a characterization of homogeneous quasi-separable groups in terms of quasi-endomorphisms. Intuitively, a group is homogeneous and quasi-separable precisely when it has "enough" quasi-endomorphisms; this is formulated in terms of density in the finite topology [7] of L(V).

Recall [9] that a group is irreducible if and only if it has no nontrivial, pure, fully invariant subgroups, that an irreducible group is homogeneous, and that G is an irreducible group if and only if V is an irreducible QE(G)-module. After Jacobson [7], call a subset S of L(V) k-fold transitive if and only if given any $j \leq k$ linearly independent vectors x_1, \dots, x_j in V and any j vectors y_1, \dots, y_j in V, there exists $f \in S$ such that $fx_i = y_i$, $i = 1, \dots, j$. Note well that G is irreducible, and thus homogeneous, if and only if QE(G) is onefold transitive.

REMARK 2.1. For a subring R of L(V) the following conditions are equivalent: (1) R is two-fold transitive; (2) R is k-fold transitive for every k; (3) R is dense in L(V). This follows immediately from Jacobson [7, p. 32].

LEMMA 2.2. Let H be a pure subgroup of G and let f be any quasi-endomorphism of G such that $f(H^*) \subseteq H^*$. Then the restriction of f to H^* is a quasi-endomorphism of H.

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Proof. Let n be a positive integer such that $n(fG) \subseteq G$; then

$$n(fH) \subseteq G \cap (fH)^* \subseteq G \cap (H^*) = H$$
.

PROPOSITION 2.3. (1) QE(G) is dense if and only if G is irreducible and Q is the centralizer of QE(G) in L(V).

(2) If QE(G) is dense, then G is homogeneous and every pure subgroup of finite rank in G is completely decomposable.

(3) QF(G) is an ideal of QE(G); if QE(G) is dense and $QF(G) \neq 0$, then QF(G) is also dense.

Proof. (1) follows from a remark of Jacobson [7, p.32] and the fact that G is irreducible if and only if V is an irreducible QE(G)-module. Let H be a pure subgroup of finite rank in G. In order to prove (2), it will suffice to show that $QE(H) = L(H^*)$ by Corollary 1.5 [4]. Let x_1, \dots, x_n be a basis of H^* and let $f \in L(H^*)$. By density and Remark 2.1, some $g \in QE(G)$ maps x_i to $fx_i, i = 1, \dots, n$, and so $g(H^*) \subseteq H^*$. By Lemma 2.2, g restricted to H^* is a quasi-endomorphism of H and so $QE(H) = L(H^*)$. In (3), it is clear that QF(G) is an ideal of QE(G); Theorem 4 [7, p.33] completes the proof.

LEMMA 2.4. If QF(G) is dense, then it contains a projection onto any finite dimensional subspace of V and thus every pure subgroup of finite rank in G is a completely decomposable quasi-summand.

Proof. Let x_1, \dots, x_n be independent in V. By density and Remark 2.1, some $f \in QF(G)$ leaves the x_i invariant. Extend x_1, \dots, x_n to a basis $x_1, \dots, x_n, y_1, \dots, y_m$ of fV. Again, some $g \in QF(G)$ leaves the x_i invariant and annihilates y_1, \dots, y_m . Now gf projects V onto the subspace spanned by x_1, \dots, x_n . Suppose H is a pure subgroup of finite rank in G; by (2) of Proposition 2.3, H is completely decomposable. We have just proved that QF(G) contains a projection e of V onto H^* . Now $G \doteq eV \cap G + (1-e)V \cap G$ and $eV \cap G = H$ because H is pure.

We are now prepared to prove

THEOREM 2.5. These are equivalent:

(1) G is homogeneous and quasi-separable.

(2) QF(G) is dense in the finite topology of L(V).

(3) QF(G) is one-fold transitive and every pure subgroup of finite rank in G is completely decomposable.

Proof. (1) implies (2). By Remark 2.1, it will be sufficient to show that QF(G) is two-fold transitive. Let x_1 and x_2 be independent

in V and let y_1 and y_2 be arbitrary elements of V. Since G is a full subgroup of V, there is some positive integer n such that nx_1 , nx_2 , ny_1 , and ny_2 are all in G; suppose these elements are contained in a completely decomposable quasi-summand, H, of finite rank. H is homogeneous because G is, so by Corollary 1.5 [4], $QE(H) = L(H^*)$. If e is an idempotent associated with H, QE(H) = eQE(G)e [8], so if $f \in L(H^*)$ sends x_i to y_i , i = 1, 2, f is induced by ege for some $g \in QE(G)$. Now $ege \in QF(G)$, so QF(G) is two-fold transitive and thus dense.

That (2) implies (3) follows from Remark 2.1 and Proposition 2.3.

(3) implies (1). G is certainly homogeneous because QE(G) is one-fold transitive. By Corollary 1.7 and Remark 1.4, it will suffice to prove that QE(G) contains a projection onto any one-dimensional subspace of V, so let x be any nonzero element in V. By hypothesis some $f \in QF(G)$ leaves x invariant; $A = fV \cap G$ is pure of finite rank in G and so is completely decomposable. $B = \{x\}^* \cap G$ is a direct summand of A [5, p. 178]. If g projects A onto B, then $gf \in QE(G)$ projects V onto $\{x\}^*$.

Under the hypothesis of Theorem 2.5, QE(G) is primitive with socle QF(G) by the Structure Theorem [7, p. 75].

3.0. Applications to separable groups. Here we prove that countable groups G with QE(G) dense in L(V) are homogeneous and completely decomposable; this is accomplished with the aid of a generalization of Pontryagin's criterion for countable free groups. k-fold transitivity of quasi-endomorphisms is interpreted in terms of endomorphisms to provide further insight into homogeneous quasi-separable groups. This suggests properties of endomorphisms both necessary and sufficient for a group to be homogeneous and separable.

LEMMA 3.1. A countable homogeneous group is completely decomposable if and only if each pure subgroup of finite rank is completely decomposable.

Proof. The necessity obtains by Theorem 46.6 [5]. For the sufficiency, let $\{a_i\}_{i=1}^{\infty}$ be an enumeration of a countable homogeneous group G, each of whose pure subgroups of finite rank is completely decomposable. Let H_n denote the pure subgroup generated by a_1, \dots, a_n and set $G_1 = H_1$. Then in general, $H_{n+1} = H_n + G_{n+1}$ [5, p.178] with G_{n+1} either 0 or of rank one. Now $G = \sum_{n=1}^{\infty} G_n$.

THEOREM 3.2. A countable group G is homogeneous and completely decomposable if and only if QE(G) is dense. *Proof.* The necessity follows from Theorem 2.5 and the sufficiency from Proposition 2.3 (2) and Lemma 3.1.

COROLLARY 3.3. A countable, homogeneous, quasi-separable group is completely decomposable.

COROLLARY 3.4. If QE(G) is dense then G is \aleph_1 -completely decomposable in the sense that every countable pure subgroup is completely decomposable.

The discussion now turns to an interpretation in terms of endomorphisms of some properties of quasi-endomorphisms encountered in §2. E(G) denotes the endomorphism ring of G and F(G) denotes those endomorphisms of G which have finite rank.

DEFINITION 3.5. A subset S of E(G) is called k-fold transitive if and only if given $j \leq k$ independent elements a_1, \dots, a_j of G and any j elements b_1, \dots, b_j of G, there exists an endomorphism $f \in S$ and a positive integer n such that $fa_i = nb_i$, $i = 1, \dots, j$.

PROPOSITION 3.6. The pure subring R of E(G) is k-fold transitive if and only if $R^*(\subseteq L(V))$ is k-fold transitive.

Proof. A straightforward computation using the fact that E(G) is full in QE(G) and using the one-to-one correspondence between pure subrings of E(G) and subalgebras of QE(G) [5, p.271].

REMARK 3.7. The above implies the following; (5) is of particular interest.

(1) G is irreducible if and only if E(G) is one-fold transitive.

(2) F(G) is a pure ideal of E(G) and $F(G)^* = QF(G)$.

(3) The pure subring R of E(G) is two-fold transitive if and only if R^* is dense.

(4) G is homogeneous and quasi-separable if and only if F(G) is two-fold transitive.

(5) A countable group G is homogeneous and completely decomposable if and only if E(G) is two-fold transitive.

A property somewhat stronger than two-fold transitivity may be required of F(G) to conclude that G is homogeneous and separable.

DEFINITION 3.8. Call a subset S of E(G) fully k-fold transitive if and only if S is k-fold transitive and in addition for any nonzero elements a and b of G such that $h(a) \leq h(b)$, some $f \in S$ maps a to b. LEMMA 3.9. Let a and b be nonzero elements of rank-one groups A and B, respectively. Then some $f \in \text{Hom}_z(A, B)$ maps a to b if and only if $h_A(a) \leq h_B(b)$.

Proof. Only the sufficiency need be checked and this can be done computationally by using the characterization of subgroups of Q found in [3].

THEOREM 3.10. The following statements about the group G are equivalent.

(1) G is homogeneous and separable.

(2) F(G) is fully two-fold transitive.

(3) F(G) is fully one-fold transitive and every pure subgroup of finite rank in G is completely decomposable.

(4) G is homogeneous, every pure subgroup of finite rank is completely decomposable, and F(G) is dense in the finite topology of E(G).

Proof. We prove that (1) and (2) are equivalent, then (2) and (3), and finally (1) and (4).

(1) implies (2). By Remark 3.7 (4), F(G) is two-fold transitive. Let a and b be any two nonzero elements of G satisfying $h(a) \leq h(b)$ and let A and B denote the pure subgroups of G generated by a and b respectively. By Lemma 3.9, some $f \in \text{Hom}_Z(A, B)$ maps a to b. By [5, p.178], there is a projection g of G onto A. Now $fg \in F(G)$ sends a to b, so F(G) is fully two-fold transitive.

(2) implies (1). By Remark 3.7 (4) and Corollary 1.7, G is homogeneous and every pure subgroup A of rank one is a quasi-summand; it will be sufficient to show that A is in fact a direct summand [5, p.178]. Write $G \doteq A + C$ with C pure in G. By [1, p.96],

$$G = B + C$$

with B isomorphic to A via some map f; let g be the projection of G onto B. Pick a nonzero element $a \in A$; $h(a) = h(f^{-1}a)$ and height is unambiguous since all relevant groups are pure subgroups of G. By hypothesis, some $r \in F(G)$ maps a to $f^{-1}a$. Let s = fgr; sa = a. $\{c \in G: sc = c\}$ is a nontrivial pure subgroup of G contained in A and so equals A, i.e., s is an idempotent.

(2) and (3) are equivalent by Remark 3.7 (4) and Theorem 2.5.

(1) implies (4). Since (1) implies (3), it will be enough to prove that F(G) is dense in the finite topology of E(G). Let f be any endomorphism of G and let a_1, \dots, a_n be arbitrary elements of G. Now $a_i, fa_i, i = 1, \dots, n$, are all contained in some direct summand of finite rank. If g is a projection associated with this summand, $gf \in F(G)$ is in the open neighborhood of f,

$$\{h \in E(G): ha_i = fa_i, i = 1, \dots, n\}$$
.

(4) implies (1). It will be sufficient to see that any pure subgroup A of finite rank in G is a direct summand [5, p.178]. By density, some $f \in F(G)$ leaves A invariant because the identity map does. Let B denote the pure subgroup of G generated by fG; by hypothesis B is completely decomposable and A is a direct summand of B by [5, p.178]. If g projects B onto A then gf projects G onto A.

REMARK 3.11. Full two-fold transitivity cannot be strengthened in the following sense. Given a_1, a_2 independent in G and b_1, b_2 arbitrary in G such that $h(a_i) \leq h(b_i)$, in general there is no endomorphism mapping a_i to b_i , i = 1, 2. Furthermore in Theorem 3.10 (3), the complete decomposability of pure subgroups of finite rank is essential, as the following discussion indicates. Let K be any subfield of the p-adic number field F_p and let $R = K \cap J_p$, J_p the subring of p-adic integers; R is a pure subring of J_p and so is indecomposable [5, p.150]. By standard arguments [5, p.212], E(R) = R, i.e., every endomorphism of the additive group of R is induced by ring multiplication. Now it is easy to see that E(R) is fully one-fold transitive, for if a and b are nonzero elements of R, $a = p^m u$, $b = p^n v$ with u and v units in J_p [6, p.225] and hence in R by purity; also

$$u^{-1} \in R = K \cap J_p$$
.

Now $h(a) \leq h(b)$ if and only if $m \leq n$ [6, p.225], so if $m \leq n$,

$$p^{n-m}vu^{-1} \in R = E(R)$$

maps a to b; otherwise vu^{-1} maps a to $p^{m-n}b$. Thus E(R) is fully one-fold transitive. In particular for K an algebraic number field [6, p.229], E(R) = F(R), F(R) is fully one-fold transitive but not (fully) two-fold transitive, and R is homogeneous but not (quasi-) separable.

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