ON THE BRAUER GROUP OF Z

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Two dimensional Amitsur cohomology is computed for certain rings of quadratic algebraic integers. Together with computations of Picard groups, this yields information on the Brauer group \( B(S/Z) \), for \( S \) quadratic algebraic integers, without resort to class field theory.

The classical Brauer group of central simple algebras over a field [10, X, Sec. 5] has been generalized to the Brauer group \( B(R) \) of central separable \( R \)-algebras over a commutative ring \( R \) [2]. One can prove, using class field theory, that the Brauer group \( B(Z) \) of the integers, is trivial. The proof is apparently well known but not in the literature, although it does appear in the dissertation of Fossum [9].

This paper is devoted to our attempt to establish this result using only an exact sequence of Chase and Rosenberg [7, p. 76]. We are able to show that if \( S \) is the integers of \( Q(\sqrt{m}) \) for \( m = \pm 3, -1, 2, \) or \( 5 \), the subgroup \( B(S/Z) \) of \( B(Z) \) consisting of elements split by \( S \), vanishes.

In § 2 we develop some technical results on norms which we use in § 3 to show that the Amitsur cohomology group \( H^2(S/Z, U) \) is zero whenever \( S \) is the ring of integers of a quadratic extension of the rationals. In § 4 we use a Mayer-Vietoris sequence of algebraic \( K \)-theory to show that the Picard group \( \text{Pic}(S \otimes \mathbb{Z}S) = 0 \) for \( S \) the integers of \( Q(\sqrt{m}) \), \( m = \pm 3, -1, 2, \) or \( 5 \). In § 5 we use this result and an exact sequence of Chase and Rosenberg [7, p. 76] to show \( B(S/Z) = 0 \) for these rings.

Dobbs [8] has results relating \( B(S/Z) \) to \( H^2(S/Z, U) \) which together with the triviality of \( B(Z) \) imply our results.

§ 2. Norms. If \( S \) is a commutative algebra over a commutative ring \( R \), \( S^n \) denotes \( S \otimes S \cdots \otimes S \), \( n \) times (here and throughout, \( \otimes \) means \( \otimes_{\mathbb{R}} \)), and \( \varepsilon_i: S^n \rightarrow S^{n+1}, i = 0, \cdots, n, \) is given by \( x_0 \otimes \cdots \otimes x_{n+1} \rightarrow x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_i \otimes \cdots \otimes x_{n+1} \). These maps satisfy \( \varepsilon_i \varepsilon_j = \varepsilon_{j+i} \varepsilon_i \) for \( i \leq j \). For any ring \( A \), \( U(A) \) denotes the group of units of \( A \). All unexplained notation and terminology is an in [7].

**Theorem 2.0.** Let \( M/K \) be a galois extension of commutative rings [6], with group \( G \), and let \( F \) be an additive functor on a full subcategory \( \mathcal{C} \) of the category of commutative \( K \)-algebras, and suppose \( M \) and \( M \otimes_K M \) lie in \( \mathcal{C} \). Then for any \( x \) in \( F(M) \), \( y = \sum_{g \in G} Fg(x) \)

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lies in $\text{Ker}(F\varepsilon_0 - F\varepsilon_1)$.

Proof. By Theorem 3.1 of [6] there are orthogonal idempotents $e_g$ (g in $G$), in $M \otimes_K M$ with $\sum e_g = 1$ and $s \otimes 1 = \sum_s (1 \otimes g(s))e_g$. In the above notation this becomes: $\varepsilon_i(s) = \sum_s \varepsilon_i(g(s))e_g$ for all $s$ in $M$.

Now $\sum e_g = 1$ implies that $M \otimes_K M = I(M \otimes_K M)e_g$ as $K$-algebras. Thus, if $\pi_g$ denotes the projection of the $g$th component, we have $\pi_g \varepsilon_i = \pi_g \varepsilon_i g$ as maps $M \to (M \otimes_K M)e_g$. Now $y = \sum_k \varepsilon_i Fh(x)$ is trivially invariant under $Fg$ so we obtain $F\pi_g F\varepsilon_i(y) = F\pi_g F\varepsilon_i g(y) = F\pi_g F\varepsilon_i(y)$ for each $g$ in $G$. By the additivity of $F$, this implies $F\varepsilon_i(y) = F\varepsilon_i(y)$ as was to be shown.

Now let $R$ be the ring of integers of an algebraic number field $K$. Let $M$ be a finite galois field extension of $K$ with group $G$ and $S$ its ring of integers and let $M: K = n$. For each $i \geq 0$ there is a map $n_i: U(S^{i+1}) \to U(S^i \otimes R)$ given by $n_i(\sum x_0 \otimes \cdots \otimes x_i) = I(\varepsilon_i \in \varepsilon_{i+1} \sum x_0 \otimes \cdots \otimes x_{i+1} \otimes g(x_i))$.

Now $S^{i+1}$ is projective, hence faithfully flat as an $S^i \otimes R$ module. By [7, Lemma 3.8] $S^i \otimes R = \text{Ker}(S^{i+1} \otimes e^{e_{i+1}-e_i} S^{i+1} \otimes_S e_{i+1} S^{i+1})$, so applying Thm. 2.0 to $M^{i+1}/(M^i \otimes_K M)$ (here $M^j = M \otimes_K M \cdots \otimes_K M$) noting that the natural map $S^i \to M^i$ is injective for all $n$ we see that the map $n_i$ indeed has its image in $S^i \otimes R$.

DEFINITION. The $i$th norm map, $N^i: U(S^{i+1}) \to U(S^i)$ is $Cn_i$ where $C: S^i \otimes R \to S^i$ is the natural isomorphism. $N^i$ is easily seen to be an abelian group map.

LEMMA 2.1. If $\varepsilon_j: U(S^{i+1}) \to U(S^{i+1})$ denote the maps defined at the beginning of the section, then $N^{i+1} \varepsilon_j(x) = \varepsilon_j N^i(x)$ for $0 \leq j < i + 1$ and $N^{i+1} \varepsilon_{i+1}(x) = x^n$, where $n = M: K$.

Proof. Clear

PROPOSITION 2.2. If $d^i: U(S^{i+1}) \to U(S^{i+2})$ is the Amitsur coboundary (given by $d^i(x) = \prod_{j=0}^{i+1} \varepsilon_j(x^{-i+1})$), then $N^{i+1} d^i(x) = [d^{i-1} N^i(x)](x^n)^{(-i+1)^{i+1}}$.

Proof.

$$N^{i+1} d^i(x) = N^{i+1} \prod_{j=0}^{i+1} \varepsilon_j(x^{-i+1}) = \prod_{j=0}^{i+1} N^{i+1} \varepsilon_j(x^{-i+1}) = [\prod_{j=0}^{i} \varepsilon_j N^i(x^{-i+1})](x^n)^{(-i)^{i+1}}$$

by Lemma 2.1. The proposition then follows from the definition of $d^{i-1}$.
**Corollary 2.3.** If for \( x \) in \( U(S^{i+1}) \) we have \( d^i(x) = 1 \), then \( (x^n)^{(-1)^i} = d^{i-1}(N^i(x)) \). In particular, \( nH^i(S/R, U) = 0 \) for \( i \geq 1 \).

**Remark.** The above are all closely parallel to results of Amitsur [1, Thm. 2.10] who defines a norm map via determinants whenever \( S/R \) is finitely generated and free. In the that case, our norm maps agree with Amitsur’s [1, Lemma 5.2].

We are primarily interested in two-cocycles:

**Corollary 2.4.** Let \( x \) in \( U(S^3) \) have \( d^3(x) = 1 \). Then \( N^3N^2(x) \) is in \( U(R) \).

**Proof.** By Corollary 2.3 with \( i = 2, x^3 = d^2(N^2(x)) \) and so \( N^2(x^3) = N^2d^2(N^2(x)) = [d^2N^2N^2(x)]N^2(x^3) \) by Proposition 2.2 with \( i = 1 \). Hence \( d^3[N^2N^2(x)] = 1 \) in \( S \otimes S \). Since \( S \) is projective, hence faithfully flat, over \( R \), it follows from Lemma 3.8 of [7] that \( N^3N^2(x) \) is in \( R \cdot 1_s \); say \( N^3N^2(x) = r \cdot 1_s \). A priori \( r \) is a unit in \( S \), but not obviously so in \( R \). Let \( t \) be the inverse in \( S \) of \( r \cdot 1_s \) and let \( t \) satisfy the integral equation \( x^m + r_1x^{m-1} + \cdots + r_m \) in \( R[x] \). So

\[
0 = (t^m + r_1t^{m-1} + \cdots + r_m)r^m \cdot 1 = 1 + r_1r + \cdots + r_mr^m.
\]

Hence \( r \) is a unit in \( R \), completing the proof.

Henceforth we will suppress the superscripts on norm maps.

Finally we give a technical lemma of general application:

**Lemma 2.5.** If \( R \) is any commutative ring and \( S \) a faithfully flat \( R \)-algebra, then \( u \) two cocycle \( x \) in \( U(S^3) \) lies in \( S \otimes S \otimes 1 \) if and only if \( x \) is in \( 1 \otimes S \otimes 1 \). In this case \( x \) is a coboundary.

**Proof.** One implication is trivial.

If \( x \) is in \( S \otimes S \otimes 1 \) we may write \( x = \varepsilon_1(a) = a \otimes 1 \) for some \( a \) in \( S \otimes S \). Then \( 1 = d^3(x) = \varepsilon_1(x)\varepsilon_1(x^{-1})\varepsilon_1(x)\varepsilon_1(x^{-1}) \). Since \( x = \varepsilon_1(a) \), it is clear that \( \varepsilon_1(x) = \varepsilon_3(x) \), so that

\[
1 = \varepsilon_3(x)\varepsilon_1(x^{-1}) = \varepsilon_0\varepsilon_2(a)\varepsilon_1(a^{-1}) = \varepsilon_2\varepsilon_0(a)\varepsilon_1(a^{-1}).
\]

Since \( \varepsilon_3 \) is a monomorphism, we have \( \varepsilon_0(a) = \varepsilon_1(a) \). As in the previous result, an application of Lemma 3.8 of [7] shows that \( a \) is in \( 1 \otimes S \) so that \( x = a \otimes 1 \) in \( 1 \otimes S \otimes 1 \). We must have \( a = 1 \otimes u \) for some unit \( u \) of \( S \) and so \( x = 1 \otimes u \otimes 1 = d^i(1 \otimes u) = d^i(a) \).
3. The cohomology of quadratic integers. In this section we use the results of the last section for explicit computations of cohomology groups. In this section $R = \mathbb{Z}$ and $S$ is the ring of integers of a quadratic field extension, $K$, of the rationals, $\mathbb{Q}$. Thus $K = \mathbb{Q}(\sqrt{m})$ for a square free integer $m$. The computations naturally divide themselves into the cases $m \equiv 2$ or $3$ and $m \equiv 1 \pmod{4}$.

**Theorem 3.0.** Let $K = \mathbb{Q}(\sqrt{m})$ with $m \equiv 2$ or $3 \pmod{4}$. If $S$ denotes the ring of integers of $K$, then $H^1(S/\mathbb{Z}, U) = 0$.

**Proof.** Let $\rho = \sqrt{m}$. Then $\{1, \rho\}$ constitutes a basis of $S$ over $\mathbb{Z}$ [12, Thm. 6-1-1]. For any $x$ and $y$ in $\mathbb{Z}$, the nontrivial $\mathbb{Q}$-automorphism takes $x + y\rho$ to $x - y\rho$, so that $N(x + y\rho) = (x + y\sqrt{m})(x - y\sqrt{m}) = x^2 - my^2$.

Now $S'$ is free over $S^{i-1}$ (acting on the first $i - 1$ factors) with generators $1, \xi$ and $1, \xi$, so that $N(x \otimes 1 + y \otimes \rho) = (x \otimes 1 + y \otimes \sqrt{m})(x \otimes 1 - y \otimes \sqrt{m}) = x^2 - my^2$ for $x$ and $y$ in $S^{i-1}$. For convenience, we call $x \otimes 1 - y \otimes \rho$ the conjugate of $x \otimes 1 + y \otimes \rho$ in $S'$.

Suppose $x$ in $U(S')$ is a two cocycle and let $y = N(x) = a \otimes 1 + b \otimes \rho$ with $a$ and $b$ in $S$. By Corollary 2.4, $a^2 - mb^2 = N(y) = \pm 1$ in $S$. We treat the two cases separately, letting $a = a_1 + a_2\rho$ and $b = b_1 + b_2\rho$ with $a_i, b_i$ in $\mathbb{Z}$.

**Case 1.** $N(y) = 1$. Here one easily sees that $y^{-1} = a \otimes 1 - b \otimes \rho$ the conjugate of $y$. Let $M$ denote the ring homomorphism $S \otimes S \to S$ defined by $M(c \otimes d) = cd$ for $c$ and $d$ in $S$. Then the unit of $S$, $M(y) = a + b\rho$ has inverse $M(y^{-1}) = a - b\rho$. Explicitly

(1) \[ M(y) = a + b\rho = a_1 + mb_2 + (a_2 + b_2)\rho \]

and

(2) \[ M(y^{-1}) = a - b\rho = a_1 - mb_2 + (a_2 - b_2)\rho. \]

Now $NM(y)$ is in $U(\mathbb{Z})$, so is $\pm 1$. If $NM(y) = 1$ we see that $M(y^{-1}) = M(y)^{-1}$ is the conjugate of $M(y)$, that is $M(y)^{-1} = (a_1 + mb_2) - (a_2 + b_2)\rho$. Using equation (2) we then have $b_1 = b_2 = 0$. Thus $y = N(x) = a_1 \otimes 1 + b_1 \otimes \rho = \varepsilon_0(c)$ where $c = a_1 + b_1\rho$ is in $U(S)$ since $y^{-1} = a \otimes 1 - b \otimes \rho = \varepsilon_0(a_1 - b_1\rho)$.

Now by Corollary 2.3 $x^2 = d^0(N(x)) = d^i(\varepsilon_0(c)) = \varepsilon_0(\varepsilon_0(c)) = \varepsilon_0(\varepsilon_0(c)\varepsilon_0(c^-)\varepsilon_0(c)\varepsilon_0(c)) = \varepsilon_0(c)\varepsilon_0(c^-)\varepsilon_0(c)\varepsilon_0(c) = \varepsilon_0(c)\otimes 1 = N(x) \otimes 1$. On the other hand, if we write $x = \alpha \otimes 1 + \beta \otimes \rho$ with $\alpha$ and $\beta$ in $S^2$, then $x^2 = (\alpha^2 + m\beta^2) \otimes 1 + 2\alpha\beta \otimes \rho$ and equating coefficients gives $2\alpha\beta = 0$ and $\alpha^2 + m\beta^2 = N(x) = \alpha^2 - m\beta^2$ (by the definition of $N$). Hence $m\beta^2 = 0$. But
since the natural map of $S^2$ into $K^2$ is injective, $S^2$ is torsion free with no nilpotents, so $\beta = 0$. Thus $x = \alpha \otimes 1$ and so is a coboundary by Lemma 2.5.

In Case 1 there remains the possibility that $NM(y) = -1$. With the notation of the previous subcase we see that $M(y)^{-1} = -(a_1 + mb_2) + (a_2 + b_1)\rho$, the negative of the conjugate of $M(y)$. Equation (2) here leads to $a_1 = b_1 = 0$ so that $y = N(x) = a_2^0 \otimes 1 + b_2^0 \otimes \rho$. Hence $NN(x) = a_2^0\rho^2 + mb_2^0\rho^2 = a_2^0m + m^2b_2^0 = m(a_2^0 + mb_2^0)$. By Corollary 2.4, this must be $\pm 1$. Since $a_2$, $b_2$ and $m$ are integers, this happens only if $m = \pm 1$. If $m = 1$, $K$ is not a proper extension (and in any case $m$ is not congruent to 2 or 3 (mod 4)). We are thus, in Case 1, reduced to considering the Gaussian integers and must consider solutions of $b_2^0 - a_2^0 = \pm 1$. Thus in this subcase, $\rho = i$. Returning to equation (1), we have $M(y) = -b_2 + a_2i$ and we have assumed $-1 = NM(y) = b_2 - a_2 = (b_2 + a_2)(b_2 - a_2)$ in $Z$. The only solutions of this are $b_2 = 0$ and $a_2 = \pm 1$. Thus by Corollary 2.3, $x^* = d^*(N(x)) = d^*(a_2i \otimes 1) = d^*(\varepsilon_2(a_2i))\varepsilon_2(a_2i) = \varepsilon_2(a_2i)$ (since $\varepsilon_2(\varepsilon_1) = \varepsilon_2\varepsilon_1$) and so $x^* = \pm 1 \otimes i \otimes 1$. But $\pm 1 \otimes i \otimes 1$ is not a square in $S^2$, else after applying the ring homomorphism $a \otimes b \otimes c \rightarrow abc$ of $S^2$ to $S$, we would have that $\pm i$, and hence $i$, is a square in $S$.

**Case 2.** $N(y) = -1$. Here $y^{-1} = -a \otimes 1 + b \otimes \rho$ and we obtain

(3) \[ M(y) = a + b\rho = a_1 + mb_2 + (a_2 + b_1)\rho \]

and

(4) \[ M(y^{-1}) = -a + b\rho = -a_1 + mb_2 + (b_1 - a_2)\rho. \]

Again $NM(y) = \pm 1$ in $Z$. As in Case 1, $NM(y) = 1$ implies $M(y^{-1}) = M(y)^{-1}$ is the conjugate of $M(y)$, that is, $M(y^{-1}) = (a_1 + mb_2) - (a_2 + b_1)\rho$. Comparing coefficients with (4) gives $a_1 = b_1 = 0$. By computations similar to the second subcase of Case 1, we are reduced to considering only $m = -1$, (the Gaussian integers) and $a_2^0 + b_2^0 = 1$ in $Z$. This equation has the solutions $a_2 = 0$ and $b_2 = \pm 1; a_2 = \pm 1, b_2 = 0$. $b_2 = 0$ and $a_2 = \pm 1$ yields, parallel to Case 1, $x^* = d^*(N(x)) = d^*(y) = d^*(a_2i \otimes 1) = -a_2(i \otimes i \otimes i)$ which again cannot be a square in $S^2$.

In the subcase $NM(y) = 1$ there remains the possibility $a_2 = 0, b_2^0 = 1$. Then again by Corollary 2.3, $x^* = d^*(N(x)) = d^*(y) = d^*(b_2i \otimes i) = b_2(i \otimes i \otimes i) = b_2(i \otimes 1 \otimes i) = -b_2(1 \otimes 1 \otimes 1) = \pm 1 \otimes 1 \otimes 1$. That $x$ is a coboundary then follows from Lemma 3.1 below, completing the subcase $NM(y) = 1$.

The subcase $NM(y) = -1$, by similar computations leads to $b_2 = a_2 = 0$. As in the first subcase of Case 1, an application of Corollary 2.3 and Lemma 2.5 shows that $x$ is a coboundary, completing Case 2.
LEMMA 3.1. Let $S$ be the Gaussian integers and $x$ in $U(S^3)$ a two cocyle. If $x^2 = \pm 1$ in $S^3$ then $x$ is a coboundary.

Proof. Consider first $x^2 = 1 \otimes 1 \otimes 1$. The following are eight solutions in $S^3$: $\pm 1 \otimes 1 \otimes 1$, $\pm i \otimes i \otimes i$, $\pm i \otimes 1 \otimes i$, and $\pm i \otimes i \otimes 1$. We claim this exhausts the solutions of the equation in $S^3$. To see this note that if $K = \mathbb{Q}(i)$, then distinct solutions in $S^3$ are also distinct in $K \otimes \mathbb{Q} K \otimes \mathbb{Q} K$, since the natural map $S \otimes S \otimes S \rightarrow K \otimes \mathbb{Q} K \otimes \mathbb{Q} K$ is monic. Since $K/\mathbb{Q}$ is galois, $K \otimes \mathbb{Q} K \otimes \mathbb{Q} K$ is isomorphic to a direct product of copies of $K$. Comparing $\mathbb{Q}$ dimensions yields $K \otimes \mathbb{Q} K \otimes \mathbb{Q} K \cong K \times K \times K \times K$. Since the only solutions in $K$ of $x^2 = 1$ are $\pm 1$, it follows that there are exactly 16 solutions in $K \otimes \mathbb{Q} K \otimes \mathbb{Q} K$. Let $x_i$ denote the eight above mentioned distinct solutions which lie in $S^3$ and let $y = (1/2)(1 \otimes 1 \otimes 1 - i \otimes i \otimes 1 - i \otimes 1 \otimes i + 1 \otimes i \otimes i)$. Then it can be seen that $y^2 = 1$ and $(x_i, x_i y)$ are solutions of $x^2 = 1$ in $K \otimes \mathbb{Q} K \otimes \mathbb{Q} K$. We claim these are distinct and that the $x_i y$ do not lie in $S$. For both claims it suffices, since the $x_i$ are in $U(S^3)$, to show that $y$ cannot lie in $S^3$. This follows easily from the fact that $1 \otimes 1 \otimes 1$, $i \otimes i \otimes 1$, $i \otimes 1 \otimes i$ and $1 \otimes i \otimes i$ are linearly independent over $\mathbb{Z}$ and that $1/2$ does not lie in $\mathbb{Z}$.

Thus the $x_i$ exhaust the solutions in $S^3$ of $x^2 = 1$. Now among these solutions a simple computation shows that the only cocycles are $1 \otimes 1 \otimes 1$ and $-1 \otimes 1 \otimes 1$ and these are, respectively $d^i(1 \otimes 1)$ and $d^i(-1 \otimes 1)$. Similarly among the solutions of $x^2 = -1 \otimes 1 \otimes 1$ only $\pm i \otimes 1 \otimes 1$, $\pm i \otimes i \otimes i$, $\pm 1 \otimes i \otimes 1$ and $1 \otimes 1 \otimes i$ lie in $S^3$ (the remaining eight comprise the multiples of these by the element $y$ given above and again cannot lie in $S$). The only cocycles are $\pm 1 \otimes i \otimes 1$ and these are coboundaries of $1 \otimes i$ and $i \otimes 1$ respectively. Thus the lemma, and so Theorem 3.0, is proved.

THEOREM 3.2. Let $K = \mathbb{Q}(\sqrt{4k+1})$. If $S$ denotes the integers of $K$, then $H^3(S/Z, U) = 0$.

Proof. Let $\rho = (1 + \sqrt{4k+1})/2$. Then $\{1, \rho\}$ is a basis of $S$ over $\mathbb{Z}[12$, Thm. 6-1-1]. The nontrivial $\mathbb{Q}$-automorphism of $K$, since it must preserve the roots of $x^2 = 4k + 1$, takes $\sqrt{4k+1}$ to $-\sqrt{4k+1}$ and so takes $a + b\rho$ to $a + b((1 - \sqrt{4k+1})/2) = a + b(1 - \rho)$. Hence, $N(x) = a^2 - b^2\rho^2 + ab + b\rho$. Since $\rho^2 = \rho + k$, we have $N(x) = a^2 - b^2k + ab$.

As in the previous theorem, the structure of $S^i$ as $S^{i-1}$-algebra is analogous to the ring structure on $S$. That is $1, s_i, 1_s, i$ and $\rho$ are a basis and $N(a \otimes 1 + b \otimes \rho) = a^2 + ab - b^2k$ for $a, b$ in $S^{i-1}$.

Computations closely paralleling those of the previous theorem show
that if \( x \) is a two cocycle in \( U(S^3) \) then \( x \) is in \( S^2 \otimes 1 \) and so by Lemma 2.5 is a coboundary. As before the computation divides itself into two cases, \( NN(x) = 1 \) or \( NN(x) = -1 \). Various subcases lead either to the desired result or to an equation in integers of the form \( 2 = a^2 - 4k \). Since a square integer is never congruent modulo four to two, the theorem is proved.

4. Pic \((S \otimes S)\). Let \( R \) be an integral domain whose quotient field, \( K \), has characteristic not 2. Let \( S \) be an integral quadratic extension of \( R \), that is, \( S = R[\rho] \) where the minimal polynomial of \( \rho \) over \( R \) is \( p(x) = x^2 + ax + b \). Let \( \bar{\rho} \) be the second (and distinct) root of \( p(x) \). Note that \( S \) is an integral domain with quotient field \( K(\rho) \), and that \( \bar{\rho} \) is in \( S \) as a consequence of the familiar formula \( \rho + \bar{\rho} = -a \). The main theorem of this section characterizes the Picard group \([5, \text{Ch. II, Sec. 4}] \) \( \text{Pic}(S \otimes_R S) \) of rank one projective \( S \otimes_R S \) modules in terms of the units of \( S \) and of \( S/(\rho - \bar{\rho})/S \). Henceforth \( \otimes \) means \( \otimes_R \) and \( S' \) denotes \( S/(\rho - \bar{\rho})S \).

**Lemma 4.0.** \( S \otimes S \cong S \times S' \). That is, in the notation of \([3, \text{IX Sec. 5, p. 478}] \), there are maps \( h_1, h_2 \) making

\[
\begin{array}{ccc}
S \otimes S & \xrightarrow{h_1} & S \\
\downarrow h_2 & & \downarrow \\
S & \longrightarrow & S'
\end{array}
\]

a cartesian square (here the unlabelled maps are the natural projections).

**Proof.** By assumption, \( S \) is free over \( R \) on 1 and \( \rho \), so \( S \otimes S \) is free on 1 and \( 1 \otimes \rho \) when regarded as an \( S \) module on the first factor. For \( s \) and \( t \) in \( S \), define \( h_1(s \otimes 1 + t \otimes \rho) = s + t\rho \) and \( h_2(s \otimes 1 + t \otimes \rho) = s + t\bar{\rho} \). Then \( h_1(a) - h_2(a) = t(\rho - \bar{\rho}) \) for any \( a = s \otimes 1 + t \otimes \rho \) in \( S \otimes S \). Conversely, suppose \( s_1 \equiv s_2 \) (mod \( (\rho - \bar{\rho})S \)), i.e., \( s_1 - s_2 = s_3(\rho - \bar{\rho}) \) for some \( s_3 \) in \( S \). Then taking \( y = s_3 \) and \( x = s_1 - s_2\rho \) gives \( s_1 = x + y\rho = h_1(x \otimes 1 + y \otimes \rho) \) and \( s_2 = x + y\bar{\rho} = h_2(x \otimes 1 + y \otimes \rho) \). Thus \( \{(s_1, s_2) \in S \times S \mid s_1 \equiv s_2 \) (mod \( (\rho - \bar{\rho})S \))\} = \( \{(h_1(a), h_2(a)) \mid a \text{ in } S \otimes S \} \). Since \( S \) is an integral domain, it follows that \( a \rightarrow (h_1(a), h_2(a)) \) is a monomorphism of \( S \otimes S \) into \( S \times S \) so the square (1) satisfies the definition of cartesian.

**Remark.** Let \( R \) be the ring of integers of an algebraic number field, \( K \), with class number 1, and \( S \) the integers of a quadratic extension, \( L \) of \( K \). \( S \) is finitely generated projective over \( R \) (cf. 12, p. 158).
If \( \{x_i, \phi_i\} \) is a projective coordinate system, the map \( f: S \rightarrow R \) given by \( f(x) = \sum \phi_i(x, x_i) \) is a split \( R \)-module epimorphism, so that \( S = R \oplus \ker f \). Since \( R \) is a PID, \( \ker f \) is free and a simple rank argument (e.g. passing to \( L \)) shows \( \ker f \cong R \cdot \rho \) for some \( \rho \) in \( S \subseteq L \). Clearly such a \( \rho \) must satisfy a quadratic monic polynomial over \( R \), so that \( S \) is quadratic over \( R \) in the above sense.

**Theorem 4.1.** Let \( S = R[\rho] \) be a commutative integral quadratic extension of an integral domain \( R \), let \( \bar{\rho} \) be the conjugate of \( \rho \) and let \( S' = S/(\rho - \bar{\rho})S \). Then the following sequence is exact:

\[
0 \rightarrow U(S \otimes S) \rightarrow U(S) \times U(S) \rightarrow U(S') \rightarrow \text{Pic}(S \otimes S) \rightarrow \text{Pic} S \times \text{Pic} S \rightarrow \text{Pic} S'.
\]

**Proof.** In view of Lemma 4.0 the above sequence is given by Theorem 5.3 [3, IX Sec. 5, p. 481].

**Remarks.** The maps of the above sequence are those of the Mayer-Vietoris sequence of [3, VII Sec. 4]. In particular, \( U(S \otimes S) \rightarrow U(S) \times U(S) \) is given by \( u \rightarrow (h_i(u), h_i(u)^{-1}) \) where \( h_i \) are the maps in Lemma 4.0, and \( U(S) \times U(S) \rightarrow U(S') \) is given by \( (s, t) \rightarrow \pi(s)\pi(t) \) where \( \pi: S \rightarrow S' \) is the natural projection. Clearly the image of \( U(S) \times U(S) \rightarrow U(S') \) is the same as the image of \( \pi \) restricted to \( U(S) \).

**Corollary 4.2.** With \( R \) and \( S \) as in Theorem 4.1, \( \text{Pic}(S \otimes S) = 0 \) if and only if \( \text{Pic} S = 0 \) and the natural projection \( U(S) \rightarrow U(S') \) is surjective.

**Proof.** The \( R \)-algebra map \( \varepsilon: S \rightarrow S \otimes S \) given by \( x \rightarrow x \otimes 1 \) is split by the map \( M: x \otimes y \rightarrow xy \). Hence \( \text{Pic} S \xrightarrow{\varepsilon^*} \text{Pic}(S \otimes S) \xrightarrow{\text{Pic} M} \text{Pic} S \) is identity, so that \( \text{Pic} \varepsilon \) is a monomorphism, i.e., \( \text{Pic} S \subseteq \text{Pic}(S \otimes S) \). The corollary is then immediate from Theorem 4.1 and remarks following it.

Now let \( K = Q(\sqrt{m}) \) be a quadratic field extension of the rationals, and \( S \) be its ring of integers. As in § 3, \( S = Z[\rho] \) where \( \rho = \sqrt{m} \) or \( (1 + \sqrt{m})/2 \) according to whether \( m \equiv 2 \) or 3 or \( m \equiv 1 \) (mod 4). We can easily compute \( S' \):

**Lemma 4.3.** If \( m \equiv 1 \) (mod 4), then \( S' \cong Z/mZ \).

**Proof.** Let \( m = 4k + 1 \) so that \( \rho + 2k = (\sqrt{m})\rho \), and write \( x + y\rho = x - 2ky + y(\rho + 2k) = x - 2ky + y\sqrt{m}(\rho - \bar{\rho}) \) where \( x, y \) lie in \( Z \). Hence \( x + y\rho \equiv x - 2ky \pmod{(\rho - \bar{\rho})S} \). Moreover, \( m = \sqrt{m}(\sqrt{m} = (\rho - \bar{\rho})^2 \equiv 0 \pmod{S(\rho - \bar{\rho})} \). Thus if, for an integer \( a, \bar{a} \) denotes the coset of a mod \( m \), we see that \( x + y\rho \rightarrow x - 2ky \) is a ring map of \( S \) onto \( Z/mZ \) whose kernel, \( J = \{x + y\rho | x - 2ky = am \} \) is con-
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If \( m \neq 1 \) (mod 4) then \( S' \cong T = \mathbb{Z}/2m\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}(\sqrt{m}) \) (where this ring has the obvious multiplication).

**Proof.** In this case \( \rho = \sqrt{m} \) and \( \rho - \bar{\rho} = 2\sqrt{m} \). Let \( \sim \) and \( \wedge \) denote reduction mod 2m and 2 respectively. Then for \( x, y \) in \( \mathbb{Z} \), \( x + y\rho \rightarrow \bar{x} + \gamma\sqrt{m} \) is a ring map whose kernel is \( \{2ma + 2b\rho | a, b \text{ are in } \mathbb{Z} \} \). Since \( 2ma + 2b\rho = 2\sqrt{m}(\sqrt{ma} + b) = (\rho - \bar{\rho})(\sqrt{ma} + b) \), this kernel is just \((\rho - \bar{\rho})S\) and the lemma is proved.

Now \( S' \) is finite in either case; It follows from Proposition 5 of [5, Ch. 2, Sec. 5, No. 4] that any semi-local ring has trivial Picard group, hence \( \text{Pic}(S') = 0 \) under the hypotheses of Lemmas 4.3 or 4.4. Suppose that \( \text{Pic} \, S = 0 \) and let \( \pi: U(S) \rightarrow U(S') \) denote the map induced by the projection \( S \rightarrow S' \). Then employing the remarks following Theorem 4.1 the exact sequence of that theorem becomes in this case

\[
0 \rightarrow \text{Im} \, \pi \rightarrow U(S') \rightarrow \text{Pic}(S \otimes S) \rightarrow 0.
\]

**Theorem 4.5.** Let \( K = \mathbb{Q}(\sqrt{m}) \) be a quadratic extension of the rational numbers, \( \mathbb{Q} \), with \( m \) a square free integer. If \( S \) denotes the integers of \( K \), then \( \text{Pic}(S \otimes S) = 0 \) for \( m = \pm 3, -1, 2, \) and 5 but for no other value of \( m \).

**Proof.** For the given values of \( m \), \( S \) is a euclidean domain [12, Propn. 6-4-1] hence a PID, or equivalently [cf. 5, Sec. 5, No. 7] \( \text{Pic} \, S = 0 \). Referring to Lemmas 4.3 and 4.4 we may easily verify the following table by direct calculation

<table>
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<tr>
<th>( m )</th>
<th>( S' )</th>
<th>( U(S') )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( \mathbb{Z}/4\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}\sqrt{2} )</td>
<td>( {\pm 1, \pm \bar{1} + \sqrt{2}} )</td>
</tr>
<tr>
<td>3</td>
<td>( \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}\sqrt{3} )</td>
<td>( {\pm 1, \pm 2 + 2\sqrt{3}} )</td>
</tr>
<tr>
<td>5</td>
<td>( \mathbb{Z}/5\mathbb{Z} )</td>
<td>( {1, 2, 3, 4} )</td>
</tr>
</tbody>
</table>

where \(-\) denotes the coset mod 4,6, or 5 respectively.

Now by the Dirichlet units theorem [12, Sec. 6-3], \( U(S) = \{\pm \varepsilon | \varepsilon \text{ in } \mathbb{Z} \} \) where the fundamental unit, \( \varepsilon \), is \( 1 + \sqrt{2}, 2 + \sqrt{3}, \) or \( (1 + \sqrt{5})/2 \) respectively [11, "Tables"]. Referring to Lemmas 4.3 and 4.4 for the definition of \( \pi \) we find in case \( m = 2 \) that \( \pi(\varepsilon) = \bar{1} + \sqrt{2} \), \( \pi(-\varepsilon) = -\bar{1} - \sqrt{2} = -\bar{1} + \sqrt{2} \) in \( S' \). In all cases \( \pi(-1) = -\bar{1} \) and \( \pi(1) = \bar{1} \). Since \( \pi \) is (the restriction of) a ring map, we see that \( \pi \) is onto when \( m = 2 \). Similarly when \( m = 3 \), \( \pi(\varepsilon) = \bar{2} + \sqrt{3} \) and \( \pi(-\varepsilon) = -\bar{2} - \sqrt{3} = -\bar{2} + \sqrt{3} \) and when \( m = 5 \) \( \pi(\varepsilon) = -\bar{2} = \bar{3} \)
which generates the cyclic group of units of $S' = \mathbb{Z}/5\mathbb{Z}$. Thus also in these cases $\pi$ is onto.

If $m = -3$ then $U(S') = U(\mathbb{Z}/3\mathbb{Z}) = \{\pm 1\}$. $\pi$ is again onto because it is the restriction of a ring map. If $m = -1$ then $U(S') = \{1, \sqrt{-1}\}$. By definition $\pi(\sqrt{-1}) = \sqrt{-1}$ so that the fact that $\pi$ is the restriction of a ring map again implies $\pi$ is onto. That $\text{Pic}(S \otimes S) = 0$ for the given $m$ now follows from Corollary 4.2.

Now suppose $m$ is not one of the listed integers. By Corollary 4.2 we need only consider integers $m$ for which $S$ is a PID. If $m \leq -5$, the Units Theorem shows $U(S) = \pm 1$. Now $S'$ contains $\mathbb{Z}/m\mathbb{Z}$ or $\mathbb{Z}/2m\mathbb{Z}$ according to whether $m \equiv 1 \pmod{4}$ or not. Let $m = -p_1p_2 \cdots p_r$ with $p_i$ distinct primes, and consider first $m \equiv 1 \pmod{4}$. Then $\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$ with $p_i$ odd primes. There being only two units in $S$, if $\pi$ is to be onto we must clearly have $r = 1$ and $p_1 = 3$, so $\pi$ is not onto. Similarly, if $-5 < m \equiv 3 \pmod{4}$, $\mathbb{Z}/2m\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$ which has the same units as $\mathbb{Z}/m\mathbb{Z}$ and, as above $\pi$ is not onto. If $m = 2$ we take $p_1 = 2$, so that $\mathbb{Z}/2m\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$. Again, if $\pi$ is to be onto there can be no factors other than $p_i$, since $\mathbb{Z}/4\mathbb{Z}$ has 2 units, so that for no $m \leq -5$ can $\pi$ be onto.

Consider now $m > 5$. For any unit $a + b\rho$ in $S$ we have that the norm

$$N(a + b\rho) = (a + b\rho)(a + b\bar{\rho})$$

is a unit in $\mathbb{Z}$, so

$$\pm 1 = (a + b\rho)(a + b\bar{\rho}) \equiv (a + b\rho)^4(\rho - \bar{\rho})S.$$

Squaring shows that for any unit $v$ in $S' = S/(\rho - \bar{\rho})S$ we have $v^4 = 1$. Now the Units Theorem shows that $U(S)$ is the direct product of the cyclic group $\langle -1 \rangle$ of order two, generated by $-1$ with an infinite cyclic group $\langle \varepsilon \rangle$ for some unit $\varepsilon$, called the fundamental unit. It then follows that $\text{Im} \pi \in U(S')$ is a group of exponent dividing four, generated by two elements, one, namely $\pi(-1)$, of order at most two. In particular $\text{Im} \pi$ has at most eight elements.

Suppose first that $m = 2p_1 \cdots p_r$ with $p_i$ distinct odd primes. Then $S' \cong \mathbb{Z}/2m\mathbb{Z} = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/p_1\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r\mathbb{Z}$. If this ring is to have at most eight units we must clearly have $p_i \leq 5$. Indeed $m = 6$ or $m = 10$ are the only possibilities, since $m = 30$ produces more than eight units. However, if $m = 6$ or 10, $S$ is not a PID [11, "Tables"] so by Corollary 4.2 we can not have $\text{Pic}(S \otimes S) = 0$. Thus in all possible remaining cases, $m = 2k$ implies $\pi$ is not onto and again Corollary 4.2 shows $\text{Pic}(S \otimes S) \neq 0$.

Consider next $m \equiv 3 \pmod{4}$ and write $m = p_1 \cdots p_r$ as the product of distinct odd primes. Then
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\[ S' \equiv \mathbb{Z}/2m\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}. \]

In order to have at most eight units we must have each \( p_i \leq 7 \). But some \( p_i = 7 \) would entail a unit of order three which can not happen. Since \( m > 5 \), we see that \( \pi \) is onto possibly only if \( m = 15 \). But in this case \( S \) is not a PID [11, "Tables"] so again we can not have \( \text{Pic}(S \otimes S) = 0 \).

Finally there remains the case \( m \equiv 1 \text{ (mod 4)} \). If \( m = p_1 p_2 \cdots p_r \), then the units of \( S = \mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z} \) are the same as those of \( \mathbb{Z}/2m\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z} \) so the same argument as above for \( m \equiv 3 \) shows that \( \text{Pic}(S \otimes S) = 0 \) only for the listed values of \( m \equiv 1 \text{ (mod 4)} \), completing the proof.

5. \( B(S/Z) \). All notation is as in [7].

**Theorem 5.0.** Let \( K = \mathbb{Q}(\sqrt{m}) \) with \( m \) a square free integer and \( \mathbb{Q} \) the rationals. Let \( S \) be the ring of integers of \( K \). Then the split Brauer group \( B(S/Z) \) is zero when \( m = -3, -1, 2, 3 \) or 5.

**Proof.** In each case \( S \) is euclidean [12, Propn. 6–4–1] hence a PID. Thus as remarked in § 4, \( \text{Pic} S = 0 \), so that \( H^1(S/R, \text{Pic}) \), being a subgroup of \( \text{Pic} S \), is zero. By Theorem 4.3, \( \text{Pic}(S \otimes S) = 0 \), hence \( H^1(S/Z, \text{Pic}) \), which is a homomorphic image of a subgroup of \( \text{Pic}(S \otimes S) \), is zero. It then follows from Theorem 7.6 of [7] that \( B(S/Z) \cong H^0(S/Z, U) \) and the result follows from Theorems 3.0 and 3.2.

Using the global class field theory, one can prove that in fact \( B(S/Z) \cong B(Z) = 0 \) [9]. Dobbs [8] has exploited this fact to obtain an improvement of our Theorems 3.0 and 3.2. Of course the conclusion of Theorem 4.5 is more than is needed to show \( B(S/Z) = 0 \). It would suffice to prove directly that \( H^1(S/Z, \text{Pic}) = 0 \) or that the map \( H^1(S/Z, \text{Pic}) \to H^2(S/Z, U) \) given in Theorem 7.6 of [7] is a monomorphism. However, \( H^1(S/Z, \text{Pic}) \) does not seem amenable to computation at the present time.

**References**


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