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FOURIER-STIELTJES TRANSFORMS AND WEAKLY ALMOST PERIODIC FUNCTIONALS FOR COMPACT GROUPS

CHARLES F. DUNKL AND DONALD EDWARD RAMIREZ

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Let G be a compact group and H a closed subgroup. A function in the Fourier algebra of H can be extended to a function in the Fourier algebra of G without increase in norm and with an arbitrarily small increase in sup-norm. For G a compact Lie group, the space of Fourier-Stieltjes transforms is not dense in the space of weakly almost periodic functionals on the Fourier algebra of G .

We let G denote an infinite compact group and \hat{G} its dual. We use the notation of [1, Chapters 7 and 8], [2], and [3]. Recall $A(G)$ denotes the Fourier algebra of G (an algebra of continuous functions on G), and $\mathcal{L}^\infty(\hat{G})$ denotes its dual space under the pairing $\langle f, \phi \rangle$ ($f \in A(G)$, $\phi \in \mathcal{L}^\infty(\hat{G})$). Further, note $\mathcal{L}^\infty(\hat{G})$ is identified with the C^* -algebra of bounded operators on $L^2(G)$ commuting with right translation. The module action of $A(G)$ on $\mathcal{L}^\infty(\hat{G})$ is defined by the following: for $f \in A(G)$, $\phi \in \mathcal{L}^\infty(\hat{G})$, $f \cdot \phi \in \mathcal{L}^\infty(\hat{G})$ by $\langle g, f \cdot \phi \rangle = \langle fg, \phi \rangle$, $g \in A(G)$. Also $\|f \cdot \phi\|_\infty \leq \|f\|_A \|\phi\|_\infty$.

Let $\phi \in \mathcal{L}^\infty(\hat{G})$. We call ϕ a weakly almost periodic functional if and only if the map $f \mapsto f \cdot \phi$ from $A(G)$ to $\mathcal{L}^\infty(\hat{G})$ is a weakly compact operator. The space of all such is denoted by $W(\hat{G})$.

Let $M(G)$ denote the measure algebra of G . For $\mu \in M(G)$, the Fourier-Stieltjes transform of μ , $\mathcal{F}\mu$, is a matrix-valued function in $\mathcal{L}^\infty(\hat{G})$ defined for $\alpha \in \hat{G}$ by

$$\alpha \mapsto (\mathcal{F}\mu)_\alpha = \int_G T_\alpha(x^{-1}) d\mu(x) \quad (T_\alpha \in \alpha).$$

We denote the closure of $\mathcal{F}M(G)$ in $\mathcal{L}^\infty(\hat{G})$ by $\mathcal{M}(\hat{G})$. In [2], we showed that $W(\hat{G})$ is a closed subspace of $\mathcal{L}^\infty(\hat{G})$, and that $\mathcal{M}(\hat{G}) \subset W(\hat{G})$ with the inclusion proper when G is a direct product of an infinite collection of nontrivial compact groups. In this paper, we show the inclusion is proper for all compact Lie groups.

We first state a standard lemma.

LEMMA 1. *Let A, B be compact subsets of a topological group G . Suppose $AB \subset U$, U an open subset of G . Then there is an open neighborhood V of the identity e of G such that $AVB \subset U$.*

PROPOSITION 2. *Let G be a compact group and H a closed subgroup. Let W be an open subset of G with $H \cap \bar{W} = \emptyset$. Then there is a continuous positive definite function p on G with $p(x) = 1, x \in H$, and $p(x) = 0, x \in W$. (Note $p \in A(G)$ and $\|p\|_A = 1$.)*

Proof. Let U be an open subset of G with $H \subset U$, and $U \cap W = \emptyset$. Choose V_1 an open neighborhood of e with $HV_1H \subset U$. Now let V be an open neighborhood of e with $VV \subset V_1$ and $V = V^{-1}$. Thus $HVVH \subset HV_1H \subset U$.

Let $p = (m_G(HV))^{-1} \chi_{HV} * \chi_{VH}$ (m_G is normalized Haar measure on G and χ_A denotes the characteristic function of A). Then $p(x) = (m_G(HV))^{-1} m_G(xHV \cap HV), x \in G$. Thus for $x \in H, p(x) = 1$. If $p(x) \neq 0$, then $xHV \cap HV \neq \emptyset$, and so $x \in HVVH \subset U$.

THEOREM 3. *Let G be a compact group and H a closed subgroup. Let $f \in A(H)$ and $\varepsilon > 0$. Then there exists $g \in A(G), \|g\|_A = \|f\|_A, g|_H = f$, and $\|g\|_\infty \leq \|f\|_\infty + \varepsilon$.*

Proof. Let h be an extension of f to G with $\|h\|_A = \|f\|_A$ (see [1, Chapter 8]). Let $V = \{x \in G: |h(x)| > \|f\|_\infty + \varepsilon\}$. Now let p be as in Proposition 2, and let $g = ph$.

We now state a characterization of $\mathcal{M}(\hat{G})$. The proof for abelian groups is in [1, Chapter 3]. The proof for nonabelian groups is analogous.

THEOREM 4. *Let G be a compact group and $\phi \in \mathcal{L}^\infty(\hat{G})$. For $\phi \in \mathcal{M}(\hat{G})$ it is necessary and sufficient that whenever $\{f_n\}$ is a sequence from $A(G)$ with $\|f_n\|_A \leq 1$ and $\|f_n\|_\infty \xrightarrow{n} 0$ we have $\langle f_n, \phi \rangle \xrightarrow{n} 0$.*

THEOREM 5. *Let G be a compact Lie group. Then $\mathcal{M}(\hat{G}) \neq W(\hat{G})$.*

Proof. Let H be a total subgroup of G ; that is, H is the circle group. Now $\mathcal{M}(\hat{H}) \neq W(\hat{H})$, (see [1, Chapter 4]).

Let π_1 denote the restriction map of $A(G)$ onto $A(H)$ and let $\hat{\pi}$ denote the adjoint map of $\mathcal{L}^\infty(\hat{H})$ into $\mathcal{L}^\infty(\hat{G})$. In [3], we showed that

$$\hat{\pi} \mathcal{M}(\hat{H}) \subset \mathcal{M}(\hat{G}) \text{ and } \hat{\pi} W(\hat{H}) \subset W(\hat{G}) .$$

Let $\phi \in W(\hat{H}) \setminus \mathcal{M}(\hat{H})$. Now $\hat{\pi}\phi \in W(\hat{G})$ so we need only show that $\hat{\pi}\phi \notin \mathcal{M}(\hat{G})$. Since $\phi \notin \mathcal{M}(\hat{H})$, there is a sequence $\{f_n\} \subset A(H), \|f_n\|_A \leq 1, \|f_n\|_\infty \xrightarrow{n} 0$ with $|\langle f_n, \phi \rangle| \geq \varepsilon$ (some $\varepsilon > 0$). Extend f_n to $g_n \in$

$A(G)$ by Theorem 3 with $\|g_n\|_A \leq 1$ and $\|g_n\|_\infty \xrightarrow{n} 0$. But $\langle g_n, \hat{\pi}\phi \rangle = \langle \pi g_n, \phi \rangle = \langle f_n, \phi \rangle$, and so $\hat{\pi}\phi \notin \mathcal{M}(\hat{G})$.

REMARK. If a compact group G has a closed subgroup H with $\mathcal{M}(\hat{H}) \neq W(\hat{H})$, then $\mathcal{M}(\hat{G}) \neq W(\hat{G})$, (in particular, if G contains an infinite abelian subgroup). Indeed, it is an open question whether an infinite compact group always contains an infinite abelian subgroup.

COROLLARY 6. *Let G be a compact group with H a closed subgroup. Then*

$$\hat{\pi}(W(\hat{H}) \setminus \mathcal{M}(\hat{H})) \subset W(\hat{G}) \setminus \mathcal{M}(\hat{G}).$$

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William O'Bannon Alltop, <i>5-designs in affine spaces</i>	547
B. G. Basmaji, <i>Real-valued characters of metacyclic groups</i>	553
Miroslav Benda, <i>On saturated reduced products</i>	557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups. II</i>	573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, <i>Fixed points and stability for a sum of two operators in locally convex spaces</i>	581
Donald Richard Chalice, <i>Restrictions of Banach function spaces</i>	593
Eugene Frank Cornelius, Jr., <i>A generalization of separable groups</i>	603
Joel L. Cunningham, <i>Primes in products of rings</i>	615
Robert Alan Morris, <i>On the Brauer group of Z</i>	619
David Earl Dobbs, <i>Amitsur cohomology of algebraic number rings</i>	631
Charles F. Dunkl and Donald Edward Ramirez, <i>Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups</i>	637
Hicham Fakhoury, <i>Structures uniformes faibles sur une classe de cônes et d'ensembles convexes</i>	641
Leslie R. Fletcher, <i>A note on $C\theta\theta$-groups</i>	655
Humphrey Sek-Ching Fong and Louis Sucheston, <i>On the ratio ergodic theorem for semi-groups</i>	659
James Arthur Gerhard, <i>Subdirectly irreducible idempotent semigroups</i>	669
Thomas Eric Hall, <i>Orthodox semigroups</i>	677
Marcel Herzog, <i>$C\theta\theta$-groups involving no Suzuki groups</i>	687
John Walter Hinrichsen, <i>Concerning web-like continua</i>	691
Frank Norris Huggins, <i>A generalization of a theorem of F. Riesz</i>	695
Carlos Johnson, Jr., <i>On certain poset and semilattice homomorphisms</i>	703
Alan Leslie Lambert, <i>Strictly cyclic operator algebras</i>	717
Howard Wilson Lambert, <i>Planar surfaces in knot manifolds</i>	727
Robert Allen McCoy, <i>Groups of homeomorphisms of normed linear spaces</i>	735
T. S. Nanjundiah, <i>Refinements of Wallis's estimate and their generalizations</i>	745
Roger David Nussbaum, <i>A geometric approach to the fixed point index</i>	751
John Emanuel de Pillis, <i>Convexity properties of a generalized numerical range</i>	767
Donald C. Ramsey, <i>Generating monomials for finite semigroups</i>	783
William T. Reid, <i>A disconjugacy criterion for higher order linear vector differential equations</i>	795
Roger Allen Wiegand, <i>Modules over universal regular rings</i>	807
Kung-Wei Yang, <i>Compact functors in categories of non-archimedean Banach spaces</i>	821
R. Grant Woods, <i>Correction to: "Co-absolutes of remainders of Stone-Čech compactifications"</i>	827
Ronald Owen Fulp, <i>Correction to: "Tensor and torsion products of semigroups"</i>	827
Bruce Alan Barnes, <i>Correction to: "Banach algebras which are ideals in a banach algebra"</i>	828