

Pacific Journal of Mathematics

***C* θ *\theta*-GROUPS INVOLVING NO SUZUKI GROUPS**

MARCEL HERZOG

$C\theta\theta$ -GROUPS INVOLVING NO SUZUKI GROUPS

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In the terminology of G. Higman, a finite group with order divisible by 3 in which centralizers of 3-elements are 3-groups is called a $C\theta\theta$ -group.

The aim of this paper is to classify simple $C\theta\theta$ -groups which involve no Suzuki simple groups.

Although simple $C\theta\theta$ -groups have been studied by several authors, their general classification remains an unsolved problem.

We will prove the following

THEOREM. *Let G be a simple $C\theta\theta$ -group and suppose that G involves no Suzuki simple groups. Then G is isomorphic to one of the following groups: $PSL(2, 4)$; $PSL(2, 8)$; $PSL(3, 4)$; $PSL(2, 3^n)$, $n > 1$ and $PSL(2, q)$, q such that $(q+1)/2$ or $(q-1)/2$ is a power of 3.*

It follows immediately from the Theorem that the following characterization of $PSL(2, 8)$ holds:

COROLLARY 1. *Let G satisfy the assumptions of the Theorem and suppose that no element of G of order 3 normalizes a nontrivial 2-subgroup of G . Then $G \cong PSL(2, 8)$.*

The Theorem leads also to a complete classification of simple $C\theta\theta$ -groups whose order is divisible by at most four distinct primes. We have

COROLLARY 2. *Let G be a simple $C\theta\theta$ -group and suppose that $|\pi(G)| = 3$. Then G is isomorphic to one of the following groups: $PSL(2, 4)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ and $PSL(2, 17)$,*

and

COROLLARY 3. *Let G be a simple $C\theta\theta$ -group and suppose that $|\pi(G)| = 4$. Then G is isomorphic to one of the following groups: $PSL(3, 4)$ and those among $PSL(2, 3^n)$, $n > 1$ and $PSL(2, q)$, $q \pm 1 = 2 \cdot 3^r$, $r > 1$, which are divisible by exactly four distinct primes.*

2. Proof of the Theorem. We will prove first the following

PROPOSITION. *Let G be a nonsolvable $C\theta\theta$ -group. Then at least one of the following statements holds.*

(i) *Whenever a section K/M of G is isomorphic to a minimal simple group L , then either L is a Suzuki group or $M = \{1\}$ and L is $PSL(2, 8)$.*

(ii) *Some nontrivial 2-subgroup of G is normalized by an element of order 3.*

Proof of the Proposition. Let G be a counter-example. Then there exist subgroups K and M of G , M normal in K , such that K/M is isomorphic to a minimal simple group L which is not of Suzuki type and if $M = \{1\}$ then L is not $PSL(2, 8)$. By Thompson's theorem [5, Corollary 1] L is one of the following: $PSL(2, 2^p)$, p any prime, $PSL(2, 3^p)$, p any odd prime; $PSL(2, p)$, p any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod{5}$ and $PSL(3, 3)$. Denote by Q the Sylow 3-subgroup of K . Since a Sylow 3-subgroup of a nonsolvable $C\theta\theta$ -group is abelian [1, Theorem 2.9], L is not $PSL(3, 3)$ and Q is the centralizer in K of each of its nonunit elements. Suppose that there exists a normal complement S of $N_K(Q)$ in K . Since M is a maximal normal subgroup of K , it follows that either $K = N_K(Q)M$ or $K = SM$, and consequently $L = K/M$ has a normal (possibly trivial) Sylow 3-subgroup, a contradiction. Thus $N_K(Q)$ has no normal complement in K and by [2, Theorem 2.3.e] the fact that 3 divides the order of L implies that 3 does not divide the order of M . It follows then by the results of Stewart¹, [4, Propositions 3.2 and 4.2] that $M = \{1\}$ if $L = PSL(2, q)$, where $q = 3^p$ or $q = p > 5$ and M is a 2-group if $L = PSL(2, q)$, where $q = 2^p$. Since no element of order 3 in G normalizes a nontrivial 2-subgroup of G , $M = \{1\}$ in all cases and L is not $PSL(2, 8)$. It is well known that the Sylow 2-subgroups of $PSL(2, q)$, where $q = 3^p$, $p > 2$ or $q = p > 3$, p a prime, are normalized by an element of order 3. Consequently, $L = PSL(2, 2^p)$, where p is a prime exceeding 3. Since the Sylow 3-subgroups in L are the centralizers of each of their nonidentity elements, it follows that $2^p \pm 1 = 3^k$ for some k . This equation has no solution for $p > 3$ and consequently G does not exist.

We proceed with a proof of the Theorem. If case (ii) of the Proposition holds, then it follows from the results of Fletcher and Gorenstein [1, Corollary 3.2] that G is isomorphic to one of the groups mentioned in the Theorem, $PSL(2, 8)$ excluded. If case (i) of the Proposition holds, but not case (ii), then we will show that G is an N -group and it follows then from Thompson's classification theorem of

¹ The author is indebted to Dr. W. B. Stewart for communication of results prior to publication.

simple N -groups [5] that only $PSL(2, 8)$ is a $C\theta\theta$ -group of the required type.

Let U be a p -subgroup of G with a nonsolvable normalizer. By our assumptions and by the Proposition $N = N_G(U)$ contains a subgroup V isomorphic to $PSL(2, 8)$. As $V \cap U = \{1\}$, VU/U is isomorphic to $PSL(2, 8)$ and consequently $U = \{1\}$. Thus G is an N -group and the proof is complete.

3. Proof of the corollaries. Since $PSL(2, 8)$ is the only group mentioned in the Theorem without an element of order 3 normalizing a nontrivial 2-subgroup, Corollary 1 immediately follows from the Theorem.

If $|\pi(G)| = 3$ then G does not involve Suzuki groups and by the Theorem and [3, Theorem 3] it is isomorphic to one of the following: $PSL(2, 4)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ and $PSL(2, 17)$.

Corollary 3 follows from the fact that 3 divides the order of G and 3 does not divide the order of the Suzuki groups. Consequently, as the Suzuki groups have orders divisible by at least 4 distinct primes, G does not involve them. Corollary 3 follows therefore immediately from the Theorem.

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Pacific Journal of Mathematics

Vol. 39, No. 3

July, 1971

William O'Bannon Alltop, <i>5-designs in affine spaces</i>	547
B. G. Basmaji, <i>Real-valued characters of metacyclic groups</i>	553
Miroslav Benda, <i>On saturated reduced products</i>	557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups. II</i>	573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, <i>Fixed points and stability for a sum of two operators in locally convex spaces</i>	581
Donald Richard Chalice, <i>Restrictions of Banach function spaces</i>	593
Eugene Frank Cornelius, Jr., <i>A generalization of separable groups</i>	603
Joel L. Cunningham, <i>Primes in products of rings</i>	615
Robert Alan Morris, <i>On the Brauer group of Z</i>	619
David Earl Dobbs, <i>Amitsur cohomology of algebraic number rings</i>	631
Charles F. Dunkl and Donald Edward Ramirez, <i>Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups</i>	637
Hicham Fakhoury, <i>Structures uniformes faibles sur une classe de cônes et d'ensembles convexes</i>	641
Leslie R. Fletcher, <i>A note on $C\theta\theta$-groups</i>	655
Humphrey Sek-Ching Fong and Louis Sucheston, <i>On the ratio ergodic theorem for semi-groups</i>	659
James Arthur Gerhard, <i>Subdirectly irreducible idempotent semigroups</i>	669
Thomas Eric Hall, <i>Orthodox semigroups</i>	677
Marcel Herzog, <i>$C\theta\theta$-groups involving no Suzuki groups</i>	687
John Walter Hinrichsen, <i>Concerning web-like continua</i>	691
Frank Norris Huggins, <i>A generalization of a theorem of F. Riesz</i>	695
Carlos Johnson, Jr., <i>On certain poset and semilattice homomorphisms</i>	703
Alan Leslie Lambert, <i>Strictly cyclic operator algebras</i>	717
Howard Wilson Lambert, <i>Planar surfaces in knot manifolds</i>	727
Robert Allen McCoy, <i>Groups of homeomorphisms of normed linear spaces</i>	735
T. S. Nanjundiah, <i>Refinements of Wallis's estimate and their generalizations</i>	745
Roger David Nussbaum, <i>A geometric approach to the fixed point index</i>	751
John Emanuel de Pillis, <i>Convexity properties of a generalized numerical range</i>	767
Donald C. Ramsey, <i>Generating monomials for finite semigroups</i>	783
William T. Reid, <i>A disconjugacy criterion for higher order linear vector differential equations</i>	795
Roger Allen Wiegand, <i>Modules over universal regular rings</i>	807
Kung-Wei Yang, <i>Compact functors in categories of non-archimedean Banach spaces</i>	821
R. Grant Woods, <i>Correction to: "Co-absolutes of remainders of Stone-Čech compactifications"</i>	827
Ronald Owen Fulp, <i>Correction to: "Tensor and torsion products of semigroups"</i>	827
Bruce Alan Barnes, <i>Correction to: "Banach algebras which are ideals in a banach algebra"</i>	828