A GENERALIZATION OF A THEOREM OF F. RIESZ

Frank Norris Huggins
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FRANK N. HUGGINS

In this paper, the concept of bounded slope variation, that of the derivative of a function with respect to an increasing function, and the Lane integral are used to obtain a generalization of a theorem of Frédéric Riesz.

In [3], R. E. Lane defined an integral which is an extension of the Stieltjes mean sigma integral defined by H. L. Smith [5]. If each of $f$ and $g$ is a real-valued function whose domain includes $[a, b]$ and $D = \{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then $S_D(f, g)$ denotes the sum

$$\sum_{i=1}^{n} \frac{1}{2} [f(x_i) + f(x_{i-1})][g(x_i) - g(x_{i-1})].$$

The concepts of singular graph, exceptional number and summability set are as in [3]. If each of $f$ and $g$ is a real-valued function whose domain includes $[a, b]$ and if there exists a summability set $G$ for $f$ and $g$ in $[a, b]$, then the Lane integral $\int_a^b f dg$ is the refinement limit

$$\lim_{D \subset G} S_D(f, g).$$

In case the entire interval $[a, b]$ is a summability set for $f$ and $g$ in $[a, b]$, the Lane integral $\int_a^b f dg$ is the Stieltjes mean sigma integral $M\int_a^b f dg$.

By Theorem 4.1 of [2], if $f$ is quasicontinuous on $[a, b]$ and $g$ is of bounded variation on $[a, b]$, then $\int_a^b f dg$ exists. (A function $f$ is said to be quasicontinuous at $(c, f(c))$ if both $f(c^+)$ and $f(c^-)$ exist.)

DEFINITION 1. The statement that $f$ has bounded slope variation with respect to $m$ over $[a, b]$ means that $f$ is a function whose domain includes $[a, b]$, $m$ is a real-valued increasing function on $[a, b]$, and there exists a nonnegative number $B$ such that if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$ with $n > 1$, then

$$\sum_{i=1}^{n-1} \left| \frac{f(x_{i+1}) - f(x_i)}{m(x_{i+1}) - m(x_i)} - \frac{f(x_i) - f(x_{i-1})}{m(x_i) - m(x_{i-1})} \right| \leq B.$$

The least such number $B$ is called the slope variation of $f$ with respect to $m$ over $[a, b]$ and is denoted by $V^s_{x}(df/dm)$. [Note: $V^s_{x}(df/dm) = 0$.]

The above sum is nondecreasing with respect to refinements.

In [4], F. Riesz proved that a necessary and sufficient condition
that a function $F$ defined on the interval $[a, b]$ be the integral of a function of bounded variation on $[a, b]$ is that $F$ have bounded slope variation with respect to $I$ over $[a, b]$, where $I$ is the function defined, for each $x$, by $I(x) = x$. In this paper, Riesz's result will be generalized using the Lane integral instead of the Riemann integral.

By Lemma 3.3 of [6], if $f$ has bounded slope variation with respect to $m$ over $[a, b]$ and $a \leq c < b$, then

$$D^+_m f(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{m(x) - m(c)}$$
exists and if $a < c \leq b$,

$$D^-_m f(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{m(x) - m(c)}$$
exists.

**Lemma 1.** If $f$ has bounded slope variation with respect to $m$ over $[a, b]$, $c$ is a number in $[a, b]$, and $m$ is continuous on the right (left) at $(c, m(c))$, then $f$ is continuous on the right (left) at $(c, f(c))$.

**Proof.** Let $\epsilon$ denote a positive number and let $c$ be a number in $[a, b]$. Suppose $m$ is continuous on the right at $(c, m(c))$. Then $a \leq c < b$ and $D^+_m f(c)$ exists. Therefore there exists a positive number $\delta_1$ such that if $c < x < c + \delta_1$, then

$$\left| \frac{f(x) - f(c)}{m(x) - m(c)} - D^+_m f(c) \right| < 1$$
from which it follows that

$$|f(x) - f(c)| < \left[ |D^+_m f(c)| + 1 \right] |m(x) - m(c)| .$$

Since $m$ is continuous on the right at $(c, m(c))$, there exists a positive number $\delta_2$ such that if $c < x < c + \delta_2$, then $|m(x) - m(c)| < \epsilon/[|D^+_m f(c)| + 1]$. Let $\delta = \min. [\delta_1, \delta_2]$. Then if $c < x < c + \delta$,

$$|f(x) - f(c)| < \left[ |D^+_m f(c)| + 1 \right] |m(x) - m(c)| < \left[ |D^+_m f(c)| + 1 \right] \cdot \epsilon/[|D^+_m f(c)| + 1] = \epsilon .$$

Therefore $f$ is continuous on the right at $(c, f(c))$.

If $m$ is continuous on the left at $(c, m(c))$, a similar argument will show that $f$ is continuous on the left at $(c, f(c))$.

**Definition 2.** Suppose $m$ is an increasing function on $[a, b]$, $f$ is
a function whose domain includes \([a, b]\) and \(c\) is a number in \([a, b]\). The statement that \(f\) has a derivative with respect to \(m\) at the point \((c, f(c))\) means that

\[
D_m f(c) = \lim_{x \to c} \frac{f(x) - f(c)}{m(x) - m(c)}
\]
exists.

**Theorem 1.** If \(f\) has bounded slope variation with respect to \(m\) over \([a, b]\), then \(D_m f(x)\) exists for each \(x\) in \([a, b] - E\), where \(E\) is a countable set.

**Proof.** Since \(f\) has bounded slope variation with respect to \(m\) over \([a, b]\), \(D^+ m f(x)\) exists for each \(x\) in \([a, b]\) and \(D^- m f(x)\) exists for each \(x\) in \((a, b]\). Let \(E_1\) denote the set of all numbers \(x\) in \([a, b]\) such that \(D^- m f(x) < D^+ m f(x)\) and let \(E_2\) denote the set of all number \(x\) in \([a, b]\) such that \(D^- m f(x) > D^+ m f(x)\). Let all rational numbers be arranged in a sequence \(r_1, r_2, r_3, \ldots\). Then if \(c\) is a number in \(E_1\) there is a smallest positive integer \(k\) such that

\[
D^- m f(c) < r_k < D^+ m f(c).
\]

There is a smallest positive integer \(h\) such that \(r_h < c\) and

\[
\frac{f(x) - f(c)}{m(x) - m(c)} < r_k
\]
for \(r_h < x < c\) and a smallest positive integer \(n\) such that \(r_n > c\) and

\[
\frac{f(x) - f(c)}{m(x) - m(c)} > r_k
\]
for \(c < x < r_n\). These two inequalities together give

(1) \(f(x) - f(c) > r_k[m(x) - m(c)]\)

for \(r_h < x < r_n, x \neq c\). Thus to every number \(c\) in \(E_1\) there corresponds a unique triad \((h, k, n)\) of positive integers. Suppose some two numbers \(x_1\) and \(x_2\) of \(E_1\) correspond to the same triad \((h, k, n)\). Then, on putting \(c = x_1\) and \(x = x_2\) in (1), we have

\[
f(x_2) - f(x_1) > r_k[m(x_2) - m(x_1)]
\]
and, on putting \(c = x_2\) and \(x = x_1\),

\[
f(x_1) - f(x_2) > r_k[m(x_1) - m(x_2)]
\]
or
This involves a contradiction. Therefore no two numbers of \( E_1 \) correspond to the same triad. Since the set of triads of positive integers is countable, it follows that \( E_1 \) is countable. A similar argument will show that \( E_2 \) is countable. Therefore \( E = E_1 \cup E_2 \) is countable.

**Theorem 2.** If the function \( m \) is increasing on \([a, b]\), each of the functions \( f \) and \( g \) is continuous on \([a, b]\) and \( D_m f(x) = D_m g(x) \) for each \( x \) in \([a, b] - H\), where \( H \) is a countable set, then \( f(x) = g(x) - g(a) + f(a) \) for each \( x \) in \([a, b]\).

**Proof.** Let \( F \) be the function defined, for each \( x \) in \([a, b]\), by \( F(x) = f(x) - g(x) \). Then \( F \) is continuous on \([a, b]\) and \( D_m F(x) = 0 \) for each \( x \) in \([a, b] - H\). Let \( \varepsilon \) denote a positive number and let \( c \) be a number in \((a, b]\). Let \( H \cap [a, c] = \{p_1, p_2, \ldots, p_n, \ldots\} \). Since \( F \) is continuous on \([a, b]\), for each positive integer \( n \) there exists a positive number \( \delta_n \) such that if \( x \) is in \((p_n - \delta_n, p_n + \delta_n) \cap [a, c]\), then

\[
|F(x) - F(p_n)| < \varepsilon/2^{n+2}.
\]

Let \( h_n = (p_n - \delta_n, p_n + \delta_n) \). It follows that if \( x_1 \) and \( x_2 \) are numbers in \( h_n \cap [a, c] \), then

\[
|F(x_1) - F(x_2)| < \varepsilon/2^{n+1}.
\]

For each \( n \), choose some particular \( h_n \) satisfying the above conditions. Now consider any number \( t \) in \([a, c] - H \cap [a, c] \). Then \( D_m F(t) = 0 \). If \( t \) is in \((a, c)\), there is a positive number \( \delta_t \) such that \((t - \delta_t, t + \delta_t)\) is a subset of \((a, c)\) and if \( x \) is in \((t - \delta_t, t + \delta_t)\) and \( x \neq t \), then

\[
\left| \frac{F(x) - F(t)}{m(x) - m(t)} \right| < \frac{\varepsilon}{12[m(c) - m(a)]}
\]

or

\[
|F(x) - F(t)| < \frac{\varepsilon |m(x) - m(t)|}{12[m(c) - m(a)]} < \frac{\varepsilon \cdot V(t)}{12[m(c) - m(a)]}
\]

where \( V(t) \) is the variation of \( m \) over \([t - \delta_t, t + \delta_t]\). If \( t = a \), there exists a positive number \( \delta_a \) such that if \( x \neq a \) and \( x \) is in \((a - \delta_a, a + \delta_a) \cap [a, c]\), then

\[
|F(x) - F(a)| < \frac{\varepsilon \cdot V(a)}{12[m(c) - m(a)]}
\]

where \( V(a) \) is the variation of \( m \) over \([a, a + \delta_a]\). If \( t = c \), there exists
a positive number $\delta_c$ such that if $x \neq c$ and $x$ is in $(c - \delta_c, c + \delta_c) \cap [a, c]$, then

$$|F(x) - F(c)| < \frac{\varepsilon \cdot V(c)}{12[m(c) - m(a)]}$$

where $V(c)$ is the variation of $m$ over $[c - \delta_c, c]$. It follows that if $t$ is in $[a, c] - H \cap [a, c]$ and $x_1$ and $x_2$ are numbers in $(t - \delta_t, t + \delta_t) \cap [a, c]$, then

$$|F(x_1) - F(x_2)| < \frac{\varepsilon \cdot V(t)}{6[m(c) - m(a)]}.$$

Let $g_t = (t - \delta_t, t + \delta_t)$. For each $t$ in $[a, c] - H \cap [a, c]$, choose some particular $g_t$ satisfying the above conditions. Let $G$ denote the collection to which $g$ belongs if and only if either (1) for some positive integer $n, g = h_n$, or (2) for some $t$ in $[a, c] - H \cap [a, c], g = g_t$. $G$ is a collection of open intervals covering $[a, c]$, hence there exists a finite sub-collection $G'$ of $G$ that covers $[a, c]$. Choose a finite chain $\{R_1, R_2, \ldots, R_k\}$ of intervals of $G'$ covering $[a, c]$ and having the property that if $E_i \neq \emptyset$, then $|i - j| = 1$. Let $a = x_n, x_i$ be a number in $R_1 \cap R_2, x_i$ be a number in $R_2 \cap R_3, \ldots, x_{k-1}$ be a number in $R_{k-1} \cap R_k$, and $x_k = c$. Note that if for every $i \leq k, R_i$ is $g_t$ for some $t$ in $[a, c] - H \cap [a, c]$ and $V_i = V(t)$ for that $t$, then

$$\sum_{i=1}^k V_i < 3[m(c) - m(a)].$$

Now

$$F(c) - F(a) = \sum_{i=1}^k [F(x_i) - F(x_{i-1})].$$

Therefore

$$|F(c) - F(a)| \leq \sum_{i=1}^k |F(x_i) - F(x_{i-1})|$$

$$= \sum_{i=1}^k |F(x_i) - F(x_{i-1})|$$

$$+ \sum_{i=2}^k |F(x_i) - F(x_{i-1})|$$

where the first sum is the sum of those terms for which $R_i$ is some $h_n$ and the second sum is the sum of those terms for which $R_i$ is some $g_t$. Now $x_{i-1}$ and $x_i$ are in $R_i$ so that

$$|F(x_i) - F(x_{i-1})| \leq \begin{cases} \frac{\varepsilon}{2^{i+1}} & \text{if } R_i = h_n \\ \frac{\varepsilon \cdot V(t)}{6[m(c) - m(a)]} & \text{if } R_i = g_t. \end{cases}$$
Hence

\[ \sum_{i} |F(x_i) - F(x_{i-1})| \leq \sum_{n=1}^{\infty} \varepsilon/2^{n+1} = \varepsilon/2 \]

and

\[ \sum_{i} |F(x_i) - F(x_{i-1})| < \frac{\varepsilon}{6[m(c) - m(a)]} \sum_{i=1}^{k} V_i \]

\[ < \frac{\varepsilon \cdot 3[m(c) - m(a)]}{6[m(c) - m(a)]} = \frac{\varepsilon}{2}. \]

Therefore \( |F(c) - F(a)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \). Thus \( F(c) = F(a) \). But \( c \) was any number in \( (a, b) \). Hence for each \( x \) in \( [a, b] \), \( F(x) = F(a) \) or \( f(x) = g(x) - g(a) + f(a) \).

**Theorem 3.** In order that the function \( F \) defined on \([a, b]\) be the Lane integral of a function \( f \) of bounded variation on \([a, b]\) with respect to a continuous, increasing function \( m \) on \([a, b]\), it is necessary and sufficient that \( F \) have bounded slope variation with respect to \( m \) over \([a, b]\).

**Proof.** It is easy to see that the condition is necessary. Suppose that \( F \) has bounded slope variation with respect to \( m \) over \([a, b]\). Then \( F \) is continuous on \([a, b]\). Let \( f \) be the function defined, for each \( x \) in \([a, b]\), by

\[
\begin{cases}
  f(x) = D_+^m F(x) & \text{for each } x \text{ in } [a, b), \\
  f(b) = D_-^m F(b). 
\end{cases}
\]

Then \( f \) is of bounded variation on \([a, b]\) and is therefore quasicontinuous on \([a, b]\). Moreover, \( D_+^m F(x) = f(x) \) for each \( x \) in \([a, b] - E \), where \( E \) is a countable set. Let \( G \) be the function defined, for each \( x \) in \([a, b]\), by \( G(x) = \int_{a}^{x} f dm \). Then \( G \) is continuous on \([a, b]\) and \( D_+^m G(x) = f(x) \) at each number \( x \) in \([a, b]\) such that \( f \) is continuous at \((x, f(x))\). Since \( f \) is quasicontinuous on \([a, b]\), \( D_+^m G(x) = f(x) \) for each \( x \) in \([a, b] - K \), where \( K \) is a countable set. Therefore \( D_+^m F(x) = D_+^m G(x) \) for each \( x \) in \([a, b] - H \), where \( H \) is a subset of \( E \cup K \). It follows from Theorem 2 that \( F(x) = \int_{a}^{x} f dm + F(a) \) for each \( x \) in \([a, b]\). That is, \( F \) is the Lane integral of a function \( f \) of bounded variation on \([a, b]\) with respect to a continuous, increasing function \( m \) over \([a, b]\).

It should be noted that if \( m = I \), then the Lane integral reduces to the Riemann integral so that Theorem 3 contains Riesz's theorem as a special case.
REFERENCES


Received November 3, 1970 and in revised form February 17, 1971.

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