REFINEMENTS OF WALLIS’S ESTIMATE AND THEIR GENERALIZATIONS

T. S. Nanjundiah
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Some refinements of Wallis’s estimate for $\pi$ noticed in the recent literature are pointed out as already contained in a certain continued fraction expansion due to Stieltjes. A property of the approximants to this continued fraction is established which yields a simple proof of the expansion and furnishes, in particular, interesting monotone sequences of rational numbers with limit $\pi$. Two estimates of the Wallis type involving quotients of gamma functions are derived. They include estimates for $\Gamma(\alpha)$ and $\pi \csc \pi \alpha (0 < \alpha < 1)$ both of which reduce for $\alpha = 1/2$ to one of the known refinements of the Wallis estimate.

O. Introduction. Let

$$g_0 = 1, \quad g_n = \frac{1.3 \cdots (2n-1)}{2.4 \cdots 2n}, \quad n = 1, 2, \ldots .$$

We have the well-known Wallis estimate

$$ng_n^2 < \frac{1}{\pi} \left( n + \frac{1}{2} \right) g_n^2 .$$

Obtaining the case $x = n + 1/2$ of the inequalities

$$(1) \quad x - \frac{1}{4} < \left[ \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x)} \right]^2 < \frac{x}{x + \frac{1}{4}}, \quad x > 0$$

by an application of a theorem in mathematical statistics, John Gurland [3] notes that

$$\left( n + \frac{1}{4} \right) g_n^2 < \frac{1}{\pi} < \frac{(n + \frac{1}{2})^2}{n + \frac{1}{4}} g_n^2 .$$

The first inequality here has been found earlier by D. K. Kazarinoff [4]. On the basis of a result of G. N. Watson, A. V. Boyd [1] has shown that one cannot have

$$\left( n + \frac{1}{4} + 1/(an + b) \right) g_n^2 < \frac{1}{\pi}, \quad a > 0, b > 0$$

for all $n$ if $a < 32$ and asserts that

$$\left( n + \frac{1}{4} + 1/(32n + b) \right) g_n^2 < \frac{1}{\pi} < \frac{(n + \frac{1}{2})^2}{n + \frac{1}{4} + 1/(32n + b)} g_n^2$$

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for all $n \geq 1$ with $b_1 = 32$ and $b_2 = 48$. All these facts are, however, overshadowed by the following continued fraction expansion due to Stieltjes [5]:

$$4 \left[ \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} \right]^2 = 4x + 1 + \frac{1}{2(4x + 1)} + \frac{3}{2(4x + 1)} + \cdots,$$

$$x > -\frac{1}{4}.$$

Indeed, this result, together with its obvious transformation

$$4 \left[ \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} \right]^2 = \frac{(4x + 2)^2}{4x + 3} + \frac{1}{2(4x + 3)} + \frac{3}{2(4x + 3)} + \cdots,$$

$$x > -\frac{1}{2},$$

suffices to dispose of (1) and the two observations made in [1], the second of which is seen to hold even with $b_1 = 12$ and $b_2 = 27$. We wish to point out a simple and informative proof of (1) which shows, in particular, that

$$(4n + 1)g_n^2 \uparrow \frac{4}{\pi}, \quad (4n + 1 + \frac{1}{2(4n + 1)})g_n^2 \downarrow \frac{4}{\pi}, \cdots.$$ 

A direct proof of (1) is easy. In fact, assuming throughout that $0 < \alpha < 1$, we prove the two generalizations

$$(II) \quad x - \frac{1 - \alpha}{2} < \left[ \frac{\Gamma(x + \alpha)}{\Gamma(x)} \right]^{1/\alpha} < \frac{1}{(1 + \alpha/x)^{1/\alpha} - 1}, \quad x > 0,$$

$$x - \alpha(1 - \alpha) < \frac{\Gamma(x + \alpha)\Gamma(x + 1 - \alpha)}{\Gamma^2(x)}$$

$$< \frac{x^\alpha}{x + \alpha(1 - \alpha)}, \quad x > 0.$$

As special cases of interest, we have estimates for $\Gamma(\alpha)$ and $\pi \csc \pi \alpha$ generalizing Gurland's estimate for $\pi$:

$$(n + \alpha/2)^{1-\alpha} g_n(\alpha) < \frac{1}{\Gamma(\alpha)} < \frac{n + \alpha}{(n + (1 + \alpha)/2)^\alpha} g_n(\alpha),$$

$$\left(1 - \frac{\alpha^2}{n + \alpha}\right) G_n(\alpha) < \frac{\sin \pi \alpha}{\pi} < \left(1 + \frac{\alpha^2}{n + 1 - \alpha}\right)^{-1} G_n(\alpha),$$

where

$$g_n(\alpha) = \left(\alpha + n - 1\right), \quad G_n(\alpha) = \alpha \prod_{k=1}^{n} \left(1 - \frac{\alpha^2}{k^2}\right).$$
One should compare (II), (III) and the inequalities

(2) \[ x - 1 + \alpha < \left[ \frac{\Gamma(x + \alpha)}{\Gamma(x)} \right]^{1/\alpha} < x, \quad x > 0, \]

which follow at once from the log-convexity of the gamma function. Wallis's estimate is the special case of (2) in which \( \alpha = 1/2 \) and \( x = n + 1/2 \) — the two together actually yield \( \Gamma(1/2) = \sqrt{\pi} \). This is a simple evaluation of \( \Gamma(1/2) \) that goes back to Stieltjes [2]; it is simple because (2) for \( \alpha = 1/2 \) requires only Schwarz's inequality for integrals.

The proofs of (I), (II) and (III) all utilize this familiar asymptotic formula implied by (2):

(3) \[ \Gamma(x + \alpha) \sim x^\alpha \Gamma(x), \quad x \to \infty. \]

1. The expansion (I). We have

\[ C_k(x) \equiv x + \frac{1^2}{2x} + \frac{3^2}{2x} + \cdots + \frac{(2k - 1)^2}{2x} = \frac{A_k(x)}{B_k(x)}, \]

\( k = 0, 1, \ldots \),

\( W_k = A_k(x) \) and \( W_k = B_k(x) \) being the two solutions of the recursion

\[ W_{k+1} = 2xW_k + (2k + 1)^2 W_{k-1} \]

defined by the initial values

\[ A_{-3}(x) = -x, \quad A_{-1}(x) = 1; \quad B_{-3}(x) = 1, \quad B_{-1}(x) = 0. \]

It is easily verified that the above recursion is equivalent to

\[ W'_{k+1} = 2(x + 2\varepsilon)W'_k + (2k + 1)^2 W'_{k-1}, \]

where

\[ W'_k = (x + (2k + 2)\varepsilon)W_k + (2k + 1)^2 W_{k-1}, \quad \varepsilon = \pm 1. \]

This establishes the matrix identity

\[ \begin{bmatrix} (x + 1)^2 B_k(x + 2) & A_k(x + 2) \\ (x - 1)^2 B_k(x - 2) & A_k(x - 2) \end{bmatrix} = \begin{bmatrix} x + 2k + 2 & (2k + 1)^2 \\ x - 2k - 2 & (2k + 1)^2 \end{bmatrix} \cdot \begin{bmatrix} A_k(x) & B_k(x) \\ A_{k-1}(x) & B_{k-1}(x) \end{bmatrix}. \]

by an induction from the cases \( k - 1 \) and \( k(\geq 0) \) to the case \( k + 1 \). Passing to determinants, we at once see that

\[ \text{sgn}\{(x - 1)^2 C_k(x + 2) - (x + 1)^2 C_k(x - 2)\} = (-1)^k, \quad x > 2, \]

which, on replacing \( x \) by \( 4x + 3 \) and introducing
\[
\gamma_k(x) = \left[ \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \right]^2 C_k(4x + 1), \quad x > -\frac{1}{4},
\]
may be written
\[
\text{sgn}\{\gamma_k(x + 1) - \gamma_k(x)\} = (-1)^k.
\]
By (3), this yields
\[
(*) \quad \gamma_{2k}(x + n) \uparrow 4, \quad \gamma_{2k+1}(x + n) \downarrow 4, \quad n \uparrow \infty.
\]
Hence \(\gamma_{2k}(x) < 4 < \gamma_{2k+1}(x)\) and so we obtain (I):
\[
\lim_{k \to \infty} \gamma_k(x) = 4.
\]
The existence of this limit is assured by a known theorem [5, p.239] on the convergence of an infinite continued fraction with positive elements.

2. The inequalities (II). Consider
\[
f(p(x), x) = (x - p) \left[ \frac{\Gamma(x)}{\Gamma(x + \alpha)} \right]^{1/\alpha}, \quad x > 0, \ -\infty < p < +\infty.
\]
We have
\[
\text{sgn}\{f(p, x + 1) - f(p, x)\} = \text{sgn}\{p - p(x)\},
\]
\[
p(x) \equiv x - \frac{1}{(1 + \alpha/x)^{1/\alpha} - 1} \uparrow \frac{1 - \alpha}{2}, \quad (0 <) \ x \uparrow \infty,
\]
\[
f(p(x), x) = f(p(x), x + 1) > f(p(x + 1), x + 1).
\]
The first of these assertions is easily checked and the last is obvious from the first two. The second, restated in the more convenient form
\[
\chi(u) \equiv p\left( \frac{\alpha}{e^{\alpha u} - 1} \right) = \frac{\alpha}{e^{\alpha u} - 1} - \frac{1}{e^{\alpha u} - 1} \uparrow \frac{1 - \alpha}{2}, \quad u \downarrow 0,
\]
follows on observing that
\[
2\chi'(u) = \frac{1}{sh^2u} - \frac{\alpha^2}{sh^3\alpha u} < 0,
\]
\((shu)/u\) being increasing in \((0, \infty)\), while
\[
\lim_{u \to 0} \chi(u) = \lim_{h \to 0} \frac{\alpha(e^{\alpha h} - 1) - (e^{\alpha h} - 1)}{\alpha h \cdot h} = \frac{1 - \alpha}{2}.
\]
Hence, by (3), we have the following limit relations which contain more than (II):
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(**) \( f((1 - \alpha)/2, x + n) \uparrow 1, \quad f(p(x + n), x + n) \downarrow 1, \quad n \uparrow \infty. \)

3. The inequalities (III). Proceeding as before, let

\[ g(q, x) = \frac{\Gamma^n(x)}{\Gamma(x + \alpha)\Gamma(x + 1 - \alpha)}, \quad x > 0, \quad -\infty < q < +\infty. \]

The readily verified facts

\[ \text{sgn}(g(q, x + 1) - g(q, x)) = \text{sgn}(q - q(x)), \]

\[ q(x) \equiv \frac{\alpha(1 - \alpha)x}{x + \alpha(1 - \alpha)} \uparrow \alpha(1 - \alpha), \quad (0 <) x \uparrow \infty, \]

\[ g(q(x), x) = g(q(x), x + 1) > g(q(x + 1), x + 1), \]

together with (3), prove more than (III):

(***) \( g(\alpha(1 - \alpha), x + n) \uparrow 1, \quad g(q(x + n), x + n) \downarrow 1, \quad n \uparrow \infty. \)

An alternative proof is given by the product expansion

\[ G(x) \equiv \frac{x\Gamma^n(x)}{\Gamma(x + \alpha)\Gamma(x + 1 - \alpha)} = \prod_{n=0}^{\infty} \left(1 + \frac{\alpha(1 - \alpha)}{(x + n)(x + n + 1)}\right), \]

which is evident from

\[ \frac{G(x)}{G(x + 1)} = 1 + \frac{\alpha(1 - \alpha)}{x(x + 1)}, \quad \lim_{x \to \infty} G(x) = 1, \]

where the limit relation is a consequence of (3). The case \( x = 1 \) of the above expansion occurs in [6].

REFERENCES


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