Pacific Journal of Mathematics

ON THE RATIO ERGODIC THEOREM FOR SEMI-GROUPS

Humphrey Sek-Ching Fong and Louis Sucheston

Vol. 39, No. 3 July 1971

PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 3, 1971

ON THE RATIO ERGODIC THEOREM FOR SEMI-GROUPS

H. FONG AND L. SUCHESTON

For a semi-group Γ of positive linear *contractions* on L_1 of a σ-finite measure space $(X, \mathscr{A}, \mu)$, strongly continuous on $(0, \infty)$, there are two ratio ergodic theorems: one, due to Chacon and Ornstein, describes the behavior at infinity; the other one, due to Krengel-Ornstein-Akcoglu-Chacon, describes the "local" behavior. In the present paper we attempt to generalize these results to the case when the semi-group is only uniformly bounded. Then the space X decomposes into two parts, Y and Z, called the *remaining* and the *disappearing* part, and both ratio theorems are shown to hold on Y. The ratio theorem at infinity fails on Z.

This generalizes the situation described in the discrete case by the second-named author, and by A. Ionescn Tulcea and M. Moretz. We have not studied the "local" behavior of the ratio on Z.

1. Definitions. Let $\Gamma = \{T_t: t \geqq 0\}$ be a semi-group of positive linear operators in L_1 of a σ-finite measure space $(X, \mathscr{A}, \mu)$. We assume that Γ is *bounded*: $\sup_{t>0} | T_t |_1 < \infty$; and that Γ is *strongly continuous* on $(0, \infty)$: i.e., for each $f \in L_1$ and each $s > 0$, we have $\lim_{t \to s} | T_t f - T_s f |_1 = 0$. It is then known (cf. [5], p. 616) that Γ is strongly integrable on every interval $[\alpha, \beta], 0 \leqq \alpha < \beta < \infty$; more precisely, for each $f \in L_1$ and $0 \leqq \alpha < \beta < \infty$, the integral $\int_\alpha^\beta T_t f dt$ is defined and is an element of $L_1(X, \mathscr{A}, \mu)$. Hence for each $f \in L_1$ there is a scalar function $T_t f(x)$, measurable with respect to the product of Lebesgue measure and μ, such that for almost all t, $T_t f(x)$, as a function of x, belongs to the equivalence class $T_t f$([5], p. 686). Moreover, there is a set $E(f)$, $\mu(E(f)) = 0$, dependent on f but independent of t, such that if $x \notin E(f)$ then $T_t f(x)$ is integrable on every finite interval $[\alpha, \beta]$ and the integral $\int_\alpha^\beta T_t f(x) dt$, as a function of x, belongs to the equivalence class $\int_\alpha^\beta T_t f dt$. Thus for each $u > 0$ and each $f \in L_1$, the integral $\int_0^u T_t f(x) dt$, denoted $S_u f(x)$, is defined for every $x \notin E(f)$.

All sets introduced in this paper are assumed measurable; all functions are measurable and extended real-valued. All relations are assumed to hold modulo sets of μ-measure zero. The indicator function of a set A is written 1_A. We write supp$f$ for the set of points at which the function f is different from zero. For a set $A \subset X$, $L_1(A)$ denotes the class of functions f in $L_1(X)$ with $\operatorname{supp} f \subset A$; A is said to be *closed* (under T) if $T\{L_1(A)\} \subset L_1(A)$.

2. Behavior at infinity. The following Theorem 2.1 is a continuous parameter version of the Chacon-Ornstein theorem; Theorem 2.1 is included in a result of Berk [3], and was also recently obtained by Akcoglu and Cunsolo [2]. The following proof shows that the result is in fact contained in that of [4].

THEOREM 2.1. *Let $\Gamma = \{T_t : t \geqq 0\}$ be a semi-group of positive linear contractions in L_1 such that Γ is strongly continuous on $(0, \infty)$. Let $f, g \in L_1$, $g \geqq 0$. Then, as $u \to \infty$, the ratio*

$$D_u(f, g)(x) \overset{\text{def}}{=} S_u f(x)/S_u g(x) \tag{2.1}$$

converges to a finite limit a.e. on the set

$$A(g) \overset{\text{def}}{=} \{x: \sup_{u>0} S_u g(x) > 0\} .$$

Proof. For $f \in L_1$, let $\bar{f}(x) = S_1 f(x)$. For each $u > 0$, write $u = n + r$, where $n = [u]$, $0 \leqq r < 1$. Writing T for T_1, we have

$$\begin{aligned} S_u f = \int_0^u T_t f dt &= \sum_{k=0}^{n-1} \int_k^{k+1} T_t f dt + \int_n^{n+r} T_t f dt \\ &= \sum_{k=0}^{n-1} T^k \int_0^1 T_t f dt + T^n \int_0^r T_t f dt \end{aligned}$$

and hence

$$S_u f(x) = \sum_{k=0}^{n-1} T^k \bar{f}(x) + T^n (S_r f)(x) . \tag{2.2}$$

We may assume that f is nonnegative; then $0 \leqq S_r f(x) \leqq \bar{f}(x)$ and $0 \leqq S_r g(x) \leqq \bar{g}(x)$ a.e., $0 \leqq r \leqq 1$. Thus, for u sufficiently large, we have on $A(g)$

$$\frac{\sum_{k=0}^{n-1} T^k \bar{f}(x)}{\sum_{k=0}^{n} T^k \bar{g}(x)} \leqq D_u(f, g)(x) \leqq \frac{\sum_{k=0}^{n} T^k \bar{f}(x)}{\sum_{k=0}^{n-1} T^k \bar{g}(x)} . \tag{2.3}$$

This completes the proof, since the Chacon-Ornstein theorem and Lemma 2 [4] imply that the first and last terms in (2.3) converge to the same finite limit on the set $\{x: \sum_{k=0}^{\infty} T^k \bar{g}(x) > 0\} = A(g)$.

For a bounded semi-group Γ, we have the following decomposition of the space X.

PROPOSITION 2.1. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1. Then the space X decomposes into Y and Z with the following properties: Z is T_t-closed for $t \geqq 0$;*

$$(2.4)\qquad \begin{cases} 0 \not\equiv f \in L_1^+(Y) \text{ implies } \liminf_{t\to\infty} \int T_t f d\mu > 0 \; ; \\ f \in L_1(Z) \text{ implies } \lim_{t\to\infty} \int T_t \mid f \mid d\mu = 0 \; . \end{cases}$$

Proof. This result in the discrete parameter case was obtained by the second author in [11]. To prove the proposition, we apply the discrete case result to $T = T_1$, obtaining the decomposition $X = Y + Z$ with the properties

$$(2.5)\qquad \begin{cases} 0 \not\equiv f \in L_1^+(Y) \text{ implies } \liminf_{n\to\infty} \int T_n f d\mu > 0 \; ; \\ f \in L_1(Z) \text{ implies } \lim_{n\to\infty} \int T_n \mid f \mid d\mu = 0 \; . \end{cases}$$

Suppose that $f \in L_1^+$ and $\liminf_{t\to\infty} \int T_t f d\mu = 0$; then given $\varepsilon \in 0$, there is an $s > 0$ such that $0 \leqq \int T_s f d\mu < \varepsilon$. For $t > s$, we have

$$\begin{aligned} 0 \leqq \int T_t f d\mu &= \int T_{t-s}(T_s f) d\mu \\ &\leqq \mid T_{t-s} \mid_1 \cdot \mid T_s f \mid_1 \leqq \varepsilon \cdot \sup_{t \geqq 0} \mid T_t \mid_1 \; , \end{aligned}$$

which shows that $\lim_{t\to\infty} \int T_t f d\mu = 0$; in view of (2.5), (2.4) is now proved. That Z is T_t-closed for each $t \geqq 0$ is an easy consequence of (2.4).

The next proposition permits us to construct a semi-group Γ' of positive linear *contractions* related to Γ; the ratio ergodic properties of Γ are then studied via Γ'.

PROPOSITION 2.2. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. Then there is a function e such that*

$$(2.6)\qquad e \in L_\infty^+, \ \operatorname{supp} e = Y, \ T_t^* e = e \text{ for } t > 0 \; .$$

Proof. We may assume that $Y \neq \phi$ for otherwise the proposition is obviously true. Let

$$H = \{h \in L_\infty \colon T_t^* h = h,\ t > 0\}\ ;$$
$$D = \{1/2^n \colon n = 0, 1, 2, \cdots\}\ ;$$
$$G = \{g \colon g = f - T_r f, f \in L_1, r \in D\}\ .$$

Let $sp(G)$ denote the linear span of G. We first show that $H \neq \{0\}$. Let $h \in L_\infty$ be such that $\int g \cdot h d\mu = 0$ for every $g \in sp(G)$. It follows from $\int (f - T_r f) \cdot h = 0$, holding for each $f \in L_1, r \in D$, that

$$T_r^* h = h, r \in D\ . \tag{2.7}$$

The strong continuity of Γ on $(0, \infty)$ now implies that (2.7) holds for any $r > 0$. Assume *ab contrario* that $H = \{0\}$; then $h = 0$, and $sp(G)$ is dense in L_1. Thus given $f \in L_1^+(Y)$, and $\varepsilon > 0$, there is a function $g \in sp(G)$ such that $|f - g|_1 < \varepsilon$. We note that g is a linear combination of functions of the form $f_j - T_{r_j} f_j$, where $f_j \in L_1, r_j \in D$, $1 \leqq j \leqq m$; hence letting $r = \min\{r_1, r_2, \cdots, r_m\}$, we have

$$\lim_n n^{-1} \cdot \left| \sum_{i=0}^{n-1} T_r^i g \right|_1 = 0\ . \tag{2.8}$$

Thus

$$\begin{aligned}\liminf_n |T_r^n f|_1 &\leqq \limsup_n n^{-1} \cdot \left| \sum_{i=0}^{n-1} T_r^i f \right|_1 \\ &\leqq \lim_n n^{-1} \left| \sum_{i=0}^{n-1} T_r^i g \right|_1 + \varepsilon \cdot \sup_t |T_t|_1 \\ &= \varepsilon \cdot \sup |T_t|_1\ .\end{aligned}$$

This contradicts relation (2.4) and the assumption $Y \neq \phi$, since $\varepsilon > 0$ is arbitrary and Γ is bounded. Now let $0 \neq h \in H$ and write $h = h^+ - h^-$, where $h^+ = \max(h, 0)$, $h^- = -\min(h, 0)$. We may assume $h^+ \neq 0$; otherwise we replace h by $-h$. We have $T_t^* h^+ \geqq h^+$ for $t > 0$. Let $h' = \lim_n T_n^* h^+$; clearly, $0 \neq h' \in L_\infty^+$ and, by the monotone continuity of T_r^* (cf. [9], p. 187), we have $T_r^* h' = h'$ for $r \in D$. It now follows from the strong continuity of Γ that $T_t^* h' = h'$ for $t > 0$. Let π be a probability measure equivalent with μ, and let s be the supremum of numbers $\pi(\operatorname{supp} h)$ where h ranges over H^+, the class of nonnegative functions in H. There exists a sequence of functions $h_n \in H^+$ with $\pi(\operatorname{supp} h_n) \to s$. If $e \in L_\infty^+$ is a proper linear combination of the h_n's, and $E = \operatorname{supp} e$, then $e \in H^+, E \subset Y$ and $\pi(E) = s$. We next show that $E = Y$. We note that E is T_t^*-closed, $t > 0$. Indeed, there are functions $f_n \uparrow 1_E$ and constants $c_n > 0$ such that $c_n f_n \leqq e$. Hence $(\operatorname{supp} T_t^* f_n) \subset E$, and by the monotone continuity of T_t^*, $(\operatorname{supp} T_t^* 1_E) \subset E$. Applying the duality relation we can now see that $E^c = X - E$ is T_t-closed, $t > 0$. If T'_t is the restriction of T_t to $L_1(E^c)$, then $\Gamma' = \{T'_t : t \geqq 0\}$ is a semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. Under Γ', E^c decomposes into sets Y' and Z' according to Proposition 2.1. Since E^c is closed under T_t for $t > 0$, we have $\int T'_t f d\mu = \int T_t f d\mu$ for $f \in L_1(E^c)$ and $t > 0$. Hence $f \in L_1(Z)$ implies $\lim_t \int T'_t |f| d\mu = \lim_t \int T_t |f| d\mu = 0$, and $0 \not\equiv f \in L_1^+(Y - E)$ implies $\liminf_t \int T'_t f d\mu = \liminf_t \int T_t f d\mu > 0$. Consequently, $Y' = Y - E$ and $Z' = Z$. Thus if $E \neq Y$, then Y' is nonnull, and hence the first part of the proof, with Γ' replacing Γ, shows that there is a function e_1, $0 \not\equiv e_1 \in L_\infty^+(E^c)$, and $T'^*_t e_1 \geqq e_1$, $t > 0$. Since $T'^*_t e_1 = 1_{E^c} \cdot T^*_t e_1$, we have $T^*_t e_1 \geqq e_1$, $t > 0$. Let $e' = \lim_n T^*_n e_1$; then $e' \in H^+$ and $(\operatorname{supp} e') \cap E^c$ is nonnull. Thus $e + e' \in H^+$ and $\pi(\operatorname{supp}(e + e')) > s$, which contradicts the definition of s. Hence $\operatorname{supp} e = Y$ and the proposition is proved.

Assume that $\Gamma = \{T_t : t \geqq 0\}$ satisfies the hypothesis of Proposition 2.2. Let e be a solution of (2.6); we may assume that $0 < e \leqq 1$ on Y. T_t may be extended to a positive linear map on $\mathscr{M}^+$, the cone of nonnegative measurable functions on $(X, \mathscr{A})$: for each fixed $t \geqq 0$, if $f \in \mathscr{M}^+$, $T_t f$ is defined as $\lim_n T_t f_n$ where $f_n \in L_1^+$, and $f_n \uparrow f$ a.e. The extended operators T_t also satisfy the semi-group property on $\mathscr{M}^+$; i.e.,

$$(2.9) \qquad T_{t+s} f = T_t(T_s f), f \in \mathscr{M}^+, t, s \geqq 0 .$$

For each $t \geqq 0$, we define an operator V_t on L_1^+ by the relation

$$(2.10) \qquad V_t f = e \cdot T_t(f/(e + 1_Z)) ,$$

and extend V_t by linearity to L_1. One shows, as in [11], that $\Gamma' = \{V_t : t \geqq 0\}$ is a family of positive linear contractions in L_1. That Γ' is a semi-group is a consequence of (2.9), (2.10), and the fact that Z is T_t-closed, $t \geqq 0$. Let $K = \{g : g = f \cdot e, f \in L_1\}$. For a fixed $s > 0$ and $g = f \cdot e \in K, f \in L_1$, we have

$$(2.11) \qquad \begin{aligned} |V_t g - V_s g|_1 &= \left| e \cdot T_t\left(\frac{g}{e + 1_Z}\right) - e \cdot T_s\left(\frac{g}{e + 1_Z}\right) \right|_1 \\ &\leqq |e|_\infty \cdot |T_t(f \cdot 1_Y) - T_s(f \cdot 1_Y)|_1 \end{aligned}$$

which, by the strong continuity of Γ, tends to zero as $t \to s$. The case of a general $g \in L_1(Y)$ follows by approximation, since K is a dense subspace of $L_1(Y)$ and $|V_t|_1 \leqq 1$. Finally, because $V_t g = V_t(g \cdot 1_Y)$ for $g \in L_1$, we conclude that Γ' is strongly continuous on $(0, \infty)$.

Theorem 2.1 may now be applied to Γ': if $f' \in L_1^+$, $g' \in L_1^+$, then

$$\lim_{u\to\infty} \int_0^u V_t f'(x)dt \Big/ \int_0^u V_t g'(x)\,dt$$

exists a.e. on the set $\{x: \sup_{u>0} \int_0^u V_t g'(x)dt > 0\}$. For arbitrary measurable nonnegative functions f and g, we write $f' = f\cdot e$, $g' = g\cdot e$. If $f' \in L_1^+$, $g' \in L_1^+$, then for sufficiently large u,

$$\text{(2.12)} \qquad \frac{\int_0^u V_t f'(x)dt}{\int_0^u V_t g'(x)dt} = \frac{\int_0^u e(x)\cdot T_t f(x)dt}{\int_0^u e(x)\cdot T_t g(x)dt} = D_u(f, g)(x)$$

on $Y \cap A(g)$, where $A(g) \overset{\text{def}}{=} \{x: \sup_{u>0} S_u g(x) > 0\}$. Thus the last ratio in (2.12) converges to a finite limit a.e. on the set $Y \cap A(g)$. The above discussion is now summarized in the following theorem:

THEOREM 2.2. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1, storongly continuous on $(0, \infty)$. If f, g are measurable functions such that $f\cdot e$, $g\cdot e \in L_1^+$, then* $\lim_{u\to\infty}(D_u(f, g)(x)$ *exists* a.e. *on the set $Y \cap A(g)$.*

We say that the ratio theorem *holds* (for Γ) on a subset B of X if whenever $f \in L_1$, $g \in L_1^+$, $\lim_{u\to\infty} D_u(f, g)(x)$ exists a.e. on the set $B \cap A(g)$; otherwise we say that the ratio theorem *fails* on B. We showed that the ratio theorem holds on Y. We now show

THEOREM 2.3. *Let $\Gamma = \{T_t: t \geqq 0\}$ be bounded semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. If there is a function $g \in L_1^+(Z)$ such that the set $C(g) \overset{\text{def}}{=} \{x: \sup_{u>0} S_u g(x) = \infty\}$ is nonnull, then the ratio theorem fails on every nonnull subset of $C(g)$.*

Proof. Theorem 2.3 in the discrete parameter case was given in [7]; (see also [11] and [6]). The method of proof in [7] extends to the continuous case. Assume that the ratio theorem holds on a nonnull subset A of $C(g)$, where $g \in L_1^+(Z)$. In particular, $\lim_{u\to\infty} D_u(f, g)(x)$ exists a.e., on A for every $f \in L_1$. Let R be the operator from L_1 into $\mathscr{M}$, the space of real-valued measurable functions on $(X, \mathscr{A})$, defined by $Rf(x) = 1_A(x)\cdot \lim_{u\to\infty} D_u(f, g)(x)$. Since $S_u g(x) \to \infty$ on A, we have for each $t > 0$

$$(2.13)\quad \begin{aligned} R(T_t g)(x) &= \lim_{u\to\infty} \frac{\int_0^u T_{s+t}g(x)ds}{\int_0^u T_s g(x)ds} \\ &= \lim_{u\to\infty}\left[\frac{\int_0^u T_s g(x)ds}{\int_0^u T_s g(x)ds} - \frac{\int_0^t T_s g(x)ds}{\int_0^u T_s g(x)ds} + \frac{\int_u^{u+t} T_s g(x)ds}{\int_0^u T_s g(x)ds}\right] \geqq 1 \end{aligned}$$

on A. On the other hand, since $|T_t g|_1 \to 0$ as $t \to \infty$, we may choose a subsequence $(T_{t_n}g)$ with $\sum_{n=1}^\infty T_{t_n}g \in L_1$. Then $0 \leqq \sum_{n=1}^\infty R(T_{t_n}g) \leqq R(\sum_{n=1}^\infty T_{t_n}g) < \infty$ μ-a.e.; hence $\lim_n R(T_{t_n}g) = 0$ μ-a.e., but this contradicts (2.13).

3. Local behavior. Akcoglu and Chacon [1] have shown that for a semi-group $\Gamma = \{T_t: t \geqq 0\}$ of positive linear contractions in $L_1(X, \mathscr{A}, \mu)$, there is a decomposition of the space X into an 'initially conservative part', C, and 'initially dissipative part', D. The set C may be defined as $\{x: S_u f(x) > 0$ for all $u > 0\}$, where f is any strictly positive function in $L_1(X, \mathscr{A}, \mu)$. We note that this decomposition remains valid for bounded semi-groups. The main result in [1] can be stated as follows:

THEOREM A. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a semi-group of positive linear contractions in $L_1(X, \mathscr{A}, \mu)$, strongly continuous on $(0, \infty)$. If $\in L_1$, $gf \in L_1^+$, then $\lim_{u\downarrow 0} (S_u f(x))/(S_u g(x))$ exists a.e. on the set $C \cap \{g>0\}$.*

We recall from § 2 that for a bounded semi-group $\Gamma = \{T_t: t \geqq 0\}$ of positive linear operators in L_1, strongly continuous on $(0, \infty)$, we can construct a semi-group $\Gamma' = V_t: t \geqq 0\}$ of positive linear *contractions* related to Γ defined by (2.10). Theorem A is thus applicable to Γ'. Let $X = C + D = C' + D'$ be the initial decompositions corresponding to Γ and Γ' respectively.

Theorem A applied to Γ' shows that if $f' \in L_1, g' \in L_1^+$, then $\lim_{u\downarrow 0}\int_0^u V_s f'(x)ds \Big/ \int_0^u V_s g'(x)ds$ exists a.e. on the set $C' \cap \{g' > 0\}$. For arbitrary measurable nonnegative functions f and g, we let $f' = f\cdot e$, $g' = g\cdot e$. If $f', g' \in L_1^+$, then

$$(3.1)\quad \frac{\int_0^u V_s f'(x)ds}{\int_0^u V_s g'(x)ds} = \frac{e(x)\cdot\int_0^u T_s f(x)ds}{e(x)\cdot\int_0^u T_s g(x)ds} = \frac{S_u f(x)}{S_u g(x)}$$

on the set $\left\{x: \int_0^u V_s g'(x)ds > 0 \text{ for } u > 0\right\}$, which contains the set $C' \cap$

$\{g' > 0\}$, as shown in [1], Lemma 2.3. Thus $\lim_{u\downarrow 0} (S_u f(x))/(S_u g(x))$ exists a.e., on the set $C' \cap (g' > 0\}$.

It is clear from $g' = g \cdot e$ that $\{g' > 0\} = \{g > 0\} \cap Y$. We next show that $C' = C \cap Y$. Let C be defined in terms of some fixed function $g \in L_1$, $g > 0$. For each $u > 0$,

$$\int_0^u T_s g(x)ds = \int_0^u T_s g_Y(x)ds + \int_0^u T_s g_Z(x)ds\ . \tag{3.2}$$

The last integral in (3.2) vanishes a.e. on Y since Z is T_s-closed, $s \geqq 0$. Hence $\int_0^u T_s g(x)ds = \int_0^u T_s g_Y(x)ds > 0$ on $C \cap Y$. Let $g' = g_Y \cdot e$. Then $\int_0^u V_s g'(x)ds = e(x) \cdot \int_0^u T_s g_Y(x)ds > 0$ for $u > 0$ on $C \cap Y$. This shows that $C' \supset C \cap Y$. Next, since $V_s g(x) = 0$ a.e. on Z for any $g \in L_1$, C' may be obtained as the set $\left\{x: \int_0^u V_s g(x)ds > 0 \text{ for } u > 0\right\}$ for any $g \in L_1^+$ such that $g > 0$ on Y. Let $g' = g \cdot e$. Then $g' > 0$ on Y and hence $\int_0^u V_s g'(x)ds > 0$ on C', $u > 0$. Since $\int_0^u V_s g'(x)ds = \int_0^u e(x) \cdot T_s g(x)ds$, we conclude that $\int_0^u T_s g(x)ds > 0$ a.e. on C', $u > 0$. Hence $C' \subset C \cap Y$. We have proved:

THEOREM 3.1. *Let $\Gamma = \{T_t: t \geqq 0\}$ be a bounded semi-group of positive linear operators in L_1, strongly continuous on $(0, \infty)$. If f, g are nonnegative measurable functions such that $f \cdot e$, $g \cdot e \in L_1^+$, then $\lim_{u\downarrow 0} (S_u f(x))/S_u g(x))$ exists a.e. on the set $\{g > 0\} \cap C \cap Y$.*

Of course, the restriction of the above statement to C is not a loss of generality, since on D the ratio D_u is of the form 0/0. The local behavior of D_u on Z does not seem to be easy to ascertain by the methods of the present paper.

REFERENCES

1. M. A. Akcoglu and R. V. Chacon, *A local ratio theorem*, To appear.
2. M. A. Akcoglu and J. Cunsolo, *An ergodic theorem for semi-groups*, To appear in Proc. Amer. Math. Soc.
3. K. N. Berk, *Ergodic theory with weighted averages.*, Ann. Math. Sta., **39** (1968). 1107-1114.
4. R. V. Chacon and D, S. Ornstein, *A general ergodic theorem*, Illinois J. Math., **4** (1960), 153-160.
5. N. Dunford and J. T. Schwartz, *Linear Operators I*, New York, Interscience Publ. 1966.
6. H. Fong, *On invariant functions for L_p*-operators, To appear in Colloquium Mathematicum, XXII.
7. A. Ionescu Tulcea and M. Moretz, *Ergodic properties of semi-Markovian operators on the Z-part.* Z. Wahrscheinlichkeitstheorie verw. Geb., **13** (1969), 119-122.
8. U. Krengel, *A local ergodic theorem*, To appear.

9. J. Neveu, *Mathematical Foundations of the calculus of probability,* San Francisco: Holden-Day, Inc. 1965.
10. D. S. Ornstein, *The sums of iterates of a positive operator,* To appear.
11. L. Sucheston, *On the ergodic theorem for positive operators I. II,* Z. Wahrscheinlichkeitstheorie verw. Geb., **8** (1967), 1-11, 353-356.

Received January 26, 1970. This research was supported by the NSF Grant GP-13692.

THE UNIVERSITY OF MICHIGAN
BOWLING GREEN STATE UNIVERSITY
AND
THE OHIO STATE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 39, No. 3 July, 1971