

# Pacific Journal of Mathematics

**CONCERNING WEB-LIKE CONTINUA**

JOHN WALTER HINRICHSEN

## CONCERNING WEB-LIKE CONTINUA

J. W. HINRICHSEN

The compact metric continuum  $M$  is said to be a web if and only if there exist two monotone upper semi-continuous decompositions  $G_1$  and  $G_2$  of  $M$  such that  $M/G_1$  and  $M/G_2$  are arcs and each element of  $G_1$  intersects each element of  $G_2$ . It is shown that there exists in Euclidean 3-space a compact continuum  $M$  that is not a web but does have two monotone upper semicontinuous decompositions  $G_1$  and  $G_2$  such that (1)  $M/G_1$  and  $M/G_2$  are simple closed curves and (2) each element of  $G_1$  intersects each element of  $G_2$ . Such continua are called pseudo-webs.

This solves a problem suggested to the author by Professor R. L. Moore. It is also shown that there do not exist pseudo-webs in the plane.

**THEOREM 1.** *Suppose  $M$  is a metric chainable continuum,  $J$  is a simple metric closed curve, and  $g_1$  and  $g_2$  are mutually exclusive subcontinua of  $M \times J$  such that if  $P$  is a point of  $M$ , then  $g_1$  and  $g_2$  intersect  $P \times J$ . Then no subcontinuum of  $M \times J$  separates  $g_1$  from  $g_2$  in  $M \times J$ .*

*Proof.* Suppose there is a subcontinuum  $g$  of  $M \times J$  which separates  $g_1$  from  $g_2$  in  $M \times J$ . Let  $\varepsilon$  denote a positive number less than the distances from  $g_1$  to  $g_2 + g$  and  $g_2$  to  $g_1 + g$ . There exists an  $\varepsilon$ -map  $f$  from  $M$  onto  $[0, 1]$ . If  $P$  is a point of  $M$  and  $j$  is in  $J$ , let  $T(P, j) = (f(P), j)$ .  $T$  is an  $\varepsilon$ -map from  $M \times J$  onto  $[0, 1] \times J$ . If  $P$  is a point of  $[0, 1]$ ,  $T(g)$ ,  $T(g_1)$ , and  $T(g_2)$  are mutually exclusive continua intersecting  $P \times J$ .

By Theorem 29 of Chapter IV of [5], there exist two mutually exclusive arcs  $\alpha_1$  and  $\alpha_2$ , each intersecting  $0 \times J$  and  $1 \times J$ , such that (1) only the endpoints of  $\alpha_1$  and  $\alpha_2$  lie on  $0 \times J$  and  $1 \times J$ , and (2)  $\alpha_1 + \alpha_2$  separates  $T(g)$  from  $T(g_1 + g_2)$  in  $[0, 1] \times J$ .  $([0, 1] \times J) - (\alpha_1 + \alpha_2)$  is the sum of two mutually separated connected point sets,  $D$  and  $D'$  containing  $T(g)$  and  $T(g_1 + g_2)$ , respectively. Let  $\beta$  denote  $\bar{D}' \cdot (0 \times J)$ .  $\beta$  is an arc of  $0 \times J$  that intersects  $T(g_1)$  and  $T(g_2)$  and does not intersect  $T(g)$ .

Let  $Z$  be a point of  $T^{-1}(\beta)$ . Let  $Z'$  denote the point of  $M$  such that  $Z$  is a point of  $Z' \times J$ . Let  $P_1$  and  $P_2$  denote points of  $g'_1 \cdot (Z' \times J)$  and  $g'_2 \cdot (Z' \times J)$ , respectively. Since  $g$  separates  $g_1$  from  $g_2$  in  $M \times J$ , there exist two points  $X_1$  and  $X_2$  of  $g$  which separate  $P_1$  from  $P_2$  in  $(Z' \times J)$ . Then  $T(X_1) + T(X_2)$  separates  $T(P_1)$  from  $T(P_2)$  in  $(0 \times J)$ . Then  $\beta$  contains

either  $T(X_1)$  or  $T(X_2)$ . This involves a contradiction. Hence  $g$  does not separate  $g_1$  from  $g_2$  on  $Z' \times J$ . Therefore, there is a connected subset of  $Z' \times J$  that intersects  $g_1$  and  $g_2$  but not  $g$ , and hence  $g$  does not separate  $g_1$  from  $g_2$  in  $M \times J$ .

**THEOREM 2.** *The Cartesian product of a metric chainable indecomposable continuum with a metric simple closed curve is a pseudo-web.*

*Proof.* Let  $M$  denote a chainable indecomposable continuum in the  $xy$ -plane of  $E^3$  and  $J$  denote a simple closed curve. It will first be shown that there exist two monotone upper semi-continuous decompositions,  $G_1$  and  $G_2$ , of  $M \times J$  such that each element of  $G_1$  intersects each element of  $G_2$  and  $(M \times J)/G_1$  and  $(M \times J)/G_2$  are simple closed curves. It will then be shown that there is no monotone upper semi-continuous decomposition of  $M \times J$  which is an arc with respect to its elements.

Let  $L$  denote a line in the  $xy$ -plane parallel to the  $y$ -axis not intersecting  $M$ .  $M \times J$  is homeomorphic to the point set  $M'$  obtained by revolving  $M$  about  $L$ . Let  $H_1$  denote the collection to which  $h$  belongs if and only if for some half-plane  $A$  with  $L$  on its boundary,  $g$  is  $M' \cdot A$ . Let  $P$  denote a point of  $L$  which is on a horizontal line intersecting  $M$ , and  $L'$  denote a line in the  $xy$ -plane distinct from  $L$  such that  $L'$  contains  $P$  and does not intersect  $M$ .  $L'$  is not perpendicular to  $L$ . Let  $H_2$  denote the collection to which  $h$  belongs if and only if for some half-plane  $A$  with  $L'$  on its boundary,  $h$  is  $M' \cdot A$ .

$M'/H_1$  and  $M'/H_2$  are simple closed curves. There exist an arc  $H'_1$  of elements of  $H_1$  and an arc  $H'_2$  of elements of  $H_2$  such that each element of  $H'_1$  intersects each element of  $H'_2$ . For each  $i = 1, 2$ , let  $G_i$  denote the collection to which  $g$  belongs if and only if  $g$  is a separating element of  $H'_i$  or  $g$  is  $\overline{(H'_i - H'_i)^*}$ .  $G_1$  and  $G_2$  are two monotone upper semi-continuous decompositions of  $M$  such that each of  $M/G_1$  and  $M/G_2$  is a simple closed curve and each element of  $G_1$  intersects each element of  $G_2$ .

Therefore, in order to prove that  $M \times J$  is a pseudo-web, it will be sufficient to show that there is no monotone upper semi-continuous decomposition of  $M$  which is an arc with respect to its elements.

Suppose there exists a monotone upper semi-continuous decomposition  $G$  of  $M \times J$  such that  $M/G$  is an arc. Suppose  $g$  is a separating element of  $G$  and there is a point  $P$  of  $M$  such that  $g$  does not intersect  $P \times J$ . Let  $M_g$  denote the set of all points  $Q$  of  $M$  such that  $g$  intersects  $Q \times J$ . Since  $g$  is closed and connected,  $M_g$  is closed and connected. Therefore, since  $M_g$  is a proper subset of  $M$ ,  $M_g$  is a subset of some component  $C$  of  $M$ . Hence,  $g$  is a subset of  $C \times J$ . But  $(M - C) \times J$  is connected and  $\overline{(M - C)} \times J$  is  $M \times J$ . Therefore,

$(M \times J) - g$  is connected. It then follows that the end elements of  $G$  intersect  $P \times J$  for each  $P$  in  $J$ .

Let  $g_1$  and  $g_2$  denote two separating elements of  $M/G$ , and let  $g$  denote an element of  $G$  between  $g_1$  and  $g_2$ .  $M, J, g_1$ , and  $g_2$  satisfy all the conditions of Theorem 1. Therefore,  $g$  is not a continuum. This involves a contradiction. Therefore, there is no monotone upper semi-continuous decomposition  $G$  of  $M \times J$  such that  $(M \times J)/G$  is an arc. Hence,  $M$  is not a web and therefore,  $M$  is a pseudo-web.

REMARKS. It can also be shown that there exists an example of a pseudo-web that contains no essential continuum of condensation. Also, in the plane, a square disc  $D$  is a web. But since  $D$  is uni-coherent, it follows that if  $G$  is a monotone upper semi-continuous decomposition of  $D$ ,  $D/G$  is not a simple closed curve.

Furthermore, a 2-torus does not have a dendratomic subset and therefore, by Theorem 48 of chapter V, part 1, of [5], a 2-torus is a web. However, one might wonder if the Cartesian product of a circularly chainable indecomposable continuum that is not chainable with a simple closed curve is a pseudo-web.

THEOREM 3. *There is no plane pseudo-web.*

*Proof.* Suppose  $M$  is a pseudo-web in the plane  $\Sigma$ . Then there exist two monotone upper semi-continuous decompositions  $G_1$  and  $G_2$  of  $M$  such that (1) each of  $M/G_1$  and  $M/G_2$  is a simple closed curve and (2) each element of  $G_1$  intersects each element of  $G_2$ .

For each point  $P$  of  $M$ , let  $g_P$  denote the component containing  $P$  of the common part of the continuum of  $G_1$  that contains  $P$  and the continuum of  $G_2$  that contains  $P$ , and  $G$  denote the collection of all continua  $g_P$  for all points  $P$  of  $M$ . Then by Theorem 7 of Chapter V, part 2, of [5],  $G$  is a continuous curve with respect to its elements.

Let  $G'$  denote the collection to which  $g'$  belongs if and only if  $g'$  is an element  $g$  of  $G$  together with all the points not in  $M$  which are separated from an element of  $G$  by  $g$ , if there are any. Let  $S'$  denote the collection of all continua  $P'$  such that  $P'$  is either a continuum of the collection  $G'$  or a point which neither belongs to a continuum of  $G'$  nor is separated by any continuum of  $G'$  from any other continuum of  $G'$ . Let  $S$  denote the set of all points of  $\Sigma$  and  $\Sigma'$  denote  $S/S'$ . Then  $\Sigma'$  is topologically equivalent to  $\Sigma$  or to a sphere.  $G'$  in  $\Sigma'$  is a continuum.

For  $i = 1, 2$ , let  $G'_i$  denote the collection to which  $g'$  belongs if and only if for some element  $g$  of  $G_i$ ,  $g'$  is the sum of all the elements of  $G'$  that intersect  $g$ . The continuum  $G'$  together with the collections

$G'_1$  and  $G'_2$  satisfy all the conditions of Theorem 1 of [1]. Hence  $G'$  is a simple plane web or simple web that is a subset of a sphere. Hence, there exist two monotone upper semi-continuous decompositions  $H'_1$  and  $H'_2$  of  $G'$  such that each of  $G'/H'_1$  and  $G'/H'_2$  is a dendron and if  $h'_1$  and  $h'_2$  are elements of  $H'_1$  and  $H'_2$ , respectively, then  $h'_1 \cdot h'_2$  exists and is totally disconnected. For each  $i = 1, 2$ , let  $H_i$  denote the collection to which  $h$  belongs if and only if for some  $h'$  in  $H'_i$ ,  $h$  is the set of all points of  $M$  in  $\Sigma$  which belong to an element of  $h'$  in  $\Sigma$ .  $H_1$  and  $H_2$  are monotone upper semi-continuous decompositions of  $M$  such that (1)  $M/H_1$  and  $M/H_2$  are dendrons and (2) each element of  $H_1$  intersects each element of  $H_2$ .  $H_1$  and  $H_2$  satisfy the conditions of an equivalent definition of a web given on page 297 of [5]. Hence, by Theorem 41 of Chapter V of [5],  $M$  is a web.

In conclusion, the following questions may be raised. Does there exist a compact metric continuum  $M$  that is not a web but does have two monotone upper semi-continuous decompositions  $G_1$  and  $G_2$  of  $M$  satisfying the conditions of a pseudo-web except that  $M/G_1$  is an arc and  $M/G_2$  is a simple closed curve? Also, does every pseudo-web contain uncountably many mutually exclusive webs? Does every web contain an indecomposable continuum?

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