A NOTE ON \( C_{\theta} \)-GROUPS

Leslie R. Fletcher
A NOTE ON $C\theta\theta$-GROUPS

L. R. FLETCHER

A $C\theta\theta$-group is a finite group of order divisible by 3 in which centralisers of 3-elements are 3-groups. Several authors have studied such groups; in particular it is known that, given the additional hypothesis that the Sylow 3-subgroups intersect trivially, a simple $C\theta\theta$-group has abelian Sylow 3-subgroups. In this note it is proved that this additional hypothesis is superfluous.

More precisely the following will be proved:

**THEOREM.** Let $G$ be a $C\theta\theta$-group in which $0^3(G) = G$ and let $M$ be a Sylow 3-subgroup of $G$. Then $M$ is a TI-set in $G$.

The proof of the theorem depends on two lemmas:

**LEMMA 1.** Let $H$ be a $C\theta\theta$-group. If any element of order 3 in $H$ is conjugate to its inverse; or, equivalently, if any 3-local subgroup of $H$ has even order; then Sylow 3-subgroups of $H$ are abelian and hence TI-sets in $H$.

**Proof.** Suppose $t$ is an element of order 3 in $H$ conjugate to its inverse. Let $T$ be a Sylow 3-subgroup of $H$ such that $t \in T$. Now the extended centraliser $C_H^+(t)$ is a Frobenius group with the 3-group $C_H(t)$ as kernel. Since $|C_H^+(t):C_H(t)| = 2$, $C_H(t)$ is abelian and every element in it is conjugate to its inverse. Now $Z(T) \leq C_H(t)$ so we may assume that $t \in Z(T)$. In this case $C_H(t) = T$ and so $T$ is abelian.

**LEMMA 2.** Let $H$ be a $C\theta\theta$-group in which $0_3(H) > 1$. Then $H$ is soluble and one of the following occurs:

(i) a Sylow 3-subgroup of $H$ is normal in $H$
(ii) $0^4(H) < H$.

**Proof.** Put $L = 0_3(H)$, $\bar{H} = H/L$. Suppose first that $|H|$ is even. Every element of $L$ is conjugate to its inverse so, by Lemma 1, Sylow 3-subgroups of $H$ are abelian. Clearly $L = C_H(L)$ is a Sylow 3-subgroup of $H$, case (i) arises, and $|\bar{H}|$ is prime to 3. $\bar{H}$ can now be regarded as a group of fixed-point-free automorphisms of $L$ so, if $p$ is odd, the Sylow $p$-subgroups of $\bar{H}$ are cyclic and the Sylow 2-subgroups are either cyclic or generalised quaternion. A group all
of whose Sylow subgroups are cyclic is soluble. (See [2] Theorem 7.6.2.) On the other hand it is not difficult to show that a group having generalised quaternion Sylow 2-subgroups either involves $A_4$, the alternating group on 4 letters, or satisfies the hypotheses of Frobenius' theorem on the existence of a normal $p$-complement for $p = 2$. If $|\bar{H}|$ is prime to 3 so $\bar{H}$ is soluble.

If $|H|$ is odd then it is well-known that $H$ is soluble. Suppose that a Sylow 3-subgroup of $H$ is not normal in $H$ i.e., $|\bar{H}|$ is divisible by 3. A Sylow 3-subgroups of $\bar{H}$ can be regarded as a group of fixed-point-free automorphisms of $0_3(H)$. Thus $\bar{H}$ has cyclic Sylow 3-subgroups. But the only 3'-automorphism of a cyclic 3-group has order 2 and $|\bar{H}|$ is odd. Hence, by Burnside's Theorem, $\bar{H}$ has a normal 3-complement; in particular $0'(\bar{H}) < \bar{H}$ and so $0'(H) < H$.

**Proof of Theorem.** Suppose $M$ is not a TI-set in $G$. Then $M$ is not abelian so, by Lemmas 1 and 2, the normaliser of every non-identity 3-subgroup of $G$ is soluble and of odd order. In the terminology of [2], this means that the normaliser of every non-identity 3-subgroup is 3-constrained and 3-stable (see [2] p. 268) and so satisfies the conditions of [2] Theorem 8.2.11. Hence $G$ satisfies the conditions of [2] Theorem 8.4.2. and 8.4.3.

Write $N = N(Z(J(M)))$. If $N$ is of type (ii) in Lemma 2 then $M \cap N'$ is a proper subgroup of $M$. By [2] Theorem 8.4.3. $M \cap G'$ is a proper subgroup of $M$ and so, by [2] Theorem 7.3.1. $0'(G)$ is a proper subgroup of $G$. This is not the case and so $N$ is of type (i) in Lemma 2.

Let $M_0$ be a maximal intersection of Sylow 3-subgroups of $G$ contained in $M$. By the maximality of $M_0$, $M_0 = 0_3(N(M_0))$; by Lemma 2, $N(M_0)$ is soluble. Hence $C(M_0) \leq M_0$; in particular, $Z(M) \leq M_0$. Let $m \in Z(M)$ and $h \in N(M_0)$. $m, m^h \in M$ so by [2] Theorem 8.4.2. there is an element $n \in N$ such that $m^h = m^n$ i.e., $n.h^{-1} \in C(m)$. Clearly then $n.h^{-1} \in M \leq N$. Hence $h \in N$ and so $N(M_0) \leq N$. But $N$ has a unique Sylow 3-subgroup, $N(M_0)$ does not. This contradiction proves that $M$ is a TI-set in $G$.

**Corollary.** A simple $\Theta\Theta$-group has abelian Sylow 3-subgroup.

**Proof.** This follows immediately from the theorem and work of Ferguson [1] and Herzog [3].

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REFERENCES


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