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REFINEMENTS OF WALLIS'S ESTIMATE AND THEIR GENERALIZATIONS

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Some refinements of Wallis's estimate for π noticed in the recent literature are pointed out as already contained in a certain continued fraction expansion due to Stieltjes. A property of the approximants to this continued fraction is established which yields a simple proof of the expansion and furnishes, in particular, interesting monotone sequences of rational numbers with limit π . Two estimates of the Wallis type involving quotients of gamma functions are derived. They include estimates for $\Gamma(\alpha)$ and $\pi \csc \pi\alpha$ ($0 < \alpha < 1$) both of which reduce for $\alpha = 1/2$ to one of the known refinements of the Wallis estimate.

0. Introduction. Let

$$g_0 = 1, \quad g_n = \frac{1.3 \cdots (2n-1)}{2.4 \cdots 2n}, \quad n = 1, 2, \dots$$

We have the well-known Wallis estimate

$$ng_n^2 < \frac{1}{\pi} < \left(n + \frac{1}{2}\right)g_n^2.$$

Obtaining the case $x = n + 1/2$ of the inequalities

$$(1) \quad x - \frac{1}{4} < \left[\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x)} \right]^2 < \frac{x^2}{x + \frac{1}{4}}, \quad x > 0$$

by an application of a theorem in mathematical statistics, John Gurland [3] notes that

$$\left(n + \frac{1}{4}\right)g_n^2 < \frac{1}{\pi} < \frac{(n + \frac{1}{2})^2}{n + \frac{3}{4}}g_n^2.$$

The first inequality here has been found earlier by D. K. Kazarinoff [4]. On the basis of a result of G. N. Watson, A. V. Boyd [1] has shown that one cannot have

$$\left(n + \frac{1}{4} + 1/(an + b)\right)g_n^2 < \frac{1}{\pi}, \quad a > 0, b > 0$$

for all n if $a < 32$ and asserts that

$$\left(n + \frac{1}{4} + 1/(32n + b_1)\right)g_n^2 < \frac{1}{\pi} < \frac{(n + \frac{1}{2})^2}{n + \frac{3}{4} + 1/(32n + b_2)}g_n^2$$

for all $n \geq 1$ with $b_1 = 32$ and $b_2 = 48$. All these facts are, however, overshadowed by the following continued fraction expansion due to Stieltjes [5]:

$$(I) \quad 4 \left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right]^2 = 4x+1 + \frac{1^2}{2(4x+1)} + \frac{3^2}{2(4x+1)} + \dots, \\ x > -\frac{1}{4}.$$

Indeed, this result, together with its obvious transformation

$$4 \left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} \right]^2 = \frac{(4x+2)^2}{4x+3} + \frac{1^2}{2(4x+3)} + \frac{3^2}{2(4x+3)} + \dots, \\ x > -\frac{1}{2},$$

suffices to dispose of (1) and the two observations made in [1], the second of which is seen to hold even with $b_1 = 12$ and $b_2 = 27$. We wish to point out a simple and informative proof of (I) which shows, in particular, that

$$(4n+1)g_n^2 \uparrow \frac{4}{\pi}, \quad \left(4n+1 + \frac{1^2}{2(4n+1)}\right)g_n^2 \downarrow \frac{4}{\pi}, \dots$$

A direct proof of (1) is easy. In fact, assuming throughout that $0 < \alpha < 1$, we prove the two generalizations

$$(II) \quad x - \frac{1-\alpha}{2} < \left[\frac{\Gamma(x+\alpha)}{\Gamma(x)} \right]^{1/\alpha} < \frac{1}{(1+\alpha/x)^{1/\alpha} - 1}, \quad x > 0,$$

$$(III) \quad x - \alpha(1-\alpha) < \frac{\Gamma(x+\alpha)\Gamma(x+1-\alpha)}{\Gamma^2(x)} \\ < \frac{x^2}{x+\alpha(1-\alpha)}, \quad x > 0.$$

As special cases of interest, we have estimates for $\Gamma(\alpha)$ and $\pi \csc \pi \alpha$ generalizing Gurland's estimate for π :

$$(n+\alpha/2)^{1-\alpha} g_n(\alpha) < \frac{1}{\Gamma(\alpha)} < \frac{n+\alpha}{(n+(1+\alpha)/2)^\alpha} g_n(\alpha), \\ \left(1 - \frac{\alpha^2}{n+\alpha}\right) G_n(\alpha) < \frac{\sin \pi \alpha}{\pi} < \left(1 + \frac{\alpha^2}{n+1-\alpha}\right)^{-1} G_n(\alpha),$$

where

$$g_n(\alpha) = \binom{\alpha+n-1}{n}, \quad G_n(\alpha) = \alpha \prod_{k=1}^n \left(1 - \frac{\alpha^2}{k^2}\right).$$

One should compare (II), (III) and the inequalities

$$(2) \quad x - 1 + \alpha < \left[\frac{\Gamma(x + \alpha)}{\Gamma(x)} \right]^{1/\alpha} < x, \quad x > 0,$$

which follow at once from the log-convexity of the gamma function. Wallis's estimate is the special case of (2) in which $\alpha = 1/2$ and $x = n + 1/2$ - the two together actually yield $\Gamma(1/2) = \sqrt{\pi}$. This is a simple evaluation of $\Gamma(1/2)$ that goes back to Stieltjes [2]; it is simple because (2) for $\alpha = 1/2$ requires only Schwarz's inequality for integrals.

The proofs of (I), (II) and (III) all utilize this familiar asymptotic formula implied by (2):

$$(3) \quad \Gamma(x + \alpha) \sim x^\alpha \Gamma(x), \quad x \rightarrow \infty.$$

1. The expansion (I). We have

$$C_k(x) \equiv x + \frac{1^2}{2x} + \frac{3^2}{2x} + \dots + \frac{(2k - 1)^2}{2x} = \frac{A_k(x)}{B_k(x)}, \quad k = 0, 1, \dots,$$

$W_k = A_k(x)$ and $W_k = B_k(x)$ being the two solutions of the recursion

$$W_{k+1} = 2x W_k + (2k + 1)^2 W_{k-1}$$

defined by the initial values

$$A_{-2}(x) = -x, \quad A_{-1}(x) = 1; \quad B_{-2}(x) = 1, \quad B_{-1}(x) = 0.$$

It is easily verified that the above recursion is equivalent to

$$W'_{k+1} = 2(x + 2\varepsilon) W'_k + (2k + 1)^2 W'_{k-1},$$

where

$$W'_k = (x + (2k + 2)\varepsilon) W_k + (2k + 1)^2 W_{k-1}, \quad \varepsilon = \pm 1.$$

This establishes the matrix identity

$$\begin{bmatrix} (x + 1)^2 B_k(x + 2) & A_k(x + 2) \\ (x - 1)^2 B_k(x - 2) & A_k(x - 2) \end{bmatrix} = \begin{bmatrix} x + 2k + 2 & (2k + 1)^2 \\ x - 2k - 2 & (2k + 1)^2 \end{bmatrix} \cdot \begin{bmatrix} A_k(x) & B_k(x) \\ A_{k-1}(x) & B_{k-1}(x) \end{bmatrix}.$$

by an induction from the cases $k - 1$ and $k (\geq 0)$ to the case $k + 1$. Passing to determinants, we at once see that

$$\text{sgn}\{(x - 1)^2 C_k(x + 2) - (x + 1)^2 C_k(x - 2)\} = (-1)^k, \quad x > 2,$$

which, on replacing x by $4x + 3$ and introducing

$$\gamma_k(x) = \left[\frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right]^2 C_k(4x + 1), \quad x > -\frac{1}{4},$$

may be written

$$\operatorname{sgn}\{\gamma_k(x + 1) - \gamma_k(x)\} = (-1)^k.$$

By (3), this yields

$$(*) \quad \gamma_{2k}(x + n) \uparrow 4, \quad \gamma_{2k+1}(x + n) \downarrow 4, \quad n \uparrow \infty.$$

Hence $\gamma_{2k}(x) < 4 < \gamma_{2k+1}(x)$ and so we obtain (I):

$$\lim_{k \rightarrow \infty} \gamma_k(x) = 4.$$

The existence of this limit is assured by a known theorem [5, p.239] on the convergence of an infinite continued fraction with positive elements.

2. The inequalities (II). Consider

$$f(p, x) = (x - p) \left[\frac{\Gamma(x)}{\Gamma(x + \alpha)} \right]^{1/\alpha}, \quad x > 0, -\infty < p < +\infty.$$

We have

$$\begin{aligned} \operatorname{sgn}\{f(p, x + 1) - f(p, x)\} &= \operatorname{sgn}\{p - p(x)\}, \\ p(x) &\equiv x - \frac{1}{(1 + \alpha/x)^{1/\alpha} - 1} \uparrow \frac{1 - \alpha}{2}, \quad (0 <) x \uparrow \infty, \\ f(p(x), x) &= f(p(x), x + 1) > f(p(x + 1), x + 1). \end{aligned}$$

The first of these assertions is easily checked and the last is obvious from the first two. The second, restated in the more convenient form

$$\chi(u) \equiv p\left(\frac{\alpha}{e^{2\alpha u} - 1}\right) = \frac{\alpha}{e^{2\alpha u} - 1} - \frac{1}{e^{2u} - 1} \uparrow \frac{1 - \alpha}{2}, \quad u \downarrow 0,$$

follows on observing that

$$2\chi'(u) = \frac{1}{\operatorname{sh}^2 u} - \frac{\alpha^2}{\operatorname{sh}^2 \alpha u} < 0,$$

$(\operatorname{sh}u)/u$ being increasing in $(0, \infty)$, while

$$\lim_{u \rightarrow 0} \chi(u) = \lim_{h \rightarrow 0} \frac{\alpha(e^h - 1) - (e^{\alpha h} - 1)}{\alpha h \cdot h} = \frac{1 - \alpha}{2}.$$

Hence, by (3), we have the following limit relations which contain more than (II):

$$(**) \quad f((1 - \alpha)/2, x + n) \uparrow 1, \quad f(p(x + n), x + n) \downarrow 1, \quad n \uparrow \infty.$$

3. The inequalities (III). Proceeding as before, let

$$g(q, x) = (x - q) \frac{\Gamma^2(x)}{\Gamma(x + \alpha)\Gamma(x + 1 - \alpha)}, \quad x > 0, -\infty < q < +\infty.$$

The readily verified facts

$$\begin{aligned} \operatorname{sgn}\{g(q, x + 1) - g(q, x)\} &= \operatorname{sgn}\{q - q(x)\}, \\ g(x) &\equiv \frac{\alpha(1 - \alpha)x}{x + \alpha(1 - \alpha)} \uparrow \alpha(1 - \alpha), \quad (0 <) x \uparrow \infty, \\ g(q(x), x) &= g(q(x), x + 1) > g(q(x + 1), x + 1), \end{aligned}$$

together with (3), prove more than (III):

$$(***) \quad g(\alpha(1 - \alpha), x + n) \uparrow 1, \quad g(q(x + n), x + n) \downarrow 1, \quad n \uparrow \infty.$$

An alternative proof is given by the product expansion

$$G(x) \equiv \frac{x\Gamma^2(x)}{\Gamma(x + \alpha)\Gamma(x + 1 - \alpha)} = \prod_{n=0}^{\infty} \left(1 + \frac{\alpha(1 - \alpha)}{(x + n)(x + n + 1)}\right),$$

which is evident from

$$\frac{G(x)}{G(x + 1)} = 1 + \frac{\alpha(1 - \alpha)}{x(x + 1)}, \quad \lim_{x \rightarrow \infty} G(x) = 1,$$

where the limit relation is a consequence of (3). The case $x = 1$ of the above expansion occurs in [6].

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