A GENERALIZATION OF A THEOREM OF F. RIESZ

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In this paper, the concept of bounded slope variation, that of the derivative of a function with respect to an increasing function, and the Lane integral are used to obtain a generalization of a theorem of Frédéric Riesz.

In [3], R. E. Lane defined an integral which is an extension of the Stieltjes mean sigma integral defined by H. L. Smith [5]. If each of \( f \) and \( g \) is a real-valued function whose domain includes \([a, b]\) and \( D = \{x_i\}_{i=0}^{n} \) is a subdivision of \([a, b]\), then \( S_\sigma(f, g) \) denotes the sum

\[
\sum_{i=1}^{n} \frac{1}{2} [f(x_i) + f(x_{i-1})][g(x_i) - g(x_{i-1})].
\]

The concepts of singular graph, exceptional number and summability set are as in [3]. If each of \( f \) and \( g \) is a real-valued function whose domain includes \([a, b]\) and if there exists a summability set \( G \) for \( f \) and \( g \) in \([a, b]\), then the Lane integral \( \int_{a}^{b} f\,dg \) is the refinement limit

\[
\lim_{b \to a} S_\sigma(f, g).
\]

In case the entire interval \([a, b]\) is a summability set for \( f \) and \( g \) in \([a, b]\), the Lane integral \( \int_{a}^{b} f\,dg \) is the Stieltjes mean sigma integral \( M\int_{a}^{b} f\,dg \).

By Theorem 4.1 of [2], if \( f \) is quasicontinuous on \([a, b]\) and \( g \) is of bounded variation on \([a, b]\), then \( \int_{a}^{b} f\,dg \) exists. (A function \( f \) is said to be quasicontinuous at \((c, f(c))\) if both \( f(c+) \) and \( f(c-) \) exist.)

DEFINITION 1. The statement that \( f \) has bounded slope variation with respect to \( m \) over \([a, b]\) means that \( f \) is a function whose domain includes \([a, b]\), \( m \) is a real-valued increasing function on \([a, b]\), and there exists a nonnegative number \( B \) such that if \( \{x_i\}_{i=0}^{n} \) is a subdivision of \([a, b]\) with \( n > 1 \), then

\[
\sum_{i=1}^{n-1} \left| \frac{f(x_{i+1}) - f(x_i)}{m(x_{i+1}) - m(x_i)} - \frac{f(x_i) - f(x_{i-1})}{m(x_i) - m(x_{i-1})} \right| \leq B.
\]

The least such number \( B \) is called the slope variation of \( f \) with respect to \( m \) over \([a, b]\) and is denoted by \( V_\sigma^m(df/dm) \). [Note: \( V_\sigma^m(df/dm) = 0 \).]

The above sum is nondecreasing with respect to refinements.

In [4], F. Riesz proved that a necessary and sufficient condition
that a function $F$ defined on the interval $[a, b]$ be the integral of a function of bounded variation on $[a, b]$ is that $F$ have bounded slope variation with respect to $I$ over $[a, b]$, where $I$ is the function defined, for each $x$, by $I(x) = x$. In this paper, Riesz's result will be generalized using the Lane integral instead of the Riemann integral.

By Lemma 3.3 of [6], if $f$ has bounded slope variation with respect to $m$ over $[a, b]$ and $a \leq c < b$, then

$$D^+_m f(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{m(x) - m(c)}$$

exists and if $a < c \leq b$,

$$D^-_m f(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{m(x) - m(c)}$$

exists.

**Lemma 1.** If $f$ has bounded slope variation with respect to $m$ over $[a, b]$, $c$ is a number in $[a, b]$, and $m$ is continuous on the right (left) at $(c, m(c))$, then $f$ is continuous on the right (left) at $(c, f(c))$.

**Proof.** Let $\varepsilon$ denote a positive number and let $c$ be a number in $[a, b]$. Suppose $m$ is continuous on the right at $(c, m(c))$. Then $a \leq c < b$ and $D^+_m f(c)$ exists. Therefore there exists a positive number $\delta_1$ such that if $c < x < c + \delta_1$, then

$$\left| \frac{f(x) - f(c)}{m(x) - m(c)} - D^+_m f(c) \right| < \varepsilon$$

from which it follows that

$$|f(x) - f(c)| < [\left| D^+_m f(c) \right| + 1] |m(x) - m(c)| .$$

Since $m$ is continuous on the right at $(c, m(c))$, there exists a positive number $\delta_2$ such that if $c < x < c + \delta_2$, then $|m(x) - m(c)| < \varepsilon/[\left| D^+_m f(c) \right| + 1]$. Let $\delta = \min. [\delta_1, \delta_2]$. Then if $c < x < c + \delta$,

$$|f(x) - f(c)| < [\left| D^+_m f(c) \right| + 1] |m(x) - m(c)|$$

$$< [\left| D^+_m f(c) \right| + 1] \cdot \varepsilon/[\left| D^+_m f(c) \right| + 1]$$

$$= \varepsilon .$$

Therefore $f$ is continuous on the right at $(c, f(c))$.

If $m$ is continuous on the left at $(c, m(c))$, a similar argument will show that $f$ is continuous on the left at $(c, f(c))$.

**Definition 2.** Suppose $m$ is an increasing function on $[a, b]$, $f$ is
A function whose domain includes \([a, b]\) and \(c\) is a number in \([a, b]\). The statement that \(f\) has a derivative with respect to \(m\) at the point \((c, f(c))\) means that

\[
D_m f(c) = \lim_{x \to c} \frac{f(x) - f(c)}{m(x) - m(c)}
\]

exists.

**Theorem 1.** If \(f\) has bounded slope variation with respect to \(m\) over \([a, b]\), then \(D_m f(x)\) exists for each \(x\) in \([a, b] - E\), where \(E\) is a countable set.

**Proof.** Since \(f\) has bounded slope variation with respect to \(m\) over \([a, b]\), \(D_+ f(x)\) exists for each \(x\) in \([a, b]\) and \(D_- f(x)\) exists for each \(x\) in \((a, b]\). Let \(E_1\) denote the set of all numbers \(x\) in \([a, b]\) such that \(D_- f(x) < D_+ f(x)\) and let \(E_2\) denote the set of all numbers \(x\) in \([a, b]\) such that \(D_- f(x) > D_+ f(x)\). Let all rational numbers be arranged in a sequence \(r_1, r_2, r_3, \ldots\). Then if \(c\) is a number in \(E_1\) there is a smallest positive integer \(k\) such that

\[
D_m f(c) < r_k < D_+ f(c)
\]

There is a smallest positive integer \(h\) such that \(r_h < c\) and

\[
\frac{f(x) - f(c)}{m(x) - m(c)} < r_k
\]

for \(r_h < x < c\) and a smallest positive integer \(n\) such that \(r_n > c\) and

\[
\frac{f(x) - f(c)}{m(x) - m(c)} > r_k
\]

for \(c < x < r_n\). These two inequalities together give

(1) \[f(x) - f(c) > r_k [m(x) - m(c)]\]

for \(r_h < x < r_n, x \neq c\). Thus to every number \(c\) in \(E_1\) there corresponds a unique triad \((h, k, n)\) of positive integers. Suppose some two numbers \(x_1\) and \(x_2\) of \(E_1\) correspond to the same triad \((h, k, n)\). Then, on putting \(c = x_1\) and \(x = x_2\) in (1), we have

\[
f(x_1) - f(x) > r_k [m(x_2) - m(x_1)]
\]

and, on putting \(c = x_2\) and \(x = x_1\),

\[
f(x_1) - f(x_2) > r_k [m(x_2) - m(x_1)]
\]

or
This involves a contradiction. Therefore no two numbers of $E_i$ correspond to the same triad. Since the set of triads of positive integers is countable, it follows that $E_i$ is countable. A similar argument will show that $E_2$ is countable. Therefore $E = E_i \cup E_2$ is countable.

**Theorem 2.** If the function $m$ is increasing on $[a, b]$, each of the functions $f$ and $g$ is continuous on $[a, b]$ and $D_m f(x) = D_m g(x)$ for each $x$ in $[a, b] - H$, where $H$ is a countable set, then $f(x) = g(x) - g(a) + f(a)$ for each $x$ in $[a, b]$.

**Proof.** Let $F$ be the function defined, for each $x$ in $[a, b]$, by $F(x) = f(x) - g(x)$. Then $F$ is continuous on $[a, b]$ and $D_m F(x) = 0$ for each $x$ in $[a, b] - H$. Let $\varepsilon$ denote a positive number and let $c$ be a number in $[a, b]$. Let $H \cap [a, c] = \{p_1, p_2, \ldots, p_n, \ldots\}$. Since $F$ is continuous on $[a, b]$, for each positive integer $n$ there exists a positive number $\delta_n$ such that if $x$ is in $(p_n - \delta_n, p_n + \delta_n) \cap [a, c]$, then

$$|F(x) - F(p_n)| < \frac{\varepsilon}{2^n}.$$ 

Let $h_n = (p_n - \delta_n, p_n + \delta_n)$. It follows that if $x_1$ and $x_2$ are numbers in $h_n \cap [a, c]$, then

$$|F(x_1) - F(x_2)| < \frac{\varepsilon}{2^{n+1}}.$$ 

For each $n$, choose some particular $h_n$ satisfying the above conditions. Now consider any number $t$ in $[a, c] - H \cap [a, c]$. Then $D_m F(t) = 0$. If $t$ is in $(a, c)$, there is a positive number $\delta_t$ such that $(t - \delta_t, t + \delta_t)$ is a subset of $(a, c)$ and if $x$ is in $(t - \delta_t, t + \delta_t)$ and $x \neq t$, then

$$|F(x) - F(t)| < \frac{\varepsilon}{12[m(c) - m(a)]}$$

or

$$|F(x) - F(t)| < \frac{\varepsilon |m(x) - m(t)|}{12[m(c) - m(a)]} < \frac{\varepsilon \cdot V(t)}{12[m(c) - m(a)]}$$

where $V(t)$ is the variation of $m$ over $[t - \delta_t, t + \delta_t]$. If $t = a$, there exists a positive number $\delta_a$ such that if $x \neq a$ and $x$ is in $(a - \delta_a, a + \delta_a) \cap [a, c]$, then

$$|F(x) - F(a)| < \frac{\varepsilon \cdot V(a)}{12[m(c) - m(a)]}$$

where $V(a)$ is the variation of $m$ over $[a, a + \delta_a]$. If $t = c$, there exists
a positive number $\delta$, such that if $x \neq c$ and $x$ is in $(c - \delta, c + \delta) \cap [a, c]$, then

$$|F(x) - F(c)| < \frac{\varepsilon \cdot V(c)}{12[m(c) - m(a)]}$$

where $V(c)$ is the variation of $m$ over $[c - \delta, c]$. It follows that if $t$ is in $[a, c] - H \cap [a, c]$ and $x_1$ and $x_2$ are numbers in $(t - \delta, t + \delta) \cap [a, c]$, then

$$|F(x_1) - F(x_2)| < \frac{\varepsilon \cdot V(t)}{6[m(c) - m(a)]}.$$ 

Let $g_t = (t - \delta, t + \delta)$. For each $t$ in $[a, c] - H \cap [a, c]$, choose some particular $g_t$ satisfying the above conditions. Let $G$ denote the collection to which $g$ belongs if and only if either (1) for some positive integer $n$, $g = h_n$ or (2) for some $t$ in $[a, c] - H \cap [a, c]$, $g = g_t$. $G$ is a collection of open intervals covering $[a, c]$, hence there exists a finite sub-collection $G'$ of $G$ that covers $[a, c]$. Choose a finite chain \( \{R_1, R_2, \ldots, R_k\} \) of intervals of $G'$ covering $[a, c]$ and having the property that if $R_i \cap R_j \neq \emptyset$, then $|i - j| = 1$. Let $a = x_n, x_1$ be a number in $R_1 \cap R_n$, $x_2$ be a number in $R_1 \cap R_n$, $\ldots$, $x_{k-1}$ be a number in $R_{k-1} \cap R_k$, and $x_k = c$. Note that if for every $i \leq k$, $R_i$ is $g_t$ for some $t$ in $[a, c] - H \cap [a, c]$ and $V_i = V(t)$ for that $t$, then

$$\sum_{i=1}^{k} V_i < 3[m(c) - m(a)].$$

Now

$$F(c) - F(a) = \sum_{i=1}^{k} [F(x_i) - F(x_{i-1})].$$

Therefore

$$|F(c) - F(a)| \leq \sum_{i=1}^{k} |F(x_i) - F(x_{i-1})|$$

$$= \sum_{i=1}^{k} |F(x_i) - F(x_{i-1})|$$

where the first sum is the sum of those terms for which $R_i$ is some $h_n$ and the second sum is the sum of those terms for which $R_i$ is some $g_t$. Now $x_{i-1}$ and $x_i$ are in $R_i$ so that

$$|F(x_i) - F(x_{i-1})| < \begin{cases} \frac{\varepsilon}{2^{i+1}} & \text{if } R_i = h_n \\ \frac{\varepsilon \cdot V(t)}{6[m(c) - m(a)]} & \text{if } R_i = g_t \end{cases}.$$
Hence

$$\sum_{i=1}^{\infty} |F(x_i) - F(x_{i-1})| < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon/2$$

and

$$\sum_{i=2}^{\infty} |F(x_i) - F(x_{i-1})| < \frac{\varepsilon}{6[m(c) - m(\alpha)]} \sum_{i=1}^{k} V_i,$$

$$< \frac{\varepsilon \cdot 3[m(c) - m(\alpha)]}{6[m(c) - m(\alpha)]} = \frac{\varepsilon}{2}.$$

Therefore $|F(c) - F(\alpha)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus $F(c) = F(\alpha)$. But $c$ was any number in $(\alpha, b]$. Hence for each $x$ in $[\alpha, b]$, $F(x) = F(\alpha)$ or $f(x) = g(x) - g(\alpha) + f(\alpha)$.

**Theorem 3.** In order that the function $F$ defined on $[a, b]$ be the Lane integral of a function $f$ of bounded variation on $[a, b]$ with respect to a continuous, increasing function $m$ on $[a, b]$, it is necessary and sufficient that $F$ have bounded slope variation with respect to $m$ over $[a, b]$.

**Proof.** It is easy to see that the condition is necessary. Suppose that $F$ has bounded slope variation with respect to $m$ over $[a, b]$. Then $F$ is continuous on $[a, b]$. Let $f$ be the function defined, for each $x$ in $[a, b]$, by

$$\begin{cases} f(x) = D_{\alpha}F(x) & \text{for each } x \text{ in } [a, b) \\ f(b) = D_{\alpha}F(b) & \end{cases}$$

Then $f$ is of bounded variation on $[a, b]$ and is therefore quasicontinuous on $[a, b]$. Moreover, $D_mF(x) = f(x)$ for each $x$ in $[a, b] - E$, where $E$ is a countable set. Let $G$ be the function defined, for each $x$ in $[a, b]$, by $G(x) = \int_{a}^{x} f dm$. Then $G$ is continuous on $[a, b]$ and $D_mG(x) = f(x)$ at each number $x$ in $[a, b]$ such that $f$ is continuous at $(x, f(x))$. Since $f$ is quasicontinuous on $[a, b]$, $D_mG(x) = f(x)$ for each $x$ in $[a, b] - K$, where $K$ is a countable set. Therefore $D_mF(x) = D_mG(x)$ for each $x$ in $[a, b] - H$, where $H$ is a subset of $E \cup K$. It follows from Theorem 2 that $F(x) = \int_{a}^{x} f dm + F(a)$ for each $x$ in $[a, b]$.

That is, $F$ is the Lane integral of a function $f$ of bounded variation on $[a, b]$ with respect to a continuous, increasing function $m$ over $[a, b]$.

It should be noted that if $m = I$, then the Lane integral reduces to the Riemann integral so that Theorem 3 contains Riesz's theorem as a special case.
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