

Pacific Journal of Mathematics

A NOTE ON $C\theta\theta$ -GROUPS

LESLIE R. FLETCHER

A NOTE ON $C\theta\theta$ -GROUPS

L. R. FLETCHER

A $C\theta\theta$ -group is a finite group of order divisible by 3 in which centralisers of 3-elements are 3-groups. Several authors have studied such groups; in particular it is known that, given the additional hypothesis that the Sylow 3-subgroups intersect trivially, a simple $C\theta\theta$ -group has abelian Sylow 3-subgroups. In this note it is proved that this additional hypothesis is superfluous.

More precisely the following will be proved:

THEOREM. *Let G be a $C\theta\theta$ -group in which $O^3(G) = G$ and let M be a Sylow 3-subgroup of G . Then M is a TI-set in G .*

The proof of the theorem depends on two lemmas:

LEMMA 1. *Let H be a $C\theta\theta$ -group. If any element of order 3 in H is conjugate to its inverse; or, equivalently, if any 3-local subgroup of H has even order; then Sylow 3-subgroups of H are abelian and hence TI-sets in H .*

Proof. Suppose t is an element of order 3 in H conjugate to its inverse. Let T be a Sylow 3-subgroup of H such that $t \in T$. Now the extended centraliser $C_H^*(t)$ is a Frobenius group with the 3-group $C_H(t)$ as kernel. Since $|C_H^*(t) : C_H(t)| = 2$, $C_H(t)$ is abelian and every element in it is conjugate to its inverse. Now $Z(T) \leq C_H(t)$ so we may assume that $t \in Z(T)$. In this case $C_H(t) = T$ and so T is abelian.

LEMMA 2. *Let H be a $C\theta\theta$ -group in which $O_3(H) > 1$. Then H is soluble and one of the following occurs:*

- (i) *a Sylow 3-subgroup of H is normal in H*
- (ii) *$O^3(H) < H$.*

Proof. Put $L = O_3(H)$, $\bar{H} = H/L$. Suppose first that $|H|$ is even. Every element of L is conjugate to its inverse so, by Lemma 1, Sylow 3-subgroups of H are abelian. Clearly $L = C_H(L)$ is a Sylow 3-subgroup of H , case (i) arises, and $|\bar{H}|$ is prime to 3. \bar{H} can now be regarded as a group of fixed-point-free automorphisms of L so, if p is odd, the Sylow p -subgroups of \bar{H} are cyclic and the Sylow 2-subgroups are either cyclic or generalised quaternion. A group all

of whose Sylow subgroups are cyclic is soluble. (See [2] Theorem 7.6.2.) On the other hand it is not difficult to show that a group having generalised quaternion Sylow 2-subgroups either involves A_4 , the alternating group on 4 letters, or satisfies the hypotheses of Frobenius' theorem on the existence of a normal p -complement for $p = 2$. $|\bar{H}|$ is prime to 3 so \bar{H} is soluble.

If $|H|$ is odd then it is well-known that H is soluble. Suppose that a Sylow 3-subgroup of H is not normal in H i.e., $|\bar{H}|$ is divisible by 3. A Sylow 3-subgroups of \bar{H} can be regarded as a group of fixed-point-free automorphisms of $0_3(\bar{H})$. Thus \bar{H} has cyclic Sylow 3-subgroups. But the only 3'-automorphism of a cyclic 3-group has order 2 and $|\bar{H}|$ is odd. Hence, by Burnside's Theorem, \bar{H} has a normal 3-complement; in particular $0^3(\bar{H}) < \bar{H}$ and so $0^3(H) < H$.

Proof of Theorem. Suppose M is not a TI -set in G . Then M is not abelian so, by Lemmas 1 and 2, the normaliser of every non-identity 3-subgroup of G is soluble and of odd order. In the terminology of [2], this means that the normaliser of every non-identity 3-subgroup is 3-constrained and 3-stable (see [2] p. 268) and so satisfies the conditions of [2] Theorem 8.2.11. Hence G satisfies the conditions of [2] Theorem 8.4.2. and 8.4.3.

Write $N = N(Z(J(M)))$. If N is of type (ii) in Lemma 2 then $M \cap N'$ is a proper subgroup of M . By [2] Theorem 8.4.3. $M \cap G'$ is a proper subgroup of M and so, by [2] Theorem 7.3.1. $0^3(G)$ is a proper subgroup of G . This is not the case and so N is of type (i) in Lemma 2.

Let M_0 be a maximal intersection of Sylow 3-subgroups of G contained in M . By the maximality of M_0 , $M_0 = 0_3(N(M_0))$; by Lemma 2, $N(M_0)$ is soluble. Hence $C(M_0) \leq M_0$; in particular, $Z(M) \leq M_0$. Let $m \in Z(M)$ and $h \in N(M_0)$. $m, m^h \in M$ so by [2] Theorem 8.4.2. there is an element $n \in N$ such that $m^h = m^n$ i.e., $n.h^{-1} \in C(m)$. Clearly then $n.h^{-1} \in M \leq N$. Hence $h \in N$ and so $N(M_0) \leq N$. But N has a unique Sylow 3-subgroup, $N(M_0)$ does not. This contradiction proves that M is a TI -set in G .

COROLLARY. *A simple $C\theta\theta$ -group has abelian Sylow 3-subgroup.*

Proof. This follows immediately from the theorem and work of Ferguson [1] and Herzog [3].

I am indebted to Mrs. Ferguson for letting me see a preliminary draft of her Ph.D. thesis.

REFERENCES

1. P. Ferguson, Ph.D. Thesis, University of Chicago, 1969.
2. D. Gorenstein, *Finite Groups*, Harper and Row, 1968.
3. M. Herzog, *On finite groups which contain a Frobenius subgroup*, J. Algebra, **6** (1967), 192-221.

Received August 14, 1970.

UNIVERSITY OF SALFORD
LANCASHIRE, ENGLAND

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. SAMELSON
Stanford University
Stanford, California 94305

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

C. R. HOBBY
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. 39. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 3

July, 1971

William O'Bannon Alltop, <i>5-designs in affine spaces</i>	547
B. G. Basmaji, <i>Real-valued characters of metacyclic groups</i>	553
Miroslav Benda, <i>On saturated reduced products</i>	557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups. II</i>	573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, <i>Fixed points and stability for a sum of two operators in locally convex spaces</i>	581
Donald Richard Chalice, <i>Restrictions of Banach function spaces</i>	593
Eugene Frank Cornelius, Jr., <i>A generalization of separable groups</i>	603
Joel L. Cunningham, <i>Primes in products of rings</i>	615
Robert Alan Morris, <i>On the Brauer group of Z</i>	619
David Earl Dobbs, <i>Amitsur cohomology of algebraic number rings</i>	631
Charles F. Dunkl and Donald Edward Ramirez, <i>Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups</i>	637
Hicham Fakhoury, <i>Structures uniformes faibles sur une classe de cônes et d'ensembles convexes</i>	641
Leslie R. Fletcher, <i>A note on $C\theta\theta$-groups</i>	655
Humphrey Sek-Ching Fong and Louis Sucheston, <i>On the ratio ergodic theorem for semi-groups</i>	659
James Arthur Gerhard, <i>Subdirectly irreducible idempotent semigroups</i>	669
Thomas Eric Hall, <i>Orthodox semigroups</i>	677
Marcel Herzog, <i>$C\theta\theta$-groups involving no Suzuki groups</i>	687
John Walter Hinrichsen, <i>Concerning web-like continua</i>	691
Frank Norris Huggins, <i>A generalization of a theorem of F. Riesz</i>	695
Carlos Johnson, Jr., <i>On certain poset and semilattice homomorphisms</i>	703
Alan Leslie Lambert, <i>Strictly cyclic operator algebras</i>	717
Howard Wilson Lambert, <i>Planar surfaces in knot manifolds</i>	727
Robert Allen McCoy, <i>Groups of homeomorphisms of normed linear spaces</i>	735
T. S. Nanjundiah, <i>Refinements of Wallis's estimate and their generalizations</i>	745
Roger David Nussbaum, <i>A geometric approach to the fixed point index</i>	751
John Emanuel de Pillis, <i>Convexity properties of a generalized numerical range</i>	767
Donald C. Ramsey, <i>Generating monomials for finite semigroups</i>	783
William T. Reid, <i>A disconjugacy criterion for higher order linear vector differential equations</i>	795
Roger Allen Wiegand, <i>Modules over universal regular rings</i>	807
Kung-Wei Yang, <i>Compact functors in categories of non-archimedean Banach spaces</i>	821
R. Grant Woods, <i>Correction to: "Co-absolutes of remainders of Stone-Čech compactifications"</i>	827
Ronald Owen Fulp, <i>Correction to: "Tensor and torsion products of semigroups"</i>	827
Bruce Alan Barnes, <i>Correction to: "Banach algebras which are ideals in a banach algebra"</i>	828