CONCERNING WEB-LIKE CONTINUA

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The compact metric continuum $M$ is said to be a web if and only if there exist two monotone upper semi-continuous decompositions $G_1$ and $G_2$ of $M$ such that $M/G_1$ and $M/G_2$ are arcs and each element of $G_1$ intersects each element of $G_2$. It is shown that there exists in Euclidean 3-space a compact continuum $M$ that is not a web but does have two monotone upper semicontinuous decompositions $G_1$ and $G_2$ such that (1) $M/G_1$ and $M/G_2$ are simple closed curves and (2) each element of $G_1$ intersects each element of $G_2$. Such continua are called pseudo-webs.

This solves a problem suggested to the author by Professor R. L. Moore. It is also shown that there do not exist pseudo-webs in the plane.

THEOREM 1. Suppose $M$ is a metric chainable continuum, $J$ is a simple metric closed curve, and $g_1$ and $g_2$ are mutually exclusive subcontinua of $M \times J$ such that if $P$ is a point of $M$, then $g_1$ and $g_2$ intersect $P \times J$. Then no subcontinuum of $M \times J$ separates $g_1$ from $g_2$ in $M \times J$.

Proof. Suppose there is a subcontinuum $g$ of $M \times J$ which separates $g_1$ from $g_2$ in $M \times J$. Let $\varepsilon$ denote a positive number less than the distances from $g_1$ to $g_2 + g$ and $g_2$ to $g_1 + g$. There exists an $\varepsilon$-map $f$ from $M$ onto $[0, 1]$. If $P$ is a point of $M$ and $j$ is in $J$, let $T(P, j) = (f(P), j)$. $T$ is an $\varepsilon$-map from $M \times J$ onto $[0, 1] \times J$. If $P$ is a point of $[0, 1]$, $T(g)$, $T(g_1)$, and $T(g_2)$ are mutually exclusive continua intersecting $P \times J$.

By Theorem 29 of Chapter IV of [5], there exist two mutually exclusive arcs $\alpha_1$ and $\alpha_2$, each intersecting $0 \times J$ and $1 \times J$, such that (1) only the endpoints of $\alpha_1$ and $\alpha_2$ lie on $0 \times J$ and $1 \times J$, and (2) $\alpha_1 + \alpha_2$ separates $T(g)$ from $T(g_1 + g_2)$ in $[0, 1] \times J$. $(0, 1] \times J - (\alpha_1 + \alpha_2)$ is the sum of two mutually separated connected point sets, $D$ and $D'$ containing $T(g)$ and $T(g_1 + g_2)$, respectively. Let $\beta$ denote $\bar{D}(0 \times J)$. $\beta$ is an arc of $0 \times J$ that intersects $T(g_1)$ and $T(g_2)$ and does not intersect $T(g)$.

Let $Z$ be a point of $T^{-1}(\beta)$. Let $Z'$ denote the point of $M$ such that $Z$ is a point of $Z' \times J$. Let $P_1$ and $P_2$ denote points of $g_1'(Z' \times J)$ and $g_2'(Z' \times J)$, respectively. Since $g$ separates $g_1$ from $g_2$ in $M \times J$, there exist two points $X_1$ and $X_2$ of $g$ which separate $P_1$ from $P_2$ in $(Z' \times J)$. Then $T(X_1) + T(X_2)$ separates $T(P_1)$ from $T(P_2)$ in $(0 \times J)$. Then $\beta$ contains
either $T(X_1)$ or $T(X_3)$. This involves a contradiction. Hence $g$ does not separate $g_1$ from $g_2$ on $Z' \times J$. Therefore, there is a connected subset of $Z' \times J$ that intersects $g_1$ and $g_2$ but not $g$, and hence $g$ does not separate $g_1$ from $g_2$ in $M \times J$.

**Theorem 2.** The Cartesian product of a metric chainable indecomposable continuum with a metric simple closed curve is a pseudo-web.

**Proof.** Let $M$ denote a chainable indecomposable continuum in the $xy$-plane of $E^3$ and $J$ denote a simple closed curve. It will first be shown that there exist two monotone upper semi-continuous decompositions, $G_1$ and $G_2$, of $M \times J$ such that each element of $G_1$ intersects each element of $G_2$ and $(M \times J)/G_1$ and $(M \times J)/G_2$ are simple closed curves. It will then be shown that there is no monotone upper semi-continuous decomposition of $M \times J$ which is an arc with respect to its elements.

Let $L$ denote a line in the $xy$-plane parallel to the $y$-axis not intersecting $M$. $M \times J$ is homeomorphic to the point set obtained by revolving $M$ about $L$. Let $H_i$ denote the collection to which $h$ belongs if and only if for some half-plane $A$ with $L$ on its boundary, $g$ is $M' \cdot A$. Let $P$ denote a point of $L$ which is on a horizontal line intersecting $M$, and $L'$ denote a line in the $xy$-plane distinct from $L$ such that $L'$ contains $P$ and does not intersect $M$. $L'$ is not perpendicular to $L$. Let $H_2$ denote the collection to which $h$ belongs if and only if for some half-plane $A$ with $U$ on its boundary, $h$ is $M' \cdot A$.

$M'/H_1$ and $M'/H_2$ are simple closed curves. There exist an arc $H'_i$ of elements of $H_i$ and an arc $H'_2$ of elements of $H_2$ such that each element of $H'_i$ intersects each element of $H'_2$. For each $i = 1, 2$, let $G_i$ denote the collection to which $g$ belongs if and only if $g$ is a separating element of $H'_i$ or $g$ is $(H_i - H'_i)$. $G_1$ and $G_2$ are two monotone upper semi-continuous decompositions of $M$ such that each of $M/G_1$ and $M/G_2$ is a simple closed curve and each element of $G_i$ intersects each element of $G_2$.

Therefore, in order to prove that $M \times J$ is a pseudo-web, it will be sufficient to show that there is no monotone upper semi-continuous decomposition of $M$ which is an arc with respect to its elements.

Suppose there exists a monotone upper semi-continuous decomposition $G$ of $M \times J$ such that $M/G$ is an arc. Suppose $g$ is a separating element of $G$ and there is a point $P$ of $M$ such that $g$ does not intersect $P \times J$. Let $M_s$ denote the set of all points $Q$ of $M$ such that $g$ intersects $Q \times J$. Since $g$ is closed and connected, $M_s$ is closed and connected. Therefore, since $M_s$ is a proper subset of $M$, $M_s$ is a subset of some composant $C$ of $M$. Hence, $g$ is a subset of $C \times J$.

But $(M - C) \times J$ is connected and $(M - C) \times J$ is $M \times J$. Therefore,


Let \( g \) and \( g_2 \) denote two separating elements of \( M/G \), and let \( g \) denote an element of \( G \) between \( g \) and \( g_2 \). \( M, J, g_2 \), and \( g \) satisfy all the conditions of Theorem 1. Therefore, \( g \) is not a continuum. This involves a contradiction. Therefore, there is no monotone upper semi-continuous decomposition \( G \) of \( M \times J \) such that \((M \times J)/G\) is an arc. Hence, \( M \) is not a web and therefore, \( M \) is a pseudo-web.

**REMARKS.** It can also be shown that there exists an example of a pseudo-web that contains no essential continuum of condensation. Also, in the plane, a square disc \( D \) is a web. But since \( D \) is unicoherent, it follows that if \( G \) is a monotone upper semi-continuous decomposition of \( D, D/G \) is not a simple closed curve.

Furthermore, a 2-torus does not have a dendratomic subset and therefore, by Theorem 48 of chapter V, part 1, of [5], a 2-torus is a web. However, one might wonder if the Cartesian product of a circularly chainable indecomposable continuum that is not chainable with a simple closed curve is a pseudo-web.

**THEOREM 3.** There is no plane pseudo-web.

**Proof.** Suppose \( M \) is a pseudo-web in the plane \( \Sigma \). Then there exist two monotone upper semi-continuous decompositions \( G_1 \) and \( G_2 \) of \( M \) such that (1) each of \( M/G_1 \) and \( M/G_2 \) is a simple closed curve and (2) each element of \( G_1 \) intersects each element of \( G_2 \).

For each point \( P \) of \( M \), let \( g_P \) denote the component containing \( P \) of the common part of the continuum of \( G_1 \) that contains \( P \) and the continuum of \( G_2 \) that contains \( P \), and \( G \) denote the collection of all continua \( g_P \) for all points \( P \) of \( M \). Then by Theorem 7 of Chapter V, part 2, of [5], \( G \) is a continuous curve with respect to its elements.

Let \( G' \) denote the collection to which \( g' \) belongs if and only if \( g' \) is an element \( g \) of \( G \) together with all the points not in \( M \) which are separated from an element of \( G \) by \( g \), if there are any. Let \( S' \) denote the collection of all continua \( P' \) such that \( P' \) is either a continuum of the collection \( G' \) or a point which neither belongs to a continuum of \( G' \) nor is separated by any continuum of \( G' \) from any other continuum of \( G' \). Let \( S \) denote the set of all points of \( \Sigma \) and \( \Sigma' \) denote \( S/S' \). Then \( \Sigma' \) is topologically equivalent to \( \Sigma \) or to a sphere. \( G' \) in \( \Sigma' \) is a continuum.

For \( i = 1, 2 \), let \( G_i \) denote the collection to which \( g' \) belongs if and only if for some element \( g \) of \( G_i \), \( g' \) is the sum of all the elements of \( G' \) that intersect \( g \). The continuum \( G' \) together with the collections
$G'$ and $G_2'$ satisfy all the conditions of Theorem 1 of [1]. Hence $G'$ is a simple plane web or simple web that is a subset of a sphere. Hence, there exist two monotone upper semi-continuous decompositions $H'_1$ and $H'_2$ of $G'$ such that each of $G'/H'_1$ and $G'/H'_2$ is a dendron and if $h'_1$ and $h'_2$ are elements of $H'_1$ and $H'_2$, respectively, then $h'_1 \cdot h'_2$ exists and is totally disconnected. For each $i = 1, 2$, let $H_i$ denote the collection to which $h$ belongs if and only if for some $h'$ in $H'_i$, $h$ is the set of all points of $M$ in $\Sigma$ which belong to an element of $h'$ in $\Sigma$. $H_1$ and $H_2$ are monotone upper semi-continuous decompositions of $M$ such that (1) $M/H_1$ and $M/H_2$ are dendrons and (2) each element of $H_i$ intersects each element of $H_2$. $H_1$ and $H_2$ satisfy the conditions of an equivalent definition of a web given on page 297 of [5]. Hence, by Theorem 41 of Chapter V of [5], $M$ is a web.

In conclusion, the following questions may be raised. Does there exist a compact metric continuum $M$ that is not a web but does have two monotone upper semi-continuous decompositions $G_1$ and $G_2$ of $M$ satisfying the conditions of a pseudo-web except that $M/G_1$ is an arc and $M/G_2$ is a simple closed curve? Also, does every pseudo-web contain uncountably many mutually exclusive webs? Does every web contain an indecomposable continuum?

REFERENCES


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