FOURIER-STIELTJES TRANSFORMS AND WEAKLY ALMOST PERIODIC FUNCTIONALS FOR COMPACT GROUPS

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Let $G$ be a compact group and $H$ a closed subgroup. A function in the Fourier algebra of $H$ can be extended to a function in the Fourier algebra of $G$ without increase in norm and with an arbitrarily small increase in sup-norm. For $G$ a compact Lie group, the space of Fourier-Stieltjes transforms is not dense in the space of weakly almost periodic functionals on the Fourier algebra of $G$.

We let $G$ denote an infinite compact group and $\hat{G}$ its dual. We use the notation of [1, Chapters 7 and 8], [2], and [3]. Recall $A(G)$ denotes the Fourier algebra of $G$ (an algebra of continuous functions on $G$), and $L^\infty(\hat{G})$ denotes its dual space under the pairing $\langle f, \phi \rangle$ ($f \in A(G), \phi \in L^\infty(\hat{G})$). Further, note $L^\infty(\hat{G})$ is identified with the $C^*$-algebra of bounded operators on $L^2(G)$ commuting with right translation. The module action of $A(G)$ on $L^\infty(\hat{G})$ is defined by the following: for $f \in A(G), \phi \in L^\infty(\hat{G}), f \cdot \phi \in L^\infty(\hat{G})$ by $\langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle, g \in A(G)$. Also $\|f \cdot \phi\|_\infty \leq \|f\|_A \|\phi\|_\infty$.

Let $\phi \in L^\infty(\hat{G})$. We call $\phi$ a weakly almost periodic functional if and only if the map $f \mapsto f \cdot \phi$ from $A(G)$ to $L^\infty(\hat{G})$ is a weakly compact operator. The space of all such is denoted by $W(\hat{G})$.

Let $M(G)$ denote the measure algebra of $G$. For $\mu \in M(G)$, the Fourier-Stieltjes transform of $\mu$, $\mathscr{F} \mu$, is a matrix-valued function in $L^\infty(\hat{G})$ defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto (\mathscr{F} \mu)_\alpha = \int_0 T_\alpha(x^{-1}) \, d\mu(x) \ (T_\alpha \in \mathcal{A}) .$$

We denote the closure of $\mathscr{F} M(G)$ in $L^\infty(\hat{G})$ by $\mathcal{M}(\hat{G})$. In [2], we showed that $W(\hat{G})$ is a closed subspace of $L^\infty(\hat{G})$, and that $\mathcal{M}(\hat{G}) \subset W(\hat{G})$ with the inclusion proper when $G$ is a direct product of an infinite collection of nontrivial compact groups. In this paper, we show the inclusion is proper for all compact Lie groups.

We first state a standard lemma.

**Lemma 1.** Let $A, B$ be compact subsets of a topological group $G$. Suppose $AB \subset U, U$ an open subset of $G$. Then there is an open neighborhood $V$ of the identity $e$ of $G$ such that $AVB \subset U$. 637
PROPOSITION 2. Let $G$ be a compact group and $H$ a closed subgroup. Let $W$ be an open subset of $G$ with $H \cap \bar{W} = \emptyset$. Then there is a continuous positive definite function $p$ on $G$ with $p(x) = 1$, $x \in H$, and $p(x) = 0$, $x \in W$. (Note $p \in A(G)$ and $\|p\|_A = 1$.)

Proof. Let $U$ be an open subset of $G$ with $H \subset U$, and $U \cap W = \emptyset$. Choose $V_1$ an open neighborhood of $e$ with $HV_1 \subset U$. Now let $V$ be an open neighborhood of $e$ with $VV \subset V_1$ and $V = V^{-1}$. Thus $HV VH \subset HV_1 H \subset U$.

Let $p = (m_\sigma(HV))^{-1} \chi_{HV}^* \chi_{VH}$ ($m_\sigma$ is normalized Haar measure on $G$ and $\chi_a$ denotes the characteristic function of $A$). Then $p(x) = (m_\sigma(HV))^{-1} m_\sigma(xHV \cap VH)$, $x \in G$. Thus for $x \in H$, $p(x) = 1$. If $p(x) \neq 0$, then $xHV \cap VH \neq \emptyset$, and so $x \in HV VH \subset U$.

THEOREM 3. Let $G$ be a compact group and $H$ a closed subgroup. Let $f \in A(H)$ and $\varepsilon > 0$. Then there exists $g \in A(G)$, $\|g\|_A = \|f\|_A$, $g|H = f$, and $\|g\|_\infty \leq \|f\|_\infty + \varepsilon$.

Proof. Let $h$ be an extension of $f$ to $G$ with $\|h\|_A = \|f\|_A$ (see [1, Chapter 8]). Let $V = \{x \in G: |h(x)| > \|f\|_\infty + \varepsilon\}$. Now let $p$ be as in Proposition 2, and let $g = ph$.

We now state a characterization of $M(\hat{G})$. The proof for abelian groups is in [1, Chapter 3]. The proof for nonabelian groups is analogous.

THEOREM 4. Let $G$ be a compact group and $\phi \in \mathcal{L}^\infty(\hat{G})$. For $\phi \in M(\hat{G})$ it is necessary and sufficient that whenever $\{f_n\}$ is a sequence from $A(G)$ with $\|f_n\|_A \leq 1$ and $\|f_n\|_\infty \xrightarrow{n} 0$ we have $\langle f_n, \phi \rangle \xrightarrow{n} 0$.

THEOREM 5. Let $G$ be a compact Lie group. Then $M(\hat{G}) \neq W(\hat{G})$.

Proof. Let $H$ be a total subgroup of $G$; that is, $H$ is the circle group. Now $M(\hat{H}) \neq W(\hat{H})$, (see [1, Chapter 4]).

Let $\pi_1$ denote the restriction map of $A(G)$ onto $A(H)$ and let $\hat{\pi}$ denote the adjoint map of $\mathcal{L}^\infty(\hat{H})$ into $\mathcal{L}^\infty(\hat{G})$. In [3], we showed that

$$\hat{\pi}M(\hat{H}) \subset M(\hat{G}) \text{ and } \hat{\pi}W(\hat{H}) \subset W(\hat{G}).$$

Let $\phi \in W(\hat{H}) \setminus M(\hat{H})$. Now $\hat{\pi}\phi \in W(\hat{G})$ so we need only show that $\hat{\pi}\phi \in M(\hat{G})$. Since $\phi \in M(\hat{H})$, there is a sequence $\{f_n\} \subset A(H)$, $\|f_n\|_A \leq 1$, $\|f_n\|_\infty \xrightarrow{n} 0$ with $|\langle f_n, \phi \rangle| \geq \varepsilon$ (some $\varepsilon > 0$). Extend $f_n$ to $g_n \in A(G)$
A(G) by Theorem 3 with \( \|g_n\|_A \leq 1 \) and \( \|g_n\|_\infty \to 0 \). But \( \langle g_n, \hat{\phi} \rangle = \langle \pi, g_n \rangle = \langle f_n, \phi \rangle \), and so \( \hat{\phi} \in \mathcal{M}(\hat{G}) \).

**Remark.** If a compact group \( G \) has a closed subgroup \( H \) with \( \mathcal{M}(\hat{H}) \neq \mathcal{M}(\hat{H}) \) and \( \mathcal{M}(\hat{G}) \neq \mathcal{M}(\hat{G}) \), (in particular, if \( G \) contains an infinite abelian subgroup). Indeed, it is an open question whether an infinite compact group always contains an infinite abelian subgroup.

**Corollary 6.** Let \( G \) be a compact group with \( H \) a closed subgroup. Then

\[
\hat{\pi}(W(\hat{H}) \setminus \mathcal{M}(\hat{H})) \subseteq W(\hat{G}) \setminus \mathcal{M}(\hat{G}).
\]

**References**


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William O’Bannon Alltop, 5-designs in affine spaces .................................................. 547
B. G. Basmaji, Real-valued characters of metacyclic groups ........................................ 553
Miroslav Benda, On saturated reduced products ............................................................... 557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, Uniquely representable semigroups. II ................................................................. 573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, Fixed points and stability for a sum of two operators in locally convex spaces ........................................... 581
Donald Richard Chalice, Restrictions of Banach function spaces .................................. 593
Eugene Frank Cornelius, Jr., A generalization of separable groups ................................ 603
Joel L. Cunningham, Primes in products of rings ............................................................ 615
Robert Alan Morris, On the Brauer group of $Z$ ............................................................... 619
David Earl Dobbs, Amitsur cohomology of algebraic number rings ................................ 631
Charles F. Dunkl and Donald Edward Ramirez, Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups ............................................. 637
Hicham Fakhoury, Structures uniformes faibles sur une classe de cônes et d’ensembles convexes ................................................................................................. 641
Leslie R. Fletcher, A note on $C\theta\theta$-groups ................................................................ 655
Humphrey Sek-Ching Fong and Louis Sucheston, On the ratio ergodic theorem for semi-groups ......................................................................................... 659
James Arthur Gerhard, Subdirectly irreducible idempotent semigroups ........................... 669
Thomas Eric Hall, Orthodox semigroups ............................................................................ 677
Marcel Herzog, $C\theta\theta$-groups involving no Suzuki groups ............................................ 687
John Walter Hinrichsen, Concerning web-like continua ................................................... 691
Frank Norris Huggins, A generalization of a theorem of F. Riesz .................................... 695
Carlos Johnson, Jr., On certain poset and semilattice homomorphisms ............................. 703
Alan Leslie Lambert, Strictly cyclic operator algebras ....................................................... 717
Howard Wilson Lambert, Planar surfaces in knot manifolds .......................................... 727
Robert Allen McCoy, Groups of homeomorphisms of normed linear spaces ................... 735
T. S. Nanjundiah, Refinements of Wallis’s estimate and their generalizations ............... 745
Roger David Nussbaum, A geometric approach to the fixed point index ......................... 751
John Emanuel de Pillis, Convexity properties of a generalized numerical range ............... 767
Donald C. Ramsey, Generating monomials for finite semigroups .................................... 783
William T. Reid, A disconjugacy criterion for higher order linear vector differential equations ........................................................................................................ 795
Roger Allen Wiegand, Modules over universal regular rings ............................................ 807
Kung-Wei Yang, Compact functors in categories of non-archimedean Banach spaces .......... 821
R. Grant Woods, Correction to: “Co-absolutes of remainders of Stone-Čech compactifications” ......................................................................................... 827
Ronald Owen Fulp, Correction to: “Tensor and torsion products of semigroups” .......... 827
Bruce Alan Barnes, Correction to: “Banach algebras which are ideals in a banach algebra” ................................................................. 828