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GENERATING MONOMIALS FOR FINITE SEMIGROUPS

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In this paper consideration is given semigroups which arise from a group (G, \cdot) by defining a binary operation \circ on G by the rule

$$x \circ y = x\phi y\psi \quad \text{for all } x, y \text{ in } G,$$

where ϕ, ψ are endomorphisms of G . In particular, the structure of such semigroups is determined. Also determined are the structure and number of semigroups that can be defined by

$$x \circ y = ax^s y^t \quad \text{for all } x, y \text{ in } G,$$

where (G, \cdot) is a finite abelian group containing a , and s, t are nonnegative integers.

1. Introduction. Let (G, \cdot) be a groupoid and let ϕ, ψ be transformations of G . A possibly different groupoid (G, \circ) is defined by the rule

$$x \circ y = x\phi y\psi \quad \text{for all } x, y \text{ in } G.$$

In §2 of this paper we assume that (G, \cdot) is a finite abelian group and define a groupoid (G, \circ) by the rule

$$x \circ y = ax^s y^t \quad \text{for all } x, y \text{ in } G,$$

where s, t are nonnegative integers and $a \in G$. Necessary and sufficient conditions on a, s , and t are found in order for (G, \circ) to be a semigroup. Also, we determine the number of nonequivalent (i.e., non-isomorphic, non-anti-isomorphic) semigroups that are defined in this manner. Whenever the rule

$$x \circ y = ax^s y^t \quad \text{for all } x, y \text{ in } G,$$

defines a semigroup, we say that (G, \circ) is generated by the monomial $ax^s y^t$ over (G, \cdot) .

In §3 it is shown that if a semigroup (G, \circ) is defined by the rule

$$x \circ y = x\phi y\psi \quad \text{for all } x, y \text{ in } G,$$

where ϕ, ψ are endomorphisms of the group (G, \cdot) , then (G, \circ) is an inflation of the direct product of a group and a rectangular band. Consequently, a semigroup generated by a monomial over a finite abelian group is an inflation of the direct product of a group

and a rectangular band. Finally, if $(F_q, +, \cdot)$ is a finite field of order q and if the rule

$$x \circ y = ax^s y^t \quad \text{for all } x, y \text{ in } F_q,$$

where $a \in F_q$ defines a semigroup (F_q, \circ) , then (F_q, \circ) is an inflation of the direct product of a cyclic group and a rectangular band, together with a zero element. This is a generalization of the results obtained in [3] by Plemmons and Yoshida.

2. Generating monomials. Throughout this section let (G, \cdot) be a finite abelian group with identity element e , and let M denote the least common multiple of the orders of the elements of G . Then M is the least positive integer q such that $x^q = e$ for all x in G . The following theorem gives necessary and sufficient conditions on a monomial $ax^s y^t$ over (G, \cdot) , in order for it to generate a semigroup.

THEOREM 1. *The monomial $ax^s y^t$ generates a semigroup over (G, \cdot) if and only if*

- (i) $a^{s-t} = e$ and
- (ii) $s^2 - s$ and $t^2 - t$ are multiples of M .

Proof. The monomial $ax^s y^t$ generates a semigroup over (G, \cdot) if and only if for all x, y, z in G

$$a(ax^s y^t)^s z^t = ax^s (ay^s z^t)^t$$

which holds if and only if for all x, y, z in G

$$a^{s+1} x^{s^2} y^{st} z^t = a^{t+1} x^s y^{st} z^{t^2}$$

which in turn holds if and only if for all x, z in G

$$(2.1) \quad a^{s-t} x^{s^2-s} = z^{t^2-t}.$$

Assuming that (i) and (ii) hold, it follows that (2.1) holds since each side of the equation reduces to e . Thus $ax^s y^t$ generates a semigroup. Conversely, if $ax^s y^t$ generates a semigroup then equation (2.1) holds for all x, z in G , and in particular when $x = z = e$, so that $a^{s-t} = e$. By letting $z = e$ in equation (2.1) and replacing a^{s-t} by e , we get that $x^{s^2-s} = e$ for all x in G , whence $s^2 - s$ is a multiple of M . In a similar fashion it can be shown that $t^2 - t$ is a multiple of M .

If $s \geq M$, then $s = qM + r$ for some integers q and r , where $q > 0$ and $0 \leq r < M$, so that

$$ax^s y^t = ax^r y^t \quad \text{for all } x, y \text{ in } G.$$

Hence, in searching for the number of nonequivalent semigroups generated by monomials over (G, \cdot) we can assume that $0 \leq s < M$ and $0 \leq t < M$. Also, since the semigroup generated by $ax^s y^s$ is anti-isomorphic to the one generated by $ax^s y^t$ we can assume that $t \leq s$. Furthermore, the following lemma shows that we need only consider monomials with $a = e$.

LEMMA 1. Suppose $ax^s y^t$ generates a semigroup (G, \circ) over (G, \cdot) . Let $(G, *)$ be the semigroup generated by $x^s y^t$ and let k denote the order of a in (G, \cdot) . Let m be the solution to the congruence

$$(2t - 1)x \equiv 1 \pmod{k}.$$

Then m is unique \pmod{k} and the mapping α from G into G defined by

$$x\alpha = a^m x \quad \text{for all } x \text{ in } G,$$

is an isomorphism of (G, \circ) onto $(G, *)$.

Proof. Since k is the order of a in (G, \cdot) , it follows that $k \mid M$. Since $ax^s y^t$ generates a semigroup, $M \mid t^2 - t$, whence $k \mid t^2 - t$. Therefore, the greatest common divisor of $2t - 1$ and k must divide $(2t - 1)^2 - 4(t^2 - t) = 1$, whence $2t - 1$ and k are relatively prime. Hence [2, Theorem 3-11, p. 34] there exists a unique solution $m \pmod{k}$ to the congruence

$$(2t - 1)x \equiv 1 \pmod{k}.$$

Therefore k is a factor of $m(2t - 1) - 1$. Now, the mapping α from G into G defined by

$$\alpha : z \rightarrow a^m z$$

is a permutation of G . Let x, y be arbitrary elements of G . Then

$$\begin{aligned} (x\alpha) * (y\alpha) &= (a^m x)^s (a^m y)^t \\ &= a^{m(s+t)} x^s y^t \\ &= a^{m+1} x^s y^t \end{aligned}$$

since

$$a^{m(s+t)-(m+1)} = a^{m(s+t-1)-1} = a^{m(s-t)+m(2t-1)-1} = e.$$

Therefore,

$$\begin{aligned} (x\alpha) * (y\alpha) &= a^{m+1} x^s y^t \\ &= (ax^s y^t)\alpha \\ &= (x \circ y)\alpha. \end{aligned}$$

Thus α is an isomorphism of (G, \circ) onto $(G, *)$.

Let n denote the order of (G, \cdot) and let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime power factorization of n , where $p_i \neq p_j$ if $i \neq j$, and $\alpha_i > 0$ for $1 \leq i \leq r$. By the fundamental theorem for finite abelian groups, G has the structure $S(p_1) \times S(p_2) \times \cdots \times S(p_r)$ where each $S(p_i)$ is the Sylow p_i -subgroup of (G, \cdot) of order $p_i^{\alpha_i}$ for $1 \leq i \leq r$. The order of any element in $S(p_i)$ is a power of the prime p_i so that for each prime p_i with $1 \leq i \leq r$, there exists an element $x_i \in G$ having order a power > 0 of p_i . Thus the prime power factorization of M is $M = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_r^{\gamma_r}$ where $0 < \gamma_i \leq \alpha_i$ for $1 \leq i \leq r$.

For each integer m let

$$G_m = \{x \in G: x^m = e\}.$$

Let s be a positive integer such that $M | s(s-1)$. Since s and $s-1$ are relatively prime, the prime factors of M which divide s do not divide $s-1$, and those dividing $s-1$ do not divide s . Assume that the indexing of the primes p_i in the factorization of M is such that $p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_j^{\gamma_j} | (s-1)$ and $p_{j+1}^{\gamma_{j+1}} p_{j+2}^{\gamma_{j+2}} \cdots p_r^{\gamma_r} | s$. Identifying the elements of G and $S(p_1) \times S(p_2) \times \cdots \times S(p_r)$ we get the following lemma.

LEMMA 2. *The set G_{s-1} is the subgroup $S(p_1) \times S(p_2) \times \cdots \times S(p_j)$ of G having order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}$.*

Proof. Let $x \in G_{s-1}$. Written as an r -tuple, $x = (x_1, x_2, \cdots, x_r)$, so $x^{s-1} = (x_1^{s-1}, x_2^{s-1}, \cdots, x_r^{s-1}) = e_r$, where e_r is the r -tuple (e, e, \cdots, e) . In particular, $x_{j+1}^{s-1} = x_{j+2}^{s-1} = \cdots = x_r^{s-1} = e$. Since the orders of $x_{j+1}, x_{j+2}, \cdots, x_r$ are relatively prime to $s-1$ it follows that $x_{j+1} = x_{j+2} = \cdots = x_r = e$. Hence $x \in S(p_1) \times S(p_2) \times \cdots \times S(p_j)$. Conversely, let $x \in S(p_1) \times S(p_2) \times \cdots \times S(p_j)$. We write

$$x = (x_1, x_2, \cdots, x_j).$$

Letting e_j denote the j -tuple (e, e, \cdots, e) , we have

$$e_j = x^{s(s-1)} = (x_1^{s(s-1)}, x_2^{s(s-1)}, \cdots, x_j^{s(s-1)}),$$

so that $x_1^{s(s-1)} = x_2^{s(s-1)} = \cdots = x_j^{s(s-1)} = e$. Since the orders of x_1, x_2, \cdots, x_j are relatively prime to s , $x_1^{s-1} = x_2^{s-1} = \cdots = x_j^{s-1} = e$, whence $x^{s-1} = e_j$ and $x \in G_{s-1}$.

LEMMA 3. *Let s and s' be positive integers less than M such that $M | s^2 - s$ and $M | s'^2 - s'$. If the order of G_{s-1} is the same as the order of $G_{s'-1}$ then $s = s'$.*

Proof. By Lemma 2 the subgroups G_{s-1} and $G_{s'-1}$ are direct products of Sylow p -subgroups of G . Since the order of G_{s-1} is the same as the order of $G_{s'-1}$, it follows that the prime powers in the factorization of M which divide $s-1$ are exactly those which divide $s'-1$. Thus $M|s(s'-1)$ and $M|s'(s-1)$, whence

$$M|[s(s'-1) - s'(s-1)] ,$$

so $M|s'-s$. Since $-M < s' - s < M$, $s' - s = 0$, whence $s' = s$.

THEOREM 2. *Suppose $x^s y^t$ and $x^{s'} y^{t'}$ generate semigroups over (G, \cdot) , where $0 \leq t \leq s < M$ and $0 \leq t' \leq s' < M$. Then these semigroups are isomorphic if and only if $s = s'$ and $t = t'$.*

Proof. Clearly if $s = s'$ and $t = t'$ then $x^s y^t$ and $x^{s'} y^{t'}$ generate the same semigroup over (G, \cdot) . Conversely, suppose that $x^s y^t$ and $x^{s'} y^{t'}$ generate semigroups (G, \circ) and $(G, *)$, respectively, and suppose (G, \circ) is isomorphic to $(G, *)$. Then the Cayley tables for (G, \circ) and $(G, *)$ must have the same number of distinct rows. That is, (G, \circ) and $(G, *)$ must have the same number of distinct inner left translations [1, p. 9]. The distinct inner left translations of (G, \circ) are determined by the distinct elements of the set $\{x^s: x \in G\}$. But

$$\{x^s: x \in G\} = G_{s-1}$$

as defined above. Thus the orders of G_{s-1} and $G_{s'-1}$ are equal, whence by Lemma 3, $s = s'$ if both s and s' are positive. If $s = 0$ then $G_{s-1} = G_{s'-1} = \{e\}$, so that $M|s'$, whence $s' = 0$. Similarly, if $s' = 0$ then $s = 0$, so that in any case $s = s'$. Dually, by considering columns in the Cayley tables of (G, \circ) and $(G, *)$, we see that $t = t'$.

We now approach the problem of determining the number of non-equivalent semigroups of order n generated by monomials over (G, \cdot) . The integers s with $0 \leq s < M$ that will serve as exponents in generating monomials are exactly those such that $M|s^2 - s$. Hence the set H of such integers is the solution set of the congruence

$$(2.2) \quad x^2 - x \equiv 0 \pmod{M} .$$

LEMMA 4. *The cardinality of the solution set H to the congruence (2.2) is 2^r , where r is the number of distinct primes in the prime power factorization of M .*

Proof. Let $M = p_1^{r_1} p_2^{r_2} \cdots p_r^{r_r}$ be the prime power factorization of M . Then x_0 is a solution to (2.2) if and only if x_0 is a simultaneous solution to the system of congruences

$$(2.3) \quad x^2 - x \equiv 0 \pmod{p_i^{r_i}} \quad 1 \leq i \leq r.$$

For each i , $1 \leq i \leq r$, suppose c_i is a solution to $x^2 - x \equiv 0 \pmod{p_i^{r_i}}$. Then, by the Chinese Remainder Theorem, there is a solution x_0 to the system

$$x \equiv c_1 \pmod{p_1^{r_1}}, x \equiv c_2 \pmod{p_2^{r_2}}, \dots, x \equiv c_r \pmod{p_r^{r_r}}$$

which is unique modulo M . Then each r -tuple (c_1, \dots, c_r) gives rise to a unique solution \pmod{M} to system (2.3). Thus the number of solutions to (2.2) is the product of the numbers of roots of the congruences in (2.3). But, by § 3.5 of [2], the solution set to each of these congruences is $\{0, 1\}$, whence the cardinality of the solution set of (2.2) is 2^r .

Finally, we have the following theorem.

THEOREM 3. *The number N_G of nonequivalent semigroups generated by monomials over (G, \cdot) is $2^{r-1}(2^r + 1)$, where r is the number of distinct primes which divide M .*

Proof. The pairs s, t of elements of H yield monomials $x^s y^t$ which generate semigroups over (G, \cdot) . Moreover, these are the only pairs modulo M which will do so. Thus to determine N_G we need only count the ways in which s and t can be picked from H with $t \leq s$. There are

$$1 + 2 + 3 + \dots + 2^r = \frac{2^r(2^r + 1)}{2} = 2^{r-1}(2^r + 1)$$

ways to do this.

3. Structure theorems. The following definition and facts are the contents of [1, p. 98, Exercise 10]. Let T be a semigroup. With each element α of T , associate a set X_α containing α such that the sets X_α are mutually disjoint. Let $S = \bigcup_{\alpha \in T} X_\alpha$, and let the product in T be extended to a product in S by defining $ab = \alpha\beta$ if $a \in X_\alpha$ and $b \in X_\beta$. Then S is a semigroup and is said to be an *inflation* of T . Now, T is a subsemigroup of S such that $S^2 \subseteq T$. If we define a mapping θ from S into T by $a\theta = \alpha$ when $a \in X_\alpha$, then

- (i) θ maps S upon T ,
- (ii) $\theta^2 = \theta$, and
- (iii) $(a\theta)(b\theta) = ab$ for all $a, b \in S$.

Let T be a subsemigroup of S such that $S^2 \subseteq T$, and let θ be a transformation of S having properties (i), (ii), and (iii) above. Then S is an inflation of T .

By a *left zero semigroup* we mean a semigroup S such that $xy = x$

for all $x, y \in S$. A *right zero semigroup* is defined dually.

THEOREM 4. *Let (S, \cdot) be a semigroup such that for some transformation ϕ of S , $xy = x\phi$ for all $x, y \in S$. Then S is an inflation of the range $S\phi$ of ϕ , and $S\phi$ is a left zero semigroup. Conversely, each inflation of a left zero semigroup is obtained in this manner.*

Proof. Since S is a semigroup, $(xy)z = x(yz)$ for all $x, y, z \in S$, so $x\phi^2 = x\phi$ for all $x \in S$, whence $\phi^2 = \phi$ on S . Since $S^2 = S\phi$, $S\phi$ is a subsemigroup of S such that $S^2 \subseteq S\phi$. Now ϕ maps S onto $S\phi$ and

$$a\phi b\phi = a\phi^2 = a\phi = ab \quad \text{for all } a, b \in S.$$

Hence, S is an inflation of $S\phi$. Let $a, b \in S\phi$. Then $a = a\phi$, so

$$ab = a\phi b = a\phi^2 = a\phi = a,$$

thus $S\phi$ is a left zero semigroup. Conversely, let (S, \cdot) be an inflation of a left zero semigroup L . Since S is an inflation of L , S is the disjoint union of subsets X_a , where $a \in L \cap X_a$. Define a transformation ϕ of S by $x\phi = a$ if and only if $x \in X_a$. Let $x, y \in S$ with $x \in X_a$ and $y \in X_b$. Then $xy = ab = a = x\phi$.

COROLLARY 1. *If (G, \circ) is generated by x^s over a finite abelian group (G, \cdot) , then (G, \circ) is an inflation of the left zero semigroup (L, \circ) , where $L = \{x^s: x \in G\}$.*

By the dual of Theorem 4 we get the following corollary.

COROLLARY 2. *If (G, \circ) is generated by y^t over the finite abelian group (G, \cdot) , then (G, \circ) is an inflation of the right zero semigroup (R, \circ) , where $R = \{y^t: y \in G\}$.*

Before investigating the structure of semigroups generated by the more general monomial $x^s y^t$ with $0 \leq t \leq s < M$, we prove the following lemma.

LEMMA 5. *Suppose the semigroup (G, \circ) is generated by $x^s y^t$ over an abelian group (G, \cdot) with $0 \leq t \leq s < M$. Then \circ is commutative if and only if $s = t$.*

Proof. Suppose $s = t$. Then for $x, y \in G$ we have

$$x \circ y = x^s y^s = y^s x^s = y \circ x.$$

Conversely, if \circ is commutative, then $x \circ y = y \circ x$ for all $x, y \in G$, so that $x^s y^t = y^s x^t$ for all $x, y \in G$. Letting $y = e$, we see that $x^s = x^t$ for all $x \in G$, so that $M|s - t$. Thus $s - t = 0$, whence $s = t$.

Given an arbitrary group (G, \cdot) and a pair of transformations ϕ, ψ of G , a groupoid (G, \circ) is defined by the rule

$$x \circ y = x\phi y\psi \quad \text{for all } x, y \text{ in } G.$$

We say that (G, \circ) is generated by the pair of transformations (ϕ, ψ) over (G, \cdot) . If we insist that the transformations ϕ and ψ be endomorphisms, the following lemma gives necessary and sufficient conditions in order for (G, \circ) to be a semigroup.

LEMMA 6. *Let (G, \cdot) be an arbitrary group with identity element e , and let ϕ, ψ be endomorphisms of (G, \cdot) . Define a groupoid (G, \circ) by the rule*

$$x \circ y = x\phi y\psi \quad \text{for all } x, y \text{ in } G.$$

Then (G, \circ) is a semigroup if and only if ϕ and ψ are idempotent and commute.

Proof. Assume that the groupoid (G, \circ) is a semigroup. Then

$$(x \circ y) \circ z = x \circ (y \circ z) \quad \text{for all } x, y, z \text{ in } G,$$

so

$$(3.1) \quad (x\phi y\psi)\phi \cdot z\psi = x\phi(y\phi z\psi)\psi \quad \text{for all } x, y, z \text{ in } G.$$

Upon setting $y = z = e$ in (3.1), we get

$$(x\phi)\phi = x\phi \quad \text{for all } x \text{ in } G,$$

since $e\phi = e\psi = e$. In a similar fashion $\psi^2 = \psi$. Letting $x = z = e$ in (3.1), we see that

$$(y\psi)\phi = (y\phi)\psi \quad \text{for all } y \text{ in } G,$$

hence $\phi\psi = \psi\phi$. Conversely, assume that $\phi^2 = \phi$, $\psi^2 = \psi$, and $\phi\psi = \psi\phi$. Then for arbitrary $x, y, z \in G$

$$\begin{aligned} (x \circ y) \circ z &= (x\phi y\psi)\phi \cdot z\psi \\ &= x\phi^2 \cdot y\psi\phi \cdot z\psi \\ &= x\phi \cdot y\phi\psi \cdot z\psi^2 \\ &= x\phi(y\phi z\psi)\psi \\ &= x \circ (y \circ z). \end{aligned}$$

Thus (G, \circ) is a semigroup.

The following definitions come from [1, p. 98, p. 25]. A semigroup S is called *stationary on the right* if for all a, b, c in S , $ab = ac$ implies $xb = xc$ for all $x \in S$. A semigroup S is called *E-inversive* if for each $a \in S$ there exists $x \in S$ such that ax is idempotent. Let a, b, x, y be elements of a semigroup S . Consider the four elements ax, ay, bx, by of S . We call S *rectangular* if, whenever three elements are equal, all four are equal. Let X and Y be any two sets, and define a binary operation in $S = X \times Y$ by

$$(x_1, y_1)(x_2, y_2) = (x_1, y_2)$$

where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then S is a semigroup called the *rectangular band* on $X \times Y$.

THEOREM 5. *Let (G, \circ) be a semigroup generated by a pair of endomorphisms (ϕ, ψ) over the group (G, \cdot) . Then (G, \circ) is an inflation of its kernel $G \circ G$ and its kernel is isomorphic to the direct product of a group and a rectangular band.*

Proof. By Lemma 6, $\phi^2 = \phi$, $\psi^2 = \psi$, and $\phi\psi = \psi\phi$. Now (G, \circ) is stationary on the right, since if $a \circ b = a \circ c$ for arbitrary $a, b, c \in G$ then $a\phi b\psi = a\phi c\psi$, so $b\psi = c\psi$. Thus $x\phi b\psi = x\phi c\psi$ for all $x \in G$, so that $x \circ b = x \circ c$ for all $x \in G$. Let $a \in G$ and denote by a^{-1} its group inverse. Then

$$a \circ a^{-1} = a\phi(a\psi)^{-1}.$$

Now,

$$\begin{aligned} (a \circ a^{-1}) \circ (a \circ a^{-1}) &= (a\phi(a\psi)^{-1})\phi \cdot (a\phi(a\psi)^{-1})\psi \\ &= a\phi^2 \cdot (a\psi\phi)^{-1} \cdot a\phi\psi \cdot (a\psi^2)^{-1} \\ &= a\phi(a\psi)^{-1} \\ &= a \circ a^{-1} \end{aligned}$$

so (G, \circ) is *E-inversive* since a was taken to be arbitrary in G . Let e denote the identity element of (G, \cdot) . Since (G, \circ) is stationary on the right it is rectangular, whence by Theorem 8 of [4], $G \circ G$ is the kernel of G and

$$G \circ G \cong H \times E$$

where E is the rectangular band consisting of the idempotents of (G, \circ) , and H is the subgroup

$$e \circ G \circ e = \{x\phi\psi: x \in G\}$$

of (G, \circ) . By [5] the mapping $\theta: G \rightarrow G \circ G$ defined by $a\theta = a \circ f$, where

f is the identity element of the maximal subgroup to which $a \circ a$ belongs, is onto, idempotent, and $a\theta \circ b\theta = a \circ b$ for all $a, b \in G$, whence (G, \circ) is an inflation of $(G \circ G, \circ)$. Thus (G, \circ) is an inflation of the direct product of a group and a rectangular band. (We note that $(H, \cdot) = (H, \circ)$.)

The structure of a semigroup (G, \circ) generated by the monomial $x^s y^t$ is revealed by the following theorem, which is a consequence of Theorem 5.

THEOREM 6. *Let (G, \circ) be a semigroup generated by the monomial $x^s y^t$ over the finite abelian group (G, \cdot) . Then (G, \circ) is an inflation of its kernel $G \circ G$, and its kernel is isomorphic to the direct product of the subgroup*

$$H = \{x^{st} : x \in G\}$$

of (G, \circ) and the rectangular band

$$E = \{x \in G : x = x^{s+t}\}.$$

Proof. Let ϕ, ψ be defined on (G, \cdot) by $x\phi = x^s$ and $y\psi = y^t$. Then ϕ, ψ are endomorphisms of (G, \cdot) since (G, \cdot) is abelian. Also, $\phi^2 = \phi$ and $\psi^2 = \psi$ since $x^{s^2} = x^s$ and $x^{t^2} = x^t$ for all $x \in G$. Since

$$(x^s)^t = x^{st} = (x^t)^s \quad \text{for all } x \in G,$$

it follows that ϕ and ψ commute. Thus ϕ and ψ as defined above satisfy the hypothesis of Theorem 5, so (G, \circ) is an inflation of its kernel $(G \circ G, \circ)$. Since $x\phi\psi = x^{st}$ for $x \in G$, and since x is an idempotent of (G, \circ) if and only if $x^{s+t} = x$, it follows that

$$G \circ G \cong H \times E$$

where H and E are as defined in the statement of the theorem.

Let (a, b) denote the greatest common divisor of integers a and b . We have the following lemma concerning certain subgroups of a cyclic group.

LEMMA 7. *Let G be a cyclic group of order n with identity element e , and let s be a nonnegative integer such that $n \mid s^2 - s$. Then $G_{s-1} = \{x \in G : x^{s-1} = e\}$ is a subgroup of G having order $(n, s-1)$.*

Proof. It follows immediately that G_{s-1} is a subgroup of G . Let m denote the order of G_{s-1} , and let $d = (n, s-1)$. Since

$$x^{s-1} = e = x^n, \quad \text{for all } x \in G_{s-1},$$

it follows that $m \mid s-1$ and $m \mid n$, whence $m \leq d$. Now, let a be a generator of G . Then $a^{n/d}$ generates a subgroup $[a^{n/d}]$ of G , of order d . But $(a^{n/d})^{s-1} = (a^n)^{(s-1)/d} = e$, so $a^{n/d} \in G_{s-1}$, whence $[a^{n/d}] \subseteq G_{s-1}$. Thus $d \leq m$, and so $m = d = (n, s-1)$.

The next theorem gives the structure of the group H in Theorem 6, whenever (G, \cdot) is a cyclic group.

THEOREM 7. *If (G, \circ) is a semigroup generated by the monomial $x^s y^t$ over the cyclic group (G, \cdot) of order n , then (G, \circ) is an inflation of its kernel $G \circ G$. Furthermore, its kernel is isomorphic to the direct product of the cyclic subgroup*

$$H = \{x^{st}: x \in G\}$$

of (G, \circ) of order $(n, st-1)$ and the rectangular band

$$E = \{x \in G: x^{s+t} = x\}.$$

Proof. Suppose $x^s y^t$ generates a semigroup over (G, \cdot) . Then the set H defined above is the same as the set

$$G_{st-1} = \{x \in G: x^{st-1} = e\}.$$

Since $n \mid s^2 - s$, and $n \mid t^2 - t$, it follows that

$$n \mid (s^2 - s)t + s^2(t^2 - t),$$

whence $n \mid (st)^2 - st$. By Lemma 7, H has order $(n, st-1)$. The remaining part of the proof follows immediately from Theorem 6.

We conclude with a corollary to Theorem 7 which extends the results obtained in [3].

COROLLARY 3. *Let $(F_q, +, \cdot)$ be a finite field of order q , and let (F_q, \circ) be a semigroup generated by $x^s y^t$ over (F_q, \cdot) . Then (F_q, \circ) is an inflation of the direct product of a cyclic group of order $(q-1, st-1)$, and a rectangular band, together with a zero element.*

Proof. Let $F_q^* = F_q \setminus \{0\}$. Then (F_q^*, \cdot) is the multiplicative group of $(F_q, +, \cdot)$, hence is a cyclic group of order $q-1$. By Theorem 7, (F_q^*, \circ) is an inflation of the direct product of a cyclic group of order $(q-1, st-1)$, and a rectangular band. Since

$$F_q = F_q^* \cup \{0\}$$

and 0 is a zero for \circ , the corollary holds.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I, Mathematical Surveys 7, Amer. Math. Soc., Providence, R.I., 1961.
2. W. J. LeVeque, *Topics in Number Theory*, Vol. I, Addison-Wesley Publishing Co., Inc., Reading, Mass., 1956.
3. R. J. Plemmons and R. Yoshida, *Generating polynomials for finite semigroups*. To appear in Mathematische Nachrichten.
4. G. Thierrin, *Demi-groupes inverse's et rectangulaires*, Acad. Roy. Belg. Cl. Sci. (5) **41** (1955), 83-92, (MR 17, 10).
5. M. Yamada, *A note on middle unitary semigroups*, Kōdai Math. Semi. Rep., **7** (1955), 49-52, (MR 17, 585).

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Pacific Journal of Mathematics

Vol. 39, No. 3

July, 1971

William O'Bannon Alltop, <i>5-designs in affine spaces</i>	547
B. G. Basmaji, <i>Real-valued characters of metacyclic groups</i>	553
Miroslav Benda, <i>On saturated reduced products</i>	557
J. T. Borrego, Haskell Cohen and Esmond Ernest Devun, <i>Uniquely representable semigroups. II</i>	573
George Lee Cain Jr. and Mohammed Zuhair Zaki Nashed, <i>Fixed points and stability for a sum of two operators in locally convex spaces</i>	581
Donald Richard Chalice, <i>Restrictions of Banach function spaces</i>	593
Eugene Frank Cornelius, Jr., <i>A generalization of separable groups</i>	603
Joel L. Cunningham, <i>Primes in products of rings</i>	615
Robert Alan Morris, <i>On the Brauer group of \mathbb{Z}</i>	619
David Earl Dobbs, <i>Amitsur cohomology of algebraic number rings</i>	631
Charles F. Dunkl and Donald Edward Ramirez, <i>Fourier-Stieltjes transforms and weakly almost periodic functionals for compact groups</i>	637
Hicham Fakhoury, <i>Structures uniformes faibles sur une classe de cônes et d'ensembles convexes</i>	641
Leslie R. Fletcher, <i>A note on $C\theta\theta$-groups</i>	655
Humphrey Sek-Ching Fong and Louis Sucheston, <i>On the ratio ergodic theorem for semi-groups</i>	659
James Arthur Gerhard, <i>Subdirectly irreducible idempotent semigroups</i>	669
Thomas Eric Hall, <i>Orthodox semigroups</i>	677
Marcel Herzog, <i>$C\theta\theta$-groups involving no Suzuki groups</i>	687
John Walter Hinrichsen, <i>Concerning web-like continua</i>	691
Frank Norris Huggins, <i>A generalization of a theorem of F. Riesz</i>	695
Carlos Johnson, Jr., <i>On certain poset and semilattice homomorphisms</i>	703
Alan Leslie Lambert, <i>Strictly cyclic operator algebras</i>	717
Howard Wilson Lambert, <i>Planar surfaces in knot manifolds</i>	727
Robert Allen McCoy, <i>Groups of homeomorphisms of normed linear spaces</i>	735
T. S. Nanjundiah, <i>Refinements of Wallis's estimate and their generalizations</i>	745
Roger David Nussbaum, <i>A geometric approach to the fixed point index</i>	751
John Emanuel de Pillis, <i>Convexity properties of a generalized numerical range</i>	767
Donald C. Ramsey, <i>Generating monomials for finite semigroups</i>	783
William T. Reid, <i>A disconjugacy criterion for higher order linear vector differential equations</i>	795
Roger Allen Wiegand, <i>Modules over universal regular rings</i>	807
Kung-Wei Yang, <i>Compact functors in categories of non-archimedean Banach spaces</i>	821
R. Grant Woods, <i>Correction to: "Co-absolutes of remainders of Stone-Čech compactifications"</i>	827
Ronald Owen Fulp, <i>Correction to: "Tensor and torsion products of semigroups"</i>	827
Bruce Alan Barnes, <i>Correction to: "Banach algebras which are ideals in a banach algebra"</i>	828