C\theta\theta\theta\theta\theta -GROUPS INVOLVING NO SUZUKI GROUPS

MARCEL HERZOG
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In the terminology of G. Higman, a finite group with order divisible by 3 in which centralizers of 3-elements are 3-groups is called a C₀₀-group.

The aim of this paper is to classify simple C₀₀-groups which involve no Suzuki simple groups.

Although simple C₀₀-groups have been studied by several authors, their general classification remains an unsolved problem.

We will prove the following

**Theorem.** Let G be a simple C₀₀-group and suppose that G involves no Suzuki simple groups. Then G is isomorphic to one of the following groups: PSL(2,4); PSL(2,8); PSL(3,4); PSL(2,3ⁿ), n > 1 and PSL(2,q), q such that (q+1)/2 or (q−1)/2 is a power of 3.

It follows immediately from the Theorem that the following characterization of PSL(2,8) holds:

**Corollary 1.** Let G satisfy the assumptions of the Theorem and suppose that no element of G of order 3 normalizes a nontrivial 2-subgroup of G. Then G ≅ PSL(2,8).

The Theorem leads also to a complete classification of simple C₀₀-groups whose order is divisible by at most four distinct primes. We have

**Corollary 2.** Let G be a simple C₀₀-group and suppose that |π(G)| = 3. Then G is isomorphic to one of the following groups: PSL(2,4), PSL(2,7), PSL(2,8), PSL(2,9) and PSL(2,17), and

**Corollary 3.** Let G be a simple C₀₀-group and suppose that |π(G)| = 4. Then G is isomorphic to one of the following groups: PSL(3,4) and those among PSL(2,3ⁿ), n > 1 and PSL(2,q), q ± 1 = 2 · 3ⁿ, r > 1, which are divisible by exactly four distinct primes.

2. Proof of the Theorem. We will prove first the following
PROPOSITION. Let $G$ be a nonsolvable $C\theta\theta$-group. Then at least one of the following statements holds.

(i) Whenever a section $K/M$ of $G$ is isomorphic to a minimal simple group $L$, then either $L$ is a Suzuki group or $M = \{1\}$ and $L$ is $PSL(2, 8)$.

(ii) Some nontrivial 2-subgroup of $G$ is normalized by an element of order 3.

Proof of the Proposition. Let $G$ be a counter-example. Then there exist subgroups $K$ and $M$ of $G$, $M$ normal in $K$, such that $K/M$ is isomorphic to a minimal simple group $L$ which is not of Suzuki type and if $M = \{1\}$ then $L$ is not $PSL(2, 8)$. By Thompson's theorem [5, Corollary 1] $L$ is one of the following: $PSL(2, 2^p)$, $p$ any prime, $PSL(2, 3^p)$, $p$ any odd prime; $PSL(2, p)$, $p$ any prime exceeding 3 such that $p^2 + 1 \equiv 0 \pmod{5}$ and $PSL(3, 3)$. Denote by $Q$ the Sylow 3-subgroup of $K$. Since a Sylow 3-subgroup of a nonsolvable $C\theta\theta$-group is abelian [1, Theorem 2.9], $L$ is not $PSL(3, 3)$ and $Q$ is the centralizer in $K$ of each of its nonunit elements. Suppose that there exists a normal complement $S$ of $N_K(Q)$ in $K$. Since $M$ is a maximal normal subgroup of $K$, it follows that either $K = N_K(Q)M$ or $K = SM$, and consequently $L = K/M$ has a normal (possibly trivial) Sylow 3-subgroup, a contradiction. Thus $N_K(Q)$ has no normal complement in $K$ and by [2, Theorem 2.3.e] the fact that 3 divides the order of $L$ implies that 3 does not divide the order of $M$. It follows then by the results of Stewart1, [4, Propositions 3.2 and 4.2] that $M = \{1\}$ if $L = PSL(2, q)$, where $q = 3^p$ or $q = p > 5$ and $M$ is a 2-group if $L = PSL(2, q)$, where $q = 2^p$. Since no element of order 3 in $G$ normalizes a nontrivial 2-subgroup of $G$, $M = \{1\}$ in all cases and $L$ is not $PSL(2, 8)$. It is well known that the Sylow 2-subgroups of $PSL(2, q)$, where $q = 3^p$, $p > 2$ or $q = p > 3$, $p$ a prime, are normalized by an element of order 3. Consequently, $L = PSL(2, 2^p)$, where $p$ is a prime exceeding 3. Since the Sylow 3-subgroups of $L$ are the centralizers of each of their nonidentity elements, it follows that $2^p + 1 = 3^k$ for some $k$. This equation has no solution for $p > 3$ and consequently $G$ does not exist.

We proceed with a proof of the Theorem. If case (ii) of the Proposition holds, then it follows from the results of Fletcher and Gorenstein [1, Corollary 3.2] that $G$ is isomorphic to one of the groups mentioned in the Theorem, $PSL(2, 8)$ excluded. If case (i) of the Proposition holds, but not case (ii), then we will show that $G$ is an $N$-group and it follows then from Thompson's classification theorem of

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1 The author is indebted to Dr. W. B. Stewart for communication of results prior to publication.
simple $N$-groups [5] that only $PSL(2, 8)$ is a $C\theta\theta$-group of the required type.

Let $U$ be a $p$-subgroup of $G$ with a nonsolvable normalizer. By our assumptions and by the Proposition $N = N_o(U)$ contains a subgroup $V$ isomorphic to $PSL(2, 8)$. As $V \cap U = \{1\}$, $VU/U$ is isomorphic to $PSL(2, 8)$ and consequently $U = \{1\}$. Thus $G$ is an $N$-group and the proof is complete.

3. Proof of the corollaries. Since $PSL(2, 8)$ is the only group mentioned in the Theorem without an element of order 3 normalizing a nontrivial 2-subgroup, Corollary 1 immediately follows from the Theorem.

If $|\pi(G)| = 3$ then $G$ does not involve Suzuki groups and by the Theorem and [3, Theorem 3] it is isomorphic to one of the following: $PSL(2, 4)$, $PSL(2, 7)$, $PSL(2, 8)$, $PSL(2, 9)$ and $PSL(2, 17)$.

Corollary 3 follows from the fact that 3 divides the order of $G$ and 3 does not divide the order of the Suzuki groups. Consequently, as the Suzuki groups have orders divisible by at least 4 distinct primes, $G$ does not involve them. Corollary 3 follows therefore immediately from the Theorem.

REFERENCES

2. M. Herzog, On finite groups which contain a Frobenius subgroup, J. Algebra, 6 (1967), 192-221.
5. J. G. Thompson, Non-Solvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc., 74 (1968), 384-437.

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TEL-AVIV UNIVERSITY
William O’Bannon Alltop, 5-designs in affine spaces ........................................ 547
B. G. Basmaji, Real-valued characters of metacyclic groups .............................. 553
Miroslav Benda, On saturated reduced products .............................................. 557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, Uniquely representable semigroups. II ................................................................. 573
William Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, Fixed points and stability for a sum of two operators in locally convex spaces ...................... 581
Donald Richard Chalice, Restrictions of Banach function spaces .......................... 593
Eugene Frank Cornelius, Jr., A generalization of separable groups ....................... 603
Joel L. Cunningham, Primes in products of rings ............................................. 615
Robert Alan Morris, On the Brauer group of Z .................................................. 619
David Earl Dobbs, Amitsur cohomology of algebraic number rings .................... 631
Charles F. Dunkl and Donald Edward Ramirez, Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups ....................... 637
Hicham Fakhoury, Structures uniformes faibles sur une classe de cônes et d’ensembles convexes ................................................................. 641
Leslie R. Fletcher, A note on $C_{\theta\theta}$-groups ............................................... 655
Humphrey Sek-Ching Fong and Louis Sucheston, On the ratio ergodic theorem for semi-groups ................................................................. 659
James Arthur Gerhard, Subdirectly irreducible idempotent semigroups .................. 669
Thomas Eric Hall, Orthodox semigroups ......................................................... 677
Marcel Herzog, $C_{\theta\theta}$-groups involving no Suzuki groups ............................... 687
John Walter Hinrichsen, Concerning web-like continua ..................................... 691
Frank Norris Huggins, A generalization of a theorem of F. Riesz ......................... 695
Carlos Johnson, Jr., On certain poset and semilattice homomorphisms .................. 703
Alan Leslie Lambert, Strictly cyclic operator algebras ........................................ 717
Howard Wilson Lambert, Planar surfaces in knot manifolds ............................... 727
Robert Allen McCoy, Groups of homeomorphisms of normed linear spaces .......... 735
T. S. Nanjundiah, Refinements of Wallis’s estimate and their generalizations ........ 745
Roger David Nussbaum, A geometric approach to the fixed point index ................ 751
John Emanuel de Pillis, Convexity properties of a generalized numerical range ....... 767
Donald C. Ramsey, Generating monomials for finite semigroups ......................... 783
William T. Reid, A disconjugacy criterion for higher order linear vector differential equations ......................................................................................... 795
Roger Allen Wiegand, Modules over universal regular rings ............................... 807
Kung-Wei Yang, Compact functors in categories of non-archimedean Banach spaces ............................................................................................. 821
R. Grant Woods, Correction to: “Co absolutes of remainders of Stone-Čech compactifications” ................................................................. 827
Ronald Owen Fulp, Correction to: “Tensor and torsion products of semigroups” .......... 827
Bruce Alan Barnes, Correction to: “Banach algebras which are ideals in a banach algebra” ......................................................................................... 828