

# Pacific Journal of Mathematics

**$C\theta\theta$ -GROUPS INVOLVING NO SUZUKI GROUPS**

MARCEL HERZOG

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In the terminology of G. Higman, a finite group with order divisible by 3 in which centralizers of 3-elements are 3-groups is called a  $C\theta\theta$ -group.

The aim of this paper is to classify simple  $C\theta\theta$ -groups which involve no Suzuki simple groups.

Although simple  $C\theta\theta$ -groups have been studied by several authors, their general classification remains an unsolved problem.

We will prove the following

**THEOREM.** *Let  $G$  be a simple  $C\theta\theta$ -group and suppose that  $G$  involves no Suzuki simple groups. Then  $G$  is isomorphic to one of the following groups:  $PSL(2, 4)$ ;  $PSL(2, 8)$ ;  $PSL(3, 4)$ ;  $PSL(2, 3^n)$ ,  $n > 1$  and  $PSL(2, q)$ ,  $q$  such that  $(q+1)/2$  or  $(q-1)/2$  is a power of 3.*

It follows immediately from the Theorem that the following characterization of  $PSL(2, 8)$  holds:

**COROLLARY 1.** *Let  $G$  satisfy the assumptions of the Theorem and suppose that no element of  $G$  of order 3 normalizes a nontrivial 2-subgroup of  $G$ . Then  $G \cong PSL(2, 8)$ .*

The Theorem leads also to a complete classification of simple  $C\theta\theta$ -groups whose order is divisible by at most four distinct primes. We have

**COROLLARY 2.** *Let  $G$  be a simple  $C\theta\theta$ -group and suppose that  $|\pi(G)| = 3$ . Then  $G$  is isomorphic to one of the following groups:  $PSL(2, 4)$ ,  $PSL(2, 7)$ ,  $PSL(2, 8)$ ,  $PSL(2, 9)$  and  $PSL(2, 17)$ ,*

and

**COROLLARY 3.** *Let  $G$  be a simple  $C\theta\theta$ -group and suppose that  $|\pi(G)| = 4$ . Then  $G$  is isomorphic to one of the following groups:  $PSL(3, 4)$  and those among  $PSL(2, 3^n)$ ,  $n > 1$  and  $PSL(2, q)$ ,  $q \pm 1 = 2 \cdot 3^r$ ,  $r > 1$ , which are divisible by exactly four distinct primes.*

2. Proof of the Theorem. We will prove first the following

**PROPOSITION.** *Let  $G$  be a nonsolvable  $C\theta\theta$ -group. Then at least one of the following statements holds.*

(i) *Whenever a section  $K/M$  of  $G$  is isomorphic to a minimal simple group  $L$ , then either  $L$  is a Suzuki group or  $M = \{1\}$  and  $L$  is  $PSL(2, 8)$ .*

(ii) *Some nontrivial 2-subgroup of  $G$  is normalized by an element of order 3.*

*Proof of the Proposition.* Let  $G$  be a counter-example. Then there exist subgroups  $K$  and  $M$  of  $G$ ,  $M$  normal in  $K$ , such that  $K/M$  is isomorphic to a minimal simple group  $L$  which is not of Suzuki type and if  $M = \{1\}$  then  $L$  is not  $PSL(2, 8)$ . By Thompson's theorem [5, Corollary 1]  $L$  is one of the following:  $PSL(2, 2^p)$ ,  $p$  any prime,  $PSL(2, 3^p)$ ,  $p$  any odd prime;  $PSL(2, p)$ ,  $p$  any prime exceeding 3 such that  $p^2 + 1 \equiv 0 \pmod{5}$  and  $PSL(3, 3)$ . Denote by  $Q$  the Sylow 3-subgroup of  $K$ . Since a Sylow 3-subgroup of a nonsolvable  $C\theta\theta$ -group is abelian [1, Theorem 2.9],  $L$  is not  $PSL(3, 3)$  and  $Q$  is the centralizer in  $K$  of each of its nonunit elements. Suppose that there exists a normal complement  $S$  of  $N_K(Q)$  in  $K$ . Since  $M$  is a maximal normal subgroup of  $K$ , it follows that either  $K = N_K(Q)M$  or  $K = SM$ , and consequently  $L = K/M$  has a normal (possibly trivial) Sylow 3-subgroup, a contradiction. Thus  $N_K(Q)$  has no normal complement in  $K$  and by [2, Theorem 2.3.e] the fact that 3 divides the order of  $L$  implies that 3 does not divide the order of  $M$ . It follows then by the results of Stewart<sup>1</sup>, [4, Propositions 3.2 and 4.2] that  $M = \{1\}$  if  $L = PSL(2, q)$ , where  $q = 3^p$  or  $q = p > 5$  and  $M$  is a 2-group if  $L = PSL(2, q)$ , where  $q = 2^p$ . Since no element of order 3 in  $G$  normalizes a nontrivial 2-subgroup of  $G$ ,  $M = \{1\}$  in all cases and  $L$  is not  $PSL(2, 8)$ . It is well known that the Sylow 2-subgroups of  $PSL(2, q)$ , where  $q = 3^p$ ,  $p > 2$  or  $q = p > 3$ ,  $p$  a prime, are normalized by an element of order 3. Consequently,  $L = PSL(2, 2^p)$ , where  $p$  is a prime exceeding 3. Since the Sylow 3-subgroups in  $L$  are the centralizers of each of their nonidentity elements, it follows that  $2^p \pm 1 = 3^k$  for some  $k$ . This equation has no solution for  $p > 3$  and consequently  $G$  does not exist.

We proceed with a proof of the Theorem. If case (ii) of the Proposition holds, then it follows from the results of Fletcher and Gorenstein [1, Corollary 3.2] that  $G$  is isomorphic to one of the groups mentioned in the Theorem,  $PSL(2, 8)$  excluded. If case (i) of the Proposition holds, but not case (ii), then we will show that  $G$  is an  $N$ -group and it follows then from Thompson's classification theorem of

<sup>1</sup> The author is indebted to Dr. W. B. Stewart for communication of results prior to publication.

simple *N*-groups [5] that only  $PSL(2, 8)$  is a *Cθθ*-group of the required type.

Let *U* be a *p*-subgroup of *G* with a nonsolvable normalizer. By our assumptions and by the Proposition  $N = N_G(U)$  contains a subgroup *V* isomorphic to  $PSL(2, 8)$ . As  $V \cap U = \{1\}$ ,  $VU/U$  is isomorphic to  $PSL(2, 8)$  and consequently  $U = \{1\}$ . Thus *G* is an *N*-group and the proof is complete.

**3. Proof of the corollaries.** Since  $PSL(2, 8)$  is the only group mentioned in the Theorem without an element of order 3 normalizing a nontrivial 2-subgroup, Corollary 1 immediately follows from the Theorem.

If  $|\pi(G)| = 3$  then *G* does not involve Suzuki groups and by the Theorem and [3, Theorem 3] it is isomorphic to one of the following:  $PSL(2, 4)$ ,  $PSL(2, 7)$ ,  $PSL(2, 8)$ ,  $PSL(2, 9)$  and  $PSL(2, 17)$ .

Corollary 3 follows from the fact that 3 divides the order of *G* and 3 does not divide the order of the Suzuki groups. Consequently, as the Suzuki groups have orders divisible by at least 4 distinct primes, *G* does not involve them. Corollary 3 follows therefore immediately from the Theorem.

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