Pacific Journal of Mathematics

ORTHODOX SEMIGROUPS

THOMAS ERIC HALL

Vol. 39, No. 3 July 1971

PACIPIC JOURNAL OF MATHEMATICS
Vol. 39, No. 3, 1971

ORTHODOX SEMIGROUPS

T. E. HALL

An orthodox semigroup is a regular semigroup in which the idempotents form a subsemigroup. The purpose of this paper is to give structure theorems for orthodox semigroups in terms of inverse semigroups and bands.

A different structure theorem for orthodox semigroups in terms of bands and inverse semigroups has already been given by Yamada in [12]; two questions posed in [12] will be answered in the negative. The present paper is the "further paper" mentioned by the author in the final paragraph of §1 [5] and in the Acknowledgement of [5].

2. **Preliminaries.** We use wherever possible, and usually without comment, the notations of Clifford and Preston [2]; further, for each element a in any semigroup S we define $V(a) = \{x \varepsilon S: axa = a$ and $xax = x\}$, the set of inverses of a in S.

RESULT 1 (from Theorem 4.6 [2]). *On any band B Green's relation $\mathscr{J}$ is the finest semilattice congruence and each $\mathscr{J}$-class is a rectangular band.*

Let $\phi: B \rightarrow Y$ be any homomorphism of B onto a semilattice Y snch that $\phi \circ \phi^{-1} = \mathscr{J}$. By denoting (for all $e \in B$) J_e by E_α where $e\phi = \alpha \in Y$ we obtain B as a semilattice Y of the rectangular bands $\{E_\alpha: \alpha \in Y\}$, i.e., $B = \bigcup_{\alpha \in Y} E_\alpha$ and for all $\alpha, \beta \in Y$ $E_\alpha \cap E_\beta = \square$ if $\alpha \neq \beta$, aud $E_\alpha E_\beta \subseteqq E_{\alpha\beta}$. It is clear that $\{(e, \alpha) \in B \times Y: e\phi = \alpha\}$ is a subband of $B \times Y$ isomorphic to B.

RESULT 2 [9, Lemma 2.2]. Let *ρ be a congruence on a regular semigroup S. Then each ρ-class which is an idempotent of S/ρ contains an idempotent of S.*

RESULT 3 (from Theorem 13 [7]). *Let ρ be any congruence contained in $\mathscr{L}$ on any semigroup S. Then any elements a and b of S are $\mathscr{L}$-related in S if and only if $a\rho$ and $b\rho$ are $\mathscr{L}$-related in S/ρ.*

Henceforth we shall let S denote an arbitrary orthodox semigroup.

The following result is part of [3, Theorem 3]; as noted in [4] it had previously been obtained by Schein [10].

RESULT 4. *The relation* $\mathscr{Y} = \{(x, y) \in S \times S: V(x) = V(y)\}$ *is the finest inverse semigroup congruence on the orthodox semigroup* S.

From [3, Remark 1] we see that the partition of S induced by $\mathscr{Y}$ is $\{V(x): x \in S\}$. Denote the band of S by B. Then we also have from [3, Remark 1] that for any $e \in B$, $e\mathscr{Y} = J_e$ (where J_e is the $\mathscr{J}$-class of B containing e) whence, from Result 2, the semilattice of $S/\mathscr{Y}$ is $B/\mathscr{J}$ ($\mathscr{J}$ being Green's relation $\mathscr{J}$ on B).

For the remainder of this section $\mathscr{L}$ and $\mathscr{R}$ shall denote Green's relations $\mathscr{L}$ and $\mathscr{R}$ on B; as usual then L_x and R_x shall denote the $\mathscr{L}$-class and $\mathscr{R}$-class respectively of B containg an element x from B.

RESULT 5 [5, Lemma 1] or [12, Footnote 5]. *For any element* $a \in S$ *and any element* $a' \in V(a)$,

$$aV(a) = R_{aa'} \text{ and } V(a)a = L_{a'a}\,.$$

RESULT 6 [5, Lemma 2] or [12, Lemma 5]. *Take any elements* a *and* b *in* S.

Then

$$aV(a)(a\mathscr{Y})V(a)a = \{a\}$$

whence $a = b$ *if and only if the triple*

$$(aV(a), a\mathscr{Y}, V(a)a) = (bV(b), b\mathscr{Y}, V(b)b)\,.$$

Henceforth, we shall identify any one element set $\{x\}$ say, with that element x, as is usual.

We shall now present two constructions appearing in [5]; one is of a representation of S by transformations of sets and the other is of a "maximal" fundamental orthodox semigroup containing B as the band of all idempotents (a semigroup T is called *fundamental* if the only congruence contained in $\mathscr{H}$ on T is the trivial congruence). This work has been generalized to regular semigroups in [6], where in fact the proofs and presentation are simpler than in [5]. For each result that we present we shall therefore refer to results in both [5] and [6].

For each element a in S define a transformation $\rho_a \in \mathscr{T}_{B/\mathscr{L}}$, the semigroup of all transformations of the set $B/\mathscr{L}$, by

$$V(x)x\rho_a = V(xa)xa \text{ for all } x \in B$$

and define also a transformation λ_a in $\mathscr{T}_{B/\mathscr{R}}$ by $xV(x)\lambda_a = axV(ax)$ for all $x \in B$.

That ρ_a and λ_a are transformations is shown in [5, Section 3] and also follows from [6, Remark 4]. Let (ρ, λ) be the mapping of S into $\mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}_{B/\mathscr{R}}^{*}$ (where $\mathscr{T}_{B/\mathscr{R}}^{*}$ is the semigroup dual to $\mathscr{T}_{B/\mathscr{R}}$) which takes each a in S to (ρ_a, λ_a).

We define now an equivalence relation $\mathscr{U}$ on B by $\mathscr{U} = \{(e, f) \in B \times B: eBe \cong fBf\}$ and for each pair $(e, f) \in \mathscr{U}$ we let $T_{e,f}$ be the set of all isomorphisms from eBe onto fBf; for each $\alpha \in T_{e,f}$ we define further transformations $\bar{\alpha} \in \mathscr{T}_{B/\mathscr{L}}$ and $\bar{\bar{\alpha}} \in \mathscr{T}_{B/\mathscr{R}}$ [6, Section 5] by

$$L_x\bar{\alpha} = L_{x\alpha} \text{ and } R_x\bar{\bar{\alpha}} = R_{x\alpha} \text{ for all } x \in eBe .$$

Further, let us consider the transformations $\rho_e\bar{\alpha}$ and $\lambda_f\overline{\overline{\alpha^{-1}}}$ (products being taken in $\mathscr{P}\mathscr{T}_{B/\mathscr{L}}$ and $\mathscr{P}\mathscr{T}_{B/\mathscr{R}}$ respectively) and let us put $(\rho_e\bar{\alpha}, \lambda_f\overline{\overline{\alpha^{-1}}}) = \phi(\alpha)$ say. Define now

$$W(B) = \bigcup_{(e,f)\in\mathscr{U}} \{(\rho_e\bar{\alpha}, \lambda_f\overline{\overline{\alpha^{-1}}}): \alpha \in T_{e,f}\} .$$

RESULT 7.

(i) *The set* $W(B)$ *is a subsemigroup of* $\mathscr{T}_{B/\mathscr{L}} \times \mathscr{T}_{B/\mathscr{R}}^{*}$.

(ii) *Further,* $W(B)$ *is a fundamental orthodox semigroup whose band of idempotents is isomorphic to* B.

(iii) *The mapping* (ρ, λ) *is a homomorphism of* S *into* $W(B)$ *which maps* B *isomorphically onto the band of idempotents of* $W(B)$.

(iv) *The congruence* $(\rho, \lambda)\circ(\rho, \lambda)^{-1}$ *is the maximum congruence contained in* $\mathscr{H}$ *on* S.

Result 7 can be obtained by the specialization to orthodox semigroups of the following results on regular semigroups from [6]: Lemma 4, Theorem 7 and Theorem 18 (vii). Alternatively, except for part (iii), Result 7 is contained in Theorems 1 and 5 of [5].

RESULT 8 [5, Theorem 2]. *Take any elements* $a, b \in S$. *Then* $a = b$ *if and only if the triple*

$$(\lambda_a, a\mathscr{Y}, \rho_a) = (\lambda_b, b\mathscr{Y}, \rho_b) .$$

3. The structure theorems.

LEMMA 1. *The mapping from* S *into* $W(B) \times (S/\mathscr{Y})$ *which maps each element* a *in* S *to* $((\rho_a, \lambda_a), a\mathscr{Y})$ *is an isomorphism.*

Proof. From Results 4 and 7 (iii) we see that the mapping is a homomorphism and from Result 8 we see that it is one-to-one.

Let now E be any band and define $W(E)$ as above. Let (ρ', λ') be the homomorphism of E into $W(E)$ which corresponds to the homomorphism (ρ, λ) of S into $W(B)$ above. From Result 7(iii) (ρ', λ') is an isomorphism from E onto the band of all idempotents of $W(E)$.

Let us denote the band of $W(E)$ by $\bar{E}$ and for each $e \in E$ let us denote $e(\rho', \lambda')$ simply by $\bar{e}$. Let $\mathscr{Y}_1$ denote the finest inverse semigroup congruence on $W(E)$, as given by Result 4.

Now let T be any inverse semigroup such that there is an idempotent-separating homomorphism ψ say, from T into $W(E)/\mathscr{Y}_1$ whose range contains all the idempotents of $W(E)/\mathscr{Y}_1$; if we let Y denote the semilattice of T then from Result 2 $\psi | Y$ maps Y isomorphically onto the semilattice of $W(E)/\mathscr{Y}_1$.

Let $\mathscr{Y}_1^\natural$ denote the natural homomorphism [2, Section 1.5] of $W(E)$ onto $W(E)/\mathscr{Y}_1$; then $x\mathscr{Y}_1^\natural = x\mathscr{Y}_1$ for any $x \in W(E)$.

Considering Green's relation $\mathscr{J}$ on $\bar{E}$ we have from § 2 that

$$(\mathscr{Y}_1^\natural | \bar{E}) \circ (\mathscr{Y}_1^\natural | \bar{E})^{-1} = \mathscr{J} \text{ whence}$$
$$[(\mathscr{Y}_1^\natural | \bar{E})(\psi | Y)^{-1}] \circ [(\mathscr{Y}_1^\natural | \bar{E})(\psi | Y)^{-1}]^{-1} = \mathscr{J}$$

and so we may index (Result 1) the $\mathscr{J}$-classes of $\bar{E}$ with the elements of Y as follows: for all $\bar{e} \in \bar{E}$ if $\bar{e}\ \mathscr{Y}_1^\natural(\psi | Y)^{-1} = \alpha \in Y$ then denote $J_{\bar{e}}$ by $\bar{E}_\alpha$.

Similarly, considering Green's relation $\mathscr{J}$ on E and denoting $(\rho', \lambda')(\mathscr{Y}_1^\natural | \bar{E})(\psi | Y)^{-1}$ by ξ we have $\xi \circ \xi^{-1} = \mathscr{J}$ whence we may index the $\mathscr{J}$-classes of E with the elements of Y as follows: for all $e \in E$ if $e\xi = \alpha \in Y$ then denote J_e by E_α. Clearly $e \in E_\alpha$ implies $\bar{e} \in \bar{E}_\alpha$ for all $e \in E$.

Define now $S_1 = S_1(E, T, \psi)$ by

$$S_1 = \{(x, t) \in W(E) \times T : x\mathscr{Y}_1 = t\psi\} .$$

THEOREM 1.

(i) *The set $S_1 = S_1(E, T, \psi)$ is an orthodox subsemigroup of $W(E) \times T$, and conversely every orthodox semigroup is obtained in this way.*

(ii) *The band of S_1 is isomorphic to E.*

(iii) *The maximum inverse semigroup homomorphic image of S_1 is isomorphic to T.*

(iv) *For each element $x \in W(E)$ let $(xV(x), x\mathscr{Y}_1, V(x)x)$ denote x. Then*

$$\mathrm{S}_1 = \{((R_{\bar{e}}, t\psi, L_{\bar{f}}), t) : t \in T, \bar{e} \in \bar{E}_{tt^{-1}}, \bar{f} \in \bar{E}_{t^{-1}t}\} ,$$

where $R_{\bar{e}}$ and $L_{\bar{f}}$ are the $\mathscr{R}$-class and $\mathscr{L}$-class respectively of $\bar{E}$ containing $\bar{e}$ and $\bar{f}$ respectively.

Proof.

(i) Take any elements (x, t), (y, u) in S_1. Then

$$(xy)\mathscr{Y}_1^\natural = (x\mathscr{Y}_1^\natural)(y\mathscr{Y}_1^\natural) = (t\psi)(u\psi) = (tu)\psi$$

whence $(x, t)(y, u) = (xy, tu) \in S_1$ and S_1 is a subsemigroup of $W(E) \times T$. Now the set of inverses of (x, t) in $W(E) \times T$ is $V(x) \times \{t^{-1}\}$ (where of course $V(x)$ denotes the set of inverses of x in $W(E)$); take any $(x', t^{-1}) \in V(x) \times \{t^{-1}\}$. Then $x'\mathscr{Y}_1$ and $t^{-1}\psi$ are both inverses of $x\mathscr{Y}_1 = t\psi$ in $W(E)/\mathscr{Y}_1$ whence $x'\mathscr{Y}_1 = t^{-1}\psi$ and $V(x) \times \{t^{-1}\} \subseteqq S_1$. In particular S_1 is regular. Since $W(E) \times T$ is orthodox we now have that S_1 is orthodox.

Conversely, consider again the orthodox semigroup S of §2. Let $\mathscr{Y}_2$ be the finest inverse semigroup congruence on $W(B)$. Then $S(\rho, \lambda)\mathscr{Y}_2^\natural$ is an inverse semigroup homomorphic image of S so

$$\mathscr{Y} \subseteqq [(\rho, \lambda)\mathscr{Y}_2^\natural]\circ[(\rho, \lambda)\mathscr{Y}_2^\natural]^{-1} .$$

Let θ be the unique homomorphism from $S/\mathscr{Y}$ onto $S(\rho, \lambda)\mathscr{Y}_2^\natural$ such that $\mathscr{Y}^\natural\theta = (\rho, \lambda)\mathscr{Y}_2^\natural$ [2, Theorem 1.6].
The semilattices of $S/\mathscr{Y}$ and $S(\rho, \lambda)\mathscr{Y}_2^\natural$ are $B\mathscr{Y}^\natural$ and $B(\rho, \lambda)\mathscr{Y}_2^\natural$ respectively (Result 2), and moreover (for $\mathscr{J}$ on B)

$$(\mathscr{Y}^\natural | B)\circ(\mathscr{Y}^\natural | B)^{-1} = \mathscr{J} = [((\rho, \lambda)\mathscr{Y}_2^\natural) | B]\circ[((\rho, \lambda)\mathscr{Y}_2^\natural) | B]^{-1}$$

so θ maps $B\mathscr{Y}^\natural$ one-to-one onto $B(\rho, \lambda)\mathscr{Y}_2^\natural$. Thus $S_1(B, S/\mathscr{Y}, \theta)$ is defined, and further, for all $a \in S$, $((\rho_a, \lambda_a), a\mathscr{Y}) \in S_1(B, S/\mathscr{Y}, \theta)$ since $(a\mathscr{Y})\theta = a(\rho, \lambda)\mathscr{Y}_2^\natural = (\rho_a, \lambda_a)\mathscr{Y}_2$.

Take now any element $(x, a\mathscr{Y}) \in S_1(B, S/\mathscr{Y}, \theta)$, where $a \in S$.
Then

$$x\mathscr{Y}_2 = (a\mathscr{Y})\theta = a(\rho, \lambda)\mathscr{Y}_2^\natural = (\rho_a, \lambda_a)\mathscr{Y}_2$$

whence $V(x) = V((\rho_a, \lambda_a))$ in $W(B)$. Take any $a' \in V(a)$ in S. Then $(\rho_{a'}, \lambda_{a'}) \in V(x)$ in $W(B)$ and from Result 7 (iii)

$$(\rho_{a'}, \lambda_{a'})x = (\rho_e, \lambda_e) \text{ and } x(\rho_{a'}, \lambda_{a'}) = (\rho_f, \lambda_f)$$

for some idempotents $e, f \in S$. Then $(\rho_e, \lambda_e)\mathscr{R}(\rho_{a'}, \lambda_{a'})\mathscr{L}(\rho_f, \lambda_f)$ in $W(B)$ whence $e\mathscr{R}a'\mathscr{L}f$ in S (from Result 7 (iv), Result 3 and the result dual to Result 3). From [2, Theorem 2.18] there is an inverse b say, of a' in S, such that $e\mathscr{L}b\mathscr{R}f$ in S. Thus $(\rho_e, \lambda_e)\mathscr{L}(\rho_b, \lambda_b)\mathscr{R}(\rho_f, \lambda_f)$ in $W(B)$; but also $(\rho_e, \lambda_e)\mathscr{L}x\mathscr{R}(\rho_f, \lambda_f)$ in $W(B)$ and both x and (ρ_b, λ_b) are inverses of $(\rho_{a'}, \lambda_{a'})$ in $W(B)$, so from [2, Theorem 2.18] $x = (\rho_b, \lambda_b)$. Note also that $b\mathscr{Y} = a\mathscr{Y}$ (since both are inverses of $a'\mathscr{Y}$ in $S/\mathscr{Y}$). Thus $(x, a\mathscr{Y}) = ((\rho_b, \lambda_b), b\mathscr{Y})$. With an observation above this gives that

$$S_1(B, S/\mathscr{Y}, \theta) = \{((\rho_a, \lambda_a), a\mathscr{Y}) \in W(B) \times (S/\mathscr{Y}) : a \in S\} .$$

From Lemma 1 we have that S is isomorphic to $S_1(B, S/\mathscr{Y}, \theta)$.

(ii) Take any idempotent (x, α) say, in $S_1 = S_1(E, T, \psi)$. Then $x^2 = x$, $\alpha^2 = \alpha$ and $x\mathscr{Y}_1 = \alpha\psi$ whence $x\mathscr{Y}_1^\natural(\psi | Y)^{-1} = \alpha$ and so $x \in \bar{E}_\alpha$. Conversely, for any $\alpha \in Y$ and $x \in \bar{E}_\alpha$ we have $x\mathscr{Y}_1^\natural(\psi \mid Y)^{-1} = \alpha$ whence $x\mathscr{Y}_1 = \alpha\psi$ and $(x, \alpha) \in S_1$. Thus the band of idempotents of S_1 is $\{(x, \alpha) \in \bar{E} \times Y: \alpha \in Y, x \in \bar{E}_\alpha\}$, which is clearly isomorphic to $\bar{E}$ (Section 2).

(iii) Let π_2: $S_1 \to T$ be the function satisfying $(x, t)\pi_2 = t$ for all $(x, t) \in S_1$, and let $\mathscr{Y}_3$ denote the finest inverse semigroup congruence on S_1. Then π_2 is a homomorphism onto T, an inverse semigroup, whence $\mathscr{Y}_3 \subseteqq \pi_2 \circ \pi_2^{-1}$.

Since from the proof of (i) the set of inverses of any element (x, t) in S_1 is $V(x) \times \{t^{-1}\}$ we have that

$\mathscr{Y}_3 = \{((x, t), (y, t)) \in S_1 \times S_1: V(x) = V(y) \text{ in } W(E)\}$. But for any (x, t), (y, t) in S_1 we have $x\mathscr{Y}_1 = t\psi = y\mathscr{Y}_1$ whence $V(x) = V(y)$ in $W(E)$. Thus $\pi_2 \circ \pi_2^{-1} \subseteqq \mathscr{Y}_3$, giving $\pi_2 \circ \pi_2^{-1} = \mathscr{Y}_3$ and $S_1/\mathscr{Y}_3$ is isomorphic to $S_1\pi_2 = T$.

(iv) We note that it is Result 6 which enables us to let $(xV(x), x\mathscr{Y}_1, V(x)x)$ denote x, for each $x \in W(E)$.

Take any element $(x, t) \in S_1$. Considering Green's relations $\mathscr{R}$ and $\mathscr{L}$ on $\bar{E}$ we have

$$(x, t) = ((xV(x), x\mathscr{Y}_1, V(x)x), t) = ((R_{xx'}, t\psi, L_{x'x}), t)$$

for any $x' \in V(x)$, from Result 5. Now $t^{-1}\psi = (t\psi)^{-1}$ and $x'\mathscr{Y}_1 = (x\mathscr{Y}_1)^{-1} = (t\psi)^{-1}$ so

$$(xx')\mathscr{Y}_1^\natural = (x\mathscr{Y}_1^\natural)(x'\mathscr{Y}_1^\natural) = (t\psi)(t\psi)^{-1} = (t\psi)(t^{-1}\psi) = (tt^{-1})\psi$$

giving that $(xx')\mathscr{Y}_1^\natural(\psi \mid Y)^{-1} = tt^{-1}$ and $xx' \in \bar{E}_{tt^{-1}}$. Similary $x'x \in \bar{E}_{t^{-1}t}$ and so

$$S_1 \subseteqq \{((R_{\bar{e}}, t\psi, L_{\bar{f}}), t): t \in T, \bar{e} \in \bar{E}_{tt^{-1}}, \bar{f} \in \bar{E}_{t^{-1}t}\} .$$

Conversely take any $t \in T$ and any $\bar{e} \in \bar{E}_{tt^{-1}}$ and $\bar{f} \in \bar{E}_{t^{-1}t}$; then $\bar{e}\mathscr{Y}_1 = (tt^{-1})\psi$ and $\bar{f}\mathscr{Y}_1 = (t^{-1}t)\psi$. Consider $((R_{\bar{e}}, t\psi, L_{\bar{f}}), t)$. Take any element $x \in W(E)$ such that $x\mathscr{Y}_1 = t\psi$. Then $(\bar{e}x\bar{f})\mathscr{Y}_1^\natural = (\bar{e}\mathscr{Y}_1^\natural)(x\mathscr{Y}_1^\natural)(\bar{f}\mathscr{Y}_1^\natural) = [(tt^{-1})\psi](t\psi)[(t^{-1}t)\psi] = t\psi$. Take any $x' \in V(x)$ and put $\bar{e}x\bar{f} = y$ and $\bar{f}x'\bar{e} = y'$. Then $y' \in V(y)$[10, Theorem 1.10], whence $y'\mathscr{Y} = (t\psi)^{-1} = t^{-1}\psi$. Thus $(yy')\mathscr{Y}_1 = (tt^{-1})\psi$ giving $yy' \in \bar{E}_{tt^{-1}}$ and similarly $y'y \in \bar{E}_{t^{-1}t}$. Now $\bar{e}, yy' \in \bar{E}_{tt^{-1}}$, a rectangular band, so

$$yy' = (\bar{e}x\bar{f})(\bar{f}x'\bar{e}) = \bar{e}yy'\bar{e} = \bar{e}$$

and similarly $y'y = \bar{f}$. Thus

$$((R_{\bar{e}}, t\psi, L_{\bar{f}}), t) = ((yV(y), y\mathscr{Y}_1, V(y)y), t) = (y, t) \in S_1 .$$

Therefore

$$S_1 = \{((R_{\bar{e}}, t\psi, L_{\bar{f}}), t): t \in T, \bar{e} \in \bar{E}_{tt^{-1}}, \bar{f} \in \bar{E}_{t_{-1}t}\} .$$

REMARK 1. Let Z denote the semilattice of $S/\mathscr{Y}$ and index the $\mathscr{J}$-classes of B with the elements of Z in the natural way. For each element $a \in S$ let $(aV(a), a\mathscr{Y}, V(a)a)$ denote a and consider the $\mathscr{R}$ and $\mathscr{L}$-classes of B. Then the method used to prove (iv) also gives that

$$S = \{(R_e, v, L_f) \in (B/\mathscr{R}) \times (S/\mathscr{Y}) \times (B/\mathscr{L})\colon v \in S/\mathscr{Y}, e \in E_{vv^{-1}} \text{ and } f \in E_{v^{-1}v}\} .$$

COROLLARY 1 (to the proof). *Consider the arbitrary band E and any inverse semigroup U. Then there exists an orthodox semigroup whose band is E and whose maximum inverse semigroup image is isomorphic to U if and only if there is a homomorphism from U into $W(E)/\mathscr{Y}_1$ which maps the idempotents of U one-to-one onto the idempotents of $W(E)/\mathscr{Y}_1$.*

Let us now define a subset $S_2 = S_2(E, T, \psi)$ of $(E/\mathscr{R}) \times T \times (E/\mathscr{L})$ by

$$S_2 = \{(R_e, t, L_f)\colon t \in T, e \in E_{tt^{-1}} \text{ and } f \in E_{t^{-1}t}\} .$$

Take any element (R_e, t, L_f) in S_2. Then $\bar{e} \in \bar{E}_{tt^{-1}}$ and $\bar{f} \in \bar{E}_{t^{-1}t}$ whence $((R_{\bar{e}}, t\psi, L_{\bar{f}}), t) \in S_1$, where $R_{\bar{e}}$ and $L_{\bar{f}}$ are the $\mathscr{R}$-class and $\mathscr{L}$-class respectively of $\bar{E}$ containing $\bar{e}$ and $\bar{f}$ respectively. Clearly now we may define a mapping Ψ of S_2 into S_1 by

$$(R_e, t, L_f)\Psi = ((R_{\bar{e}}, t\psi, L_{\bar{f}}), t)$$

for any element $(R_e, t, L_f) \in S_2$. It is also clear that Ψ is one-to-one and it is routine to show that Ψ is onto S_1. Thus Ψ is a one-to-one correspondence between S_2 and S_1.

Let us denote by juxtaposition the unique multiplication on S_2 which makes Ψ an isomorphism from S_2 onto S_1; then for any elements (R_e, t, L_f) and (R_g, u, L_h) in S_2

$$(R_e, t, L_f)(R_g, u, L_h) = [(R_e, t, L_f)\Psi(R_g, u, L_h)\Psi]\Psi^{-1} .$$

From Result 6 and Theorem 1 (iv) $((R_{\bar{e}}, t\psi, L_{\bar{f}}), t)$ denotes the element $(R_{\bar{e}}(t\psi)L_{\bar{f}}, t)$ of S_1; thus

$$(R_e, t, L_f)\Psi = (R_{\bar{e}}(t\psi)L_{\bar{f}}, t)$$

for any element (R_e, t, L_f) in S_2.

For each idempotent $x \in W(E)$ let $\tilde{x}$ denote $x(\rho', \lambda')^{-1}$; then $\bar{\tilde{x}} = x$ for all $x \in \bar{E}$ and $\tilde{\bar{e}} = e$ for all $e \in E$. Then for any elements (R_e, t, L_f) and (R_g, u, L_h) in S_2

$$(R_e, t, L_f)(R_g, u, L_h) = (R_{\widetilde{zz'}}, tu, L_{\widetilde{z'z}})$$

where (in $W(E)$) $R_{\bar{e}}(u\psi)L_{\bar{f}} = x$, $R_{\bar{g}}(u\psi)L_{\bar{h}} = y$, $xy = z$ and $z' \in V(z)$; this is because $(tu)\psi = (xy)\mathscr{Y}_1 = z\mathscr{Y}_1$ and

$$(R_{\widetilde{zz'}}, tu, L_{\widetilde{z'z}})\Psi = ((R_{zz'}, (tu)\psi, L_{z'z}), tu) = (z, tu) = (xy, tu) .$$

We restate these facts in the next theorem.

THEOREM 2. *Let $S_2 = S_2(E, T, \psi)$ be the subset of $(E/\mathscr{R}) \times T \times (E/\mathscr{L})$ given by*

$S_2 = \{(R_e, t, L_f)\colon t \in T, e \in E_{tt^{-1}}$ and $f \in E_{t^{-1}t}\}$ and let a multiplication on S_2 be given by (for any elements (R_e, t, L_f) and (R_g, u, L_h) in S_2)

$$(R_e, t, L_f)(R_g, u, L_h) = (R_{\widetilde{zz'}}, tu, L_{\widetilde{z'z}})$$

where (for the $\mathscr{R}$ and $\mathscr{L}$-classes of $\bar{E}$ we have) $R_{\bar{e}}(t\psi)L_{\bar{f}} = x$, $R_{\bar{g}}(u\psi)L_{\bar{h}} = y$, $xy = z$ and $z' \in V(x)$(all in $W(E)$). Then $S_2(E, T, \psi)$ is a semigroup isomorphic to $S_1(E, T, \psi)$.

4. Some counter-examples.

4.1. Let T denote the bicyclic semigroup [2, Section 1.12]. We shall construct a band B which is an ω-chain of rectangular bands and such that there is no orthodox semigroup S with band B and with T as a homomorphic image.

Let Y be the semilattice of T; then Y is an ω-chain. For each $\alpha \in Y$ let E_α be a rectangular band such that, for all $\alpha, \beta \in Y$, if $\alpha \neq \beta$ then $E_\alpha \cap E_\beta = \square$ and $|E_\alpha| \neq |E_\beta|$. Put $B = \bigcup_{\alpha \in Y} E_\alpha$ and, following Clifford [1] extend the multiplications of the bands $\{E_\alpha\colon \alpha \in Y\}$ to a multiplication for B as follows: for any $e, f \in B$, where $e \in E_\alpha$ and $f \in E_\beta$ say, define

$$ef = \begin{cases} e \text{ if } \alpha < \beta \\ ef \text{ as in } E_\alpha \text{ if } \alpha = \beta \\ f \text{ if } \alpha > \beta. \end{cases}$$

Note that if $\alpha > \beta$ then $ef = fe = f$. It is routine to show that this multiplication is associative (alternatively see [8]) and that then the band B is an ω-chain Y of the rectangular bands $\{E_\alpha\colon \alpha \in Y\}$. Also, if $e \in E_\alpha$ and $f \in E_\beta$ $(\alpha, \beta \in Y)$ then $eBe = \{e\} \cup (\bigcup_{\gamma<\alpha} E_\gamma)$ whence eBe is isomorphic to fBf if and only if $\alpha = \beta$. From [5, Main Theorem] any orthodox semigroup, S say, with band B is a union of groups. But any homomorphic image of a semigroup which is a union of groups is also a union of groups; thus T is not the maximum inverse semigroup homomorphic image of S.

REMARK 2. The band B just defined is one of a class of bands called, by the author, almost commutative bands; a band E is called *almost commutative* if, for any $e, f \in E$, $J_e \neq J_f$ implies $ef = fe$. It is easily shown (See [8])) that a band E is almost commutative if and only if, for $e, f \in E$, $J_e > J_f$ implies $e > f$ (where $J_e > J_f$ means that $E^1eE^1 \supset E^1fE^1$ [2, Section 2.1] and $e > f$ means that $ef = fe = f \neq e$[2, Section 1.8]). A determination of the structure of almost commutative bands in terms of semilattices is given in [8].

REMARK 3. The band B and inverse semigroup T above answer in the negative the first question posed on page 269 [12]. We now briefly give alternative examples of a different nature. Let E consist of the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let T_1 consist of the matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Under matrix multiplication E is a band, T_1 is an inverse semigroup with semilattice isomorphic to $E/\mathscr{J}$, and there is no orthodox semigroup S say, with band E and such that $S/\mathscr{Y}$ is isomorphic to T_1.

4.2. We now give two non-isomorphic orthodox semigroups S_1 and S_2 whose bands are isomorphic and whose maximum inverse semigroup homomorphic images are isomorphic. This answers the second question on page 269 [12] in the negative. The referee has pointed out that this question has also been essentially answered in the last remark of Yamada [13].

Let S_1 consist of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

and let S_2 consist of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Under matrix multiplication S_1 and S_2 are orthodox semigroups.

The bands of S_1 and S_2 are both two-element left zero semigroups with an identity adjoined and the maximum inverse semigroup homomorphic images are both two-element groups with a zero adjoined. But $\mathscr{H}$ is a congruence on S_2 and not on S_1, so S_1 and S_2 are not isomorphic.

REFERENCES

1. A. H. Clifford, *Naturally totally ordered commutative semigroups*, Amer. J. Math., **76** (1954), 631-646.

2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vols. I and II, Amer. Math. Soc., Providence, R. I., 1961 and 1967.

3. T. E. Hall, *On regular semigroups whose idempotents form a subsemigroup*, Bull. Austral. Math. Soc., **1** (1969), 195-208.

4. ———, *On regular semigroups whose idempotents form a subsemigroup: Addenda*, Bull. Austral. Math. Soc., **3** (1970), 287-288.

5. ———, *On orthodox semigroups and uniform and antiuniform bands*, J. Algebra, **16** (1970), 204-217.

6. ———, *On regular semigroups*, J. Algebra, (to appear).

7. ———, *Congruences and Green's relations on regular semigroups*, Glasgow Math. J., (to appear).

8. ———, *Almost commutative bands*, Glasgow Math. J., (to appear).

9. G. Lallement, *Congruences et équivalences de Green sur un demi-groupe régulier*, C. R. Acad. Sci. Paris, Série A, **262** (1966), 613-616.

10. B. M. Schein, *On the theory of generalized groups and generalized heaps*, (Russian), *Theory of semigroups and appl. I* (Russian), 286-324, (Izdat. Saratov. Univ., Saratov, 1965).

11. M. Yamada, *Regular semigroups whose idempotents satisfy permutation identities*, Pacific J. Math., **21** (1967), 371-392.

12. ———, *On a regular semigroup in which the idempotents form a band*, Pacific J. Math., **33** (1970), 261-272.

13. ———, *Construction of inversive semigroups*, Mem. Fac. Lit. & Sci., Shimane Univ., Nat. Sci., **4** (1971), 1-9.

Received February 12, 1971 and in revised form August 2, 1971. This research was supported by a Nuffield Travelling Fellowship. The author thanks the referee for references [1] and [13] and for suggesting several improvements.

UNIVERSITY OF STIRLING

SCOTLAND

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. The editorial "we" must not be used in the synopsis, and items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. Index to Vol. **39**. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 270, 3-chome Totsuka-cho, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 39, No. 3 July, 1971