A DISCONJUGACY CRITERION FOR HIGHER ORDER LINEAR VECTOR DIFFERENTIAL EQUATIONS

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For a higher order linear quasi-differential equation which is non-self-adjoint there is presented a disconjugacy criterion that is a consequence of the disconjugacy of an associated self-adjoint quasi-differential equation. In particular, there is considered the specific form of this criterion for a higher order differential equation of the canonical form which has been presented by the author, Transactions of the American Mathematical Society, 85 (1957), 446-461.

1. Introduction. For self-adjoint Hamiltonian differential systems which satisfy a condition of definiteness that in the case of accessory systems for variational problems is the strengthened Legendre or Clebsch condition, it is well-known, (see, for example, Bliss [1, Secs. 89, 90], Morse [5; 6, Ch. IV], Reid [7; 9; 11, Sec. VII. 5]), that the condition of disconjuguacy is equivalent to the positive definiteness of the associated (Dirichlet) hermitian functional. In turn, for non-self-adjoint differential systems one may derive a sufficient condition for disconjuguacy as a consequence of the disconjuguacy of certain associated self-adjoint systems. An example of this procedure involving a linear homogeneous vector differential equation of the second order is given in Reid [7, Sec. 5]; see also, Hartman and Wintner [3]. The purpose of the present paper is to present corresponding results for more sophisticated differential systems of higher order.

Matrix notation is used throughout; in particular, one column matrices are called vectors. The \( n \times n \) identity matrix is denoted by \( E_n \), or merely by \( E \) when there is no ambiguity, and \( 0 \) is used indiscriminately for the zero matrix of any dimensions. The conjugate transpose of a matrix \( M \) is denoted by \( M^* \). The symbols \( M \geq N \), \( \{M > N\} \), are used to signify that \( M \) and \( N \) are hermitian matrices of the same dimensions and \( M - N \) is a nonnegative, (positive), definite matrix. A matrix function is termed continuous, integrable, etc., when each element of the matrix possesses the specified property.

If a matrix function \( M(t) \) is a.c., (absolutely continuous), on a compact interval \([a, b]\), then \( M'(t) \) signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if \( M(t) \) is (Lebesgue) integrable on \([a, b]\), then \( \int_a^b M(t) \, dt \) denotes the matrix of integrals of respective elements of \( M(t) \). For a given interval \([a, b]\), the symbols \( C_{pq}[a, b], C_{pq}^*[a, b], C_{pq}[a, b], C_{pq}[a, b], C_{pq}^*[a, b], C_{pq}^*[a, b], C_{pq}^*[a, b], C_{pq}[a, b], C_{pq}^*[a, b], C_{pq}^*[a, b], C_{pq}^*[a, b], C_{pq}^*[a, b] \) are used to denote the class of \( p \times q \) matrix functions...
\[ M(t) = [M_{\alpha \beta}(t)], \quad (\alpha = 1, \cdots, p; \beta = 1, \cdots, q) \] which on \([a, b]\) are respectively continuous, continuous and possessing continuous derivatives of the first \(n\) orders, (Lebesgue) integrable, (Lebesgue) measurable and \(|M_{\alpha \beta}(t)|^k\) integrable, measurable and essentially bounded, a.e., of class \(C_{pq}[a, b]\) with \(M^{(n-1)}(t) \in \mathbb{R}_{pq}[a, b]\). For brevity, the double subscript \(pq\) is reduced to merely \(p\) for the \(p\)-dimensional vector case specified by \(p, q = 1\), and both subscripts are omitted in the scalar case \(p = 1, q = 1\). For \(n \geq 1\), the subclass of vector functions \(y \in \mathbb{V}_n[a, b]\) for which \(y^{(n)}(t) \in \mathbb{L}_n[a, b]\) is denoted by \(\mathbb{V}_n[a, b]\). Also for \(n \geq 1\) the subclasses of vector functions \(y\) belonging to \(\mathbb{C}_p[a, b], \mathbb{V}_n[a, b], \mathbb{V}_n[a, b]\) for which \(y^{(n)}(a) = 0 = y^{(n)}(b), (\alpha = 1, \cdots, n)\), are denoted by \(\mathbb{C}_p[a, b], \mathbb{V}_n[a, b], \mathbb{V}_n[a, b]\), \(\mathbb{V}_n[a, b]\), respectively. If matrix functions \(M(t)\) and \(N(t)\) are equal a.e. (almost everywhere) on their interval of definition we write simply \(M(t) = N(t)\).

2. Preliminary results. Let \(F_{i:j}(t) = [F_{\sigma:\tau_{ij}}(t)], (i, j = 0, 1, \cdots, n)\), be \(r \times r\) matrix functions defined on an interval \(I\) on the real line, and satisfying the following hypothesis.

\(F_{nn}(t)\) is nonsingular for \(t \in I\), and for arbitrary compact subintervals \([a, b] \subset I\), and \(\alpha, \beta = 0, 1, \cdots, n - 1\) we have:

\[(\S) \quad \begin{align*}
(\alpha) \quad & F_{\sigma \alpha}, F^{-1}_{\sigma \alpha}, F_{\tau \alpha}, F^{-1}_{\tau \alpha} \text{ and } F_{\alpha \beta}F^{-1}_{\alpha \beta} \text{ belong to } L^\infty_{rr}[a, b]; \\
(\beta) \quad & F_{n \alpha} \text{ and } F_{\alpha \alpha} \text{ belong to } L^2_{rr}[a, b].
\end{align*}\]

The \((n + 1)r \times (n + 1)r\) matrix which for \(i, j = 0, 1, \cdots, n\) and \(\sigma, \tau = 1, \cdots, r\) has the element in the \((ir + \sigma)\)th row and \((jr + \tau)\)th column equal to \(F_{\sigma:\tau_{ij}}(t)\) will be denoted by \(F(t)\), and for \(k = 0, 1, \cdots, n\) the \(r \times (n + 1)r\) matrix whose element in the \(\sigma\)th row and \((jr + \tau)\)th column is \(F_{\sigma:\tau_{ik}}(t)\) will be denoted by merely \(F_k(t)\). If \([a, b] \subset I\) we shall denote by \(\mathbb{D}[a, b]\) the linear vector space of \(r\)-dimensional vector functions \(y \in \mathbb{V}_n[a, b]\), and by \(\mathbb{D}_c[a, b]\) the subspace consisting of those \(y \in \mathbb{D}[a, b]\) with \(y^{(n)}(a) = 0 = y^{(n)}(b), (\alpha = 0, 1, \cdots, n - 1)\). Also, if \(y \in \mathbb{D}[a, b]\) we shall denote by \(\hat{y}\) the \((n + 1)r\)-dimensional vector function with \(\hat{y}_{j+r+1}(t) = y^{(j)}(t), (j = 0, 1, \cdots, n; \tau = 1, \cdots, r)\).

If \([a, b] \subset I\) and \(y \in \mathbb{D}[a, b], \ z \in \mathbb{D}[a, b]\) then the integral

\[(2.1) \quad J[y, z | a, b] = \int_a^b \hat{z}^*(t)F(t)\hat{y}(t)dt\]

is well defined, and is a sesquilinear form on \(\mathbb{D}[a, b] \times \mathbb{D}[a, b]\).

**Lemma 2.1.** If \(y \in \mathbb{D}[a, b]\), then

\[(2.1') \quad J[y, z | a, b] = 0, \text{ for } z \in \mathbb{D}_c[a, b]\]

if and only if \(y\) is a solution on \([a, b]\) of the vector quasi-differential...
equation

(2.2) \[ \mathcal{L}[y; F](t) = F_0(t)\dot{y}(t) - \{F_1(t)\dot{y}(t) - \cdots - \{F_p(t)\dot{y}(t)\}'\}' = 0. \]

In conformity with usual terminology, (see, for example, Bradley [2], Reid [9, Sec. 4]), an \( r \)-dimensional vector function \( y(t) \) is a solution of (2.2) if \( y \in \mathcal{D}[a, b] \) and the \( r \)-dimensional vector functions \( v_n(t) = (v_{nk}(t)), (\sigma = 1, \ldots, r; k = 1, \ldots, n) \), defined recursively as

(2.3)

\[
\begin{align*}
v_0(t) &= F_0(t)\dot{y}(t) \\
v_{n-p}(t) &= F_{n-p}(t)\dot{y}(t) - v'_{n-p+1}(t), p = 1, \ldots, n - 1,
\end{align*}
\]

all belong to \( \mathbb{V}_r[a, b] \) and on \([a, b]\),

(2.4) \[ \mathcal{L}[y; F](t) = F_0(t)\dot{y}(t) - v'_1(t) = 0. \]

The result of Lemma 2.1 follows by the classical proof of the fundamental lemma of the calculus of variations, (see, for example, Bliss [1, Sec. 5] for simplest instance; Reid [11, Probs. III. 2:1-8] for more general cases). Indeed, if for an integrable vector function \( w(t) \) on \([a, b]\) we introduce \( I[w](t) \) for \( \int_a^t w(s)ds \), and for \( y \in \mathcal{D}[a, b] \) we set

(2.5)

\[
\begin{align*}
w_0(t) &= F_0(t)\dot{y}(t) \\
w_{1+p}(t) &= F_{1+p}(t)\dot{y}(t) - I[w_p](t), \quad p = 1, \ldots, n - 1,
\end{align*}
\]

then upon suitable integration by parts condition (2.1) becomes

(2.6) \[ \int_a^b \mathcal{L}[w](s)F_n(s)\dot{y}(s) - I[w_n](s)ds = 0 \quad \text{for} \quad z \in \mathbb{D}_0[a, b]. \]

By the more familiar form of the fundamental lemma we obtain the existence of a vector polynomial \( P_{n-1}(t) \) of degree at most \( n - 1 \) such that on \([a, b]\) we have

(2.7) \[ F_n(t)\dot{y}(t) - I[w_n](t) = P_{n-1}(t). \]

Relation (2.7) clearly implies that \( v_n(t) = I[w_n](t) + P_{n-1}(t) \) is a vector function of class \( \mathbb{V}_r[a, b] \) such that \( v_n = F_n\dot{y} \) and

\[
\begin{align*}
v'_{n}(t) &= w_n(t) + P'_{n-1}(t) \\
&= F_{n-1}(t)\dot{y}(t) - I[w_{n-1}](t) + P'_{n-1}(t).
\end{align*}
\]

Then \( v_{n-1}(t) = I[w_{n-1}](t) - P'_{n-1}(t) \) is a vector function of class \( \mathbb{V}_r[a, b] \) such that \( v_{n-1}(t) = F_{n-1}(t)\dot{y}(t) - v'_n(t) \), and iteration of this procedure leads successively to vector functions \( v_{n-p}(t) = I[w_{n-p}](t) + (-1)^p P_{n-p}^{[p]}(t) \)

of class \( \mathbb{V}_r[a, b] \) and satisfying the equations (2.3). In particular, \( v_1(t) = I[w_1](t) + (-1)^{n-1}P_n^{[n]}(t) \) is a vector function of class \( \mathbb{V}_r[a, b] \) satisfying \( v_1(t) = F_1(t)\dot{y}(t) - v'_1(t) \). Since \( P_n^{[n]}(t) \) is constant it then
follows that \( 0 = w_1(t) - v'_1(t) = F_1(t)g(t) - v'_1(t) \), which is the equation (2.2).

Conversely, if \( v_1(t), \ldots, v_n(t) \) are vector functions of class \( \mathcal{D}[a, b] \) satisfying with a vector function \( y \in \mathcal{D}[a, b] \) the system of equations (2.3), (2.4), then

\[
\tilde{z}^*F \tilde{g} = z^*v'_1 + \sum_{j=1}^{n-1} z^*[v_j + v'_{j+1}] + z^*[\alpha]v_n
\]

and consequently (2.1) holds.

For a vector function \( y \in \mathcal{D}[a, b] \), let the \( r \)-dimensional vector functions \( u_1(t), \ldots, u_n(t) \) be defined as

\[
u_k(t) = y^{k-1}(t) = \left(u_{1,k}(t), \ldots, u_{r,k}(t)\right), \quad (k = 1, \ldots, n).
\]

Finally, let \( u(t) \) and \( v(t) \) denote the \( nr \)-dimensional vector functions \((u_\rho(t)), (v_\rho(t)), (\rho = 1, \ldots, nr)\), with

\[
u_{ir+\sigma}(t) = y^{[i]}_\sigma(t) = u_{r;i+1}(t), \quad (i = 0, 1, \ldots, n-1; \sigma = 1, \ldots, r).
\]

The above quasi-differential equation (2.2), or the associated system (2.3), (2.4), may then be written in the matrix form

\[
\mathcal{L}[u; v](t) = -v'(t) + C(t)u(t) - D(t)v(t) = 0,
\]

where \( A(t), B(t), C(t), D(t) \) are \((nr) \times (nr)\) matrix functions which will be written as partitioned matrices in \( r \times r \) matrices as \( A(t) = [A_{hk}(t)], B(t) = [B_{hk}(t)], C(t) = [C_{hk}(t)], D(t) = [D_{hk}(t)], (h, k = 1, \ldots, n) \), with

\[
\begin{align*}
(a) & \quad A_{hk}(t) = \delta_{h,k+1}E_r, \quad (h = 1, \ldots, n-1, k = 1, \ldots, n) \\
(b) & \quad B_{hk}(t) = \delta_{h,k}\delta_{nk}F_{nk}^{-1}(t), \quad (h, k = 1, \ldots, n); \\
(c) & \quad C_{hk}(t) = F_{h-1,k-1}(t) - F_{h-1,n}(t)F_{nk}^{-1}(t)F_{nk-1}(t), \quad (h, k = 1, \ldots, n) \\
(d) & \quad D_{hk}(t) = \delta_{h,k+1}E_r, \quad (k = 1, \ldots, n-1, h = 1, \ldots, n), \\
& \quad D_{hn}(t) = -F_{h-1,n}(t)F_{nn}^{-1}(t), \quad (h = 1, \ldots, n).
\end{align*}
\]

It is to be noted that whenever hypothesis (2) is satisfied the differential system (2.10) in \((u; v)\) is identically normal; that is, if \( u(t) \equiv 0, v(t) \) is a solution of (2.10) on a nondegenerate subinterval \( I_0 \) of \( I \) then \( u(t) \equiv 0, v(t) \equiv 0 \) throughout \( I \). Indeed, if \( u(t) \equiv 0, v(t) \) is a solution of (2.10) on \( I_0 \), then from the equation \( \mathcal{L}[u; v](t) = 0 \) it follows that \( v_n(t) \equiv 0 \) on \( I_0 \). In turn, from \( \mathcal{L}[u, v](t) = 0 \) it follows
that \(-v'_{h+1} + v_h = 0, (h = 1, \ldots, n - 1)\), and consequently also \(v_h(t) \equiv 0\) on \(I_0\) for \(h = 1, \ldots, n - 1\). From the condition \(u(t) \equiv 0, v(t) \equiv 0\) on \(I_0\) it then follows that \(u(t) \equiv 0, v(t) \equiv 0\) on \(I\), thus establishing the identical normality of (2.10) on \(I\).

Two distinct points \(t_1\) and \(t_2\) on \(I\) are said to be (mutually) conjugate with respect to (2.2), or with respect to (2.10), if there exists a solution \((u(t); v(t))\) of this latter system with \(u(t) \neq 0\) on the subinterval with endpoints \(t_1\) and \(t_2\), while \(u(t_1) = 0 = u(t_2)\). Since \(u_h(t) = y^{(h-1)}(t), (h = 1, \ldots, n)\), this condition states that \(t_1 = t_2\) and \(t = t_2\) are zeros of the vector function \(y(t)\) of order greater than or equal to \(n\). Moreover, if \(t_1 \in I\) and \(U(t), V(t)\) are \((nr) \times (nr)\) matrix functions whose column vectors are solutions of (2.10), and satisfying the initial matrix conditions

\[
U(t_1) = 0, V(t_1) = E_{nr},
\]

then a value \(t_2 \neq t_1\) is conjugate to \(t_1\) if and only if \(U(t_2)\) is singular. If \(U(t_2)\) has rank \(nr - q\), so that there are \(q\) linearly independent solutions \((u^{(\rho)}(t); v^{(\rho)}(t)), (\rho = 1, \ldots, q),\) of (2.10) satisfying \(u^{(\rho)}(t_1) = 0 = u^{(\rho)}(t_2)\), then \(t_2\) is said to be a conjugate point to \(t_1\) of order \(q\).

If \(I_0\) is a nondegenerate subinterval of \(I\) such that no two distinct points of \(I_0\) are conjugate with respect to (2.2), or (2.10), then this quasi-differential equation or differential system is said to be disconjugate or non-oscillatory on \(I_0\).

Finally, it is to be noted that \(y \in \mathbb{D}[a, b]\) if and only if the \((nr)\)-dimensional vector function

\[
\eta(t) = (\eta_\rho(t)), \quad \text{with} \quad \eta_{r+i \sigma}(t) = y^{(r+i \sigma)}(t),
\]

\[
(\sigma = 1, \ldots, r; i = 0, 1, \ldots, n - 1),
\]

has an associated \((nr)\)-dimensional vector function \(\zeta(t) = (\zeta_\rho(t)) \in \mathbb{H}^{nr}[a, b]\) such that \(\mathcal{L}[\eta, \zeta](t) = 0\) on \([a, b]\). In view of the form of \(B(t)\), clearly only the last \(r\) components of \(\zeta(t)\) are uniquely determined, with values

\[
\zeta_{(m-1)r+\sigma}(t) = \sum_{r=1}^{\tilde{r}} F_{\sigma;m}(t) y^{(s)}_{r+\sigma}(t), \quad (\sigma = 1, \ldots, r).
\]

3. Self-adjoint systems. The quasi-differential system (2.2), or the equivalent first order system (2.10), is self-adjoint when the coefficient matrix function satisfies in addition to \((\tilde{S})\) the further condition

\[
(F(t) \text{ is hermitian for } t \in I).
\]

The hermitian character of \(F(t)\) is equivalent to the condition that
the component \( r \times r \) matrix functions \( F_{ij} \) are such that \([F_{ij}(t)]^* = F_{ji}(t)\) for \( t \in I \). In particular, the diagonal component matrix functions \( F_{ii}(t) \) are hermitian on \( I \). It follows readily that under hypotheses (\( \mathfrak{G} \)) and (\( \mathfrak{G}_1 \)) the coefficient matrices of (2.10) are such that

\[ A(t) = D^*(t), B(t) = B^*(t), C(t) = C^*(t), \]

and (2.10) is of the canonical form of a linear Hamiltonian system for which one has a generalization of the Sturmian theory for real scalar linear homogeneous differential equations of the second order, (see, in particular, references [5]—[11] of the Bibliography).

Corresponding to the class \( \mathcal{D}[a, b] \) we shall denote by \( D[a, b] \) the linear vector space of \((nr)\)-dimensional vector functions \( \eta(t) \) which are of class \( \mathfrak{M}_n[a, b] \), and for which there are corresponding \((nr)\)-dimensional vector functions \( \zeta(t) \in \mathfrak{M}_n[a, b] \) such that \( L_2[\eta, \zeta](t) = 0 \) on this interval. The subspace of \( D[a, b] \) on which \( \eta(a) = 0 = \eta(b) \) will be denoted by \( D_0[a, b] \). The fact that a \( \zeta(t) \in \mathfrak{M}_n[a, b] \) is thus associated with \( \eta(t) \in \mathfrak{M}_n[a, b] \) is denoted by the respective symbols \( \eta \in D[a, b]; \zeta \) and \( \eta \in D_0[a, b]; \zeta \).

When hypotheses (\( \mathfrak{G} \)) and (\( \mathfrak{G}_1 \)) hold, and \( y^{(p)}(t) \in \mathcal{D}[a, b], (p = 1, 2) \), let \( \gamma^{(p)}(t) = (\gamma_{p}^{(p)}(t)), (p = 1, 2) \), be defined by corresponding equations (2.12), and \( \zeta^{(p)}(t) = (\zeta_{p}^{(p)}(t)) \) associated vector functions of class \( \mathfrak{M}_n[a, b] \) whose last \( r \) components are specified by equations corresponding to (2.13). The functional \( J[y^{(1)}, y^{(2)} | a, b] \) defined by (2.1) is then expressible in terms of \( \gamma^{(p)}(t), \zeta^{(p)}(t) \) as

\[ J[\gamma^{(1)}, \gamma^{(2)} | a, b] = \int_a^b \{ \zeta^{(2)*} B \zeta^{(1)} + \gamma^{(2)*} C \gamma^{(1)} \} dt, \]

with the defining relations now equivalent to the condition that \( \gamma(t) = \gamma^{(p)}(t), \zeta(t) = \zeta^{(p)}(t), (p = 1, 2) \) satisfy the differential equation of restraint

\[ L_2[\eta, \zeta](t) = \gamma'(t) - A(t)\eta(t) - B(t)\zeta(t) = 0. \]

As pointed out at the end of the preceding section, if \( \eta \in D[a, b]; \zeta \) the vector function \( \zeta \) corresponding to a given \( \eta \) is not uniquely determined; however, the vector function \( B \zeta \) is uniquely determined. Consequently if \( \gamma^{(p)} \in D[a, b], (p = 1, 2) \), then the value of the integral in (3.1) is independent of the particular corresponding \( \zeta^{(p)} \), so that this integral does indeed define a functional of \( \gamma^{(1)}, \gamma^{(2)} \). Moreover, in view of the hermitian character of the coefficient matrix functions \( B \) and \( C \), \( J[\gamma^{(1)}, \gamma^{(2)} | a, b] \) is an hermitian functional on \( D[a, b] \times D[a, b] \). In particular, \( J[\eta | a, b] = J[\eta, \eta | a, b] \) given as

\[ J[\eta | a, b] = \int_a^b \{ \zeta^* B \zeta + \eta^* C \eta \} dt \]
is a real-valued functional on $D\lfloor a, b\rfloor$.

For a system (2.10) which satisfies hypotheses $(\phi)$ and $(\phi_1)$ it follows readily that if $y^{(p)} = (u^{(p)}; v^{(p)})$, $(p = 1, 2)$, are solutions of this system then the function

$$(u^{(1)}, v^{(1)} \mid u^{(2)}, v^{(2)})(t) = u^*_x(t)u_x(t) - u^*_v(t)v_x(t)$$

is constant on $I$. If two solutions of this system are such that this constant is zero, these solutions are said to be (mutually) conjoined. If $Y(t) = (U(t); V(t))$ is a $(2nr) \times q$ matrix whose column vectors are linearly independent solutions of (2.10) which are mutually conjoined, then these solutions form a basis for a conjoined family of solutions of dimension $q$, consisting of these solution of (2.10) which are linear combinations of the column vector functions. In general, (see, for example, Reid [7, Sec. 2; 11, Sec. VII. 2]), the maximal dimension of a conjoined family of solutions of (2.10) is $nr$, and a given conjoined family of dimension less than $nr$ is contained in a conjoined family of dimension $nr$.

If $[a, b]$ is a nondegenerate compact subinterval of $I$, then the symbol $\mathcal{S}_+[a, b]$ will signify the condition that the functional $J[y \mid a, b]$ is positive definite on $D_0[a, b]$; that is, for $y \in D_0[a, b]$ we have $J[y \mid a, b] \geq 0$, with the equality sign holding only if $y(t) = 0$ on $[a, b]$. This condition may be equally well stated as the nonnegativeness of the functional (3.3) on the vector space $D_0[a, b]$, with $J[\eta \mid a, b] = 0$ for an $\eta \in D_0[a, b]; \zeta$ only if $\eta(t) = 0$ and $B(t)\zeta(t) = 0$ on $[a, b]$.

From the basic result for canonical Hamiltonian systems concerning disconjugacy on a compact interval, (see, for example, Reid [10, Theorem 5.1] or Reid [11, Sec. VII. 4]), we have the following criterion.

**Theorem 3.1.** If hypotheses $(\phi)$ and $(\phi_1)$ are satisfied, and $[a, b]$ is a nondegenerate compact subinterval of $I$, then $\mathcal{S}_+[a, b]$ holds if and only if $F_{ns}(t) > 0$ for $t$ a.e. on $[a, b]$, together with one of the following conditions:

(i) (2.10) is disconjugate on $[a, b]$;

(ii) there exists a conjoined family of solutions $Y(t) = (U(t); V(t))$ of (2.10) of dimension $nr$ with $U(t)$ nonsingular on $[a, b]$.

4. A disconjugacy criterion for (2.2). Suppose that hypothesis $(\phi)$ is satisfied by the coefficient matrix function $F(t)$ of (2.2) on an interval $I$, and that $[a, b]$ is a nondegenerate subinterval of $I$ such that $t = a$ and $t = b$ are mutually conjugate with respect to the equation (2.2). Let $y(t)$ be a solution of (2.2) such that $y(t) \neq 0$ on $[a, b]$, and $y^{(\alpha)}(a) = 0 = y^{(\alpha)}(b)$, $(\alpha = 0, 1, \ldots, n - 1)$. Then $y \in D_0[a, b]$, and in view of Lemma 2.1 we have that
(4.1) \[ 0 = J[y, y | a, b] = \int_a^b \hat{y}^*(t)F(t)\hat{y}(t)dt. \]

From this relation it follows that \( \Re F(t) = \frac{1}{2}[F(t) + F^*(t)] \) and \( \Im F(t) = \frac{1}{2i}[-1(F^*(t) - F(t))] \) are hermitian matrix functions. If \( \lambda_0, \lambda_i \) are real constants then

(4.2) \[ F(t; \lambda) = \lambda_0 \Re F(t) + \lambda_i \Im F(t) \]
is an hermitian matrix function such that the given solution \( y(t) \) of (2.2) satisfies the condition

(4.3) \[ \int_a^b \hat{y}^*(t)F(t; \lambda)\hat{y}(t)dt = 0. \]

Now if \( F(t; \lambda) \) has the partitioned representation \( \{F_{ij}(t; \lambda)\}, (i, j = 0, 1, \cdots, n) \) in terms of \( r \times r \) matrix functions, and \( F(t; \lambda) \) satisfies hypothesis (\$) with \( F_{nn}(t; \lambda) > 0 \) for \( t \) a.e. on \([a, b]\), then the conclusion (i) of Theorem 3.1 applied to the self-adjoint matrix differential equation \( \mathcal{L}[y; F(\cdot; \lambda)](t) = 0 \) implies that this equation fails to be disconjugate on \([a, b]\). Consequently, we have the following result, corresponding to that of §5 of Reid [7] for a second order linear homogeneous matrix differential equation. The reader is also referred to Hartman and Wintner [3] for a similar treatment of disconjugacy criteria for second order vector differential systems. For a consideration of non-self-adjoint differential equations of even order by a method which is similar in basic idea, but different in specific detail, see Kreith [4].

**Theorem 4.1.** Suppose that hypothesis (\$) is satisfied by the coefficient matrix function \( F(t) \) of (2.2) on an interval \( I \), and for a given nondegenerate subinterval \([a, b]\) of \( I \) there exist real constants \( \lambda_0, \lambda_i \) such that on \([a, b]\) the matrix function \( F(t; \lambda) = \{F_{ij}(t; \lambda)\}, (i, j = 0, 1, \cdots, n) \), of (4.2) satisfies hypothesis (\$) and \( F_{nn}(t; \lambda) > 0 \) for \( t \) a.e. on \([a, b]\). Then whenever the self-adjoint quasi-differential equation \( \mathcal{L}[y; F(\cdot; \lambda)](t) = 0 \) is disconjugate on \([a, b]\), the system (2.2) is also disconjugate on \([a, b]\).

It is to be emphasized that in the above theorem the constant multipliers \( \lambda_0, \lambda_i \) may depend upon the subinterval \([a, b]\), and that any criterion of disconjugacy for the associated self-adjoint equation \( \mathcal{L}[y; F(\cdot; \lambda)](t) = 0 \) yields a sufficient condition for disconjugacy of the original equation (2.2). In particular, the results of Reid [9, Sec. 4] for scalar quasi-differential equations of even order, and their analogues for vector equations, provide sufficient conditions for (2.2) to be disconjugate on a non-compact interval \( (t_1, \infty) \).
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5. A special canonical form. Attention will be directed now to a linear differential expression of order \( m \) in the \( r \)-dimensional vector function \( y(t) = (y_\sigma(t)) \) of the form

\[
\mathcal{L}[y](t) = \sum_{\mu=0}^{\infty} P_\mu(t)y^{(\mu)}(t)
\]

where the \( r \times r \) coefficient matrix functions \( P_\mu(t) = [P_{\sigma\tau}(t)] \) are supposed to be of class \( \mathbb{C}_{\sigma\tau}[a, b] \) for arbitrary compact subintervals \([a, b]\) of a given interval \( I \) on the real line. It is to be emphasized that in the discussion leading to the result of Theorem 5.1 we do not require the leading coefficient matrix \( P_m(t) \) to be nonsingular, or even to be nonzero. The purpose of this section is to present for vector differential operators of the form (5.1) an analogue of the results of Reid [8] for linear scalar differential equations, and to note the particular form of the disconjugacy criterion of § 4 for the involved canonical form.

For a given compact subinterval \([a, b]\) of \( I \), let \( T_0 \) denote a corresponding differential operator with domain \( \mathcal{C}_{\sigma\tau}[a, b] \) and value \( T_0y = \mathcal{L}[y] \). If \( \mathcal{D}^* \) denotes the totality of \( r \)-dimensional vector functions \( z \in \mathcal{C}_{\sigma\tau}[a, b] \) with \( P_\mu(t)z(t) \in \mathcal{C}_{\sigma\tau}[\alpha, \delta], (\mu = 0, 1, \ldots, m) \), and for which there exists a corresponding \( f_z \in \mathcal{C}_{\sigma\tau}[\alpha, \delta] \) such that

\[
\int_a^b z^* \mathcal{L}[y] dt = \int_a^b f_z^* y dt, \quad y \in \mathcal{C}_{\sigma\tau}[a, b],
\]

then the operator \( T_0^* \) with domain \( \mathcal{D}^* \) and value \( T_0^*z = f_z \) is termed the adjoint of \( T_0 \). In particular, if \( P_{\mu} \in \mathcal{C}_{\sigma\tau}[a, b] \) and \( P_m(t) \) is nonsingular for \( t \in [a, b] \), then by classical results, (see, for example, Reid [11, Sec. III. 9]) we have that \( \mathcal{D}^* = \mathcal{H}_{\sigma\tau}[a, b] \), and for \( z \in \mathcal{H}_{\sigma\tau}[a, b] \) the value of \( T_0^*z \) is given by the Lagrange adjoint \( \sum_{\mu=0}^{m} (-1)^{\mu}\{P_\mu z\}^{(\mu)} \). Of special importance is the Hilbert space case that occurs when \( P_m \in \mathcal{H}_{\sigma\tau}[a, b], (\mu = 0, 1, \ldots, m) \), and analogous to the above defined \( T_0 \) one considers the operator with values \( \mathcal{L}[y] \) on the domain of functions \( y \in \mathcal{H}_{\sigma\tau}[a, b] \) such that \( \mathcal{L}[y] \in \mathcal{L}_{\sigma\tau}[a, b] \).

Of particular significance for the present considerations are differential expressions \( \mathcal{L}[y] = \Lambda_{\sigma}[y; P] \) where \( P \) is an \( r \times r \) matrix function, and

\[
\begin{align*}
\Lambda_0[y; P](t) &= P(t)y(t), \\
\Lambda_{2p}[y; P](t) &= \{P(t)y^{(p)}(t)\}^{(p)} , \\
\Lambda_{2p-1}[y; P](t) &= \{P(t)y^{(p-1)}(t)\}^{(p)} + \{P(t)y^{(p)}(t)\}^{(p-1)} , \quad (p = 1, 2, \ldots),
\end{align*}
\]

with the understanding that in the definition of \( \Lambda_{2p} \) and \( \Lambda_{2p-1} \) the involved matrix function \( P \) is of class \( \mathcal{H}_{\sigma\tau}[a, b] \). If for (5.1) we have \( \mathcal{L}[y] = \Lambda_m[y; P], (m \geq 1) \), then the fact that \( \mathcal{H}_{\sigma\tau}[a, b] \subset \mathcal{D}^* \) and \( T_0^*z = \Lambda_m[z; (-1)^mP^*] \) for \( z \in \mathcal{H}_{\sigma\tau}[a, b] \) is a direct consequence of the well-
known equation

\[ z^*A_m[y; P] - (-1)^m[A_m[z; P^*]]^*y = \{K_n[y, z; P]\}' \]

for arbitrary \( y, z \) of \( \mathbb{R}^n[a, b] \), where \( K_n[y, z; P] \) is the so-called bilinear concomitant of the form \( \sum_{\mu, \nu=0}^{m} z^{*\nu-1}(t)K_{\nu}(t; P)P_{\mu}(t) \).

Let \( e^{(k)} \) denote the \( r \)-dimensional unit vector \( e^{(h)} = (\delta_{hh}), (h = 1, \ldots, r) \), and designate by \( g_\lambda(t) \) (\( \lambda = 0, 1, \ldots \)) the particular scalar polynomials \( g_\lambda(t) = t^\lambda \) \((\lambda = 1, 2, \ldots)\). Moreover, let \( k_j \) equal \( j/2 \) or \((j + 1)/2\) according as \( j \) is even or odd. Corresponding to Theorem 3.2 of Reid [8], we now have the following representation theorem.

**Theorem 5.1.** Suppose that \( L[y] \) is given by (5.1) with \( P_\mu \in \mathcal{L}_r[a, b] \), \((\mu = 0, 1, \ldots, m)\), and the differential operator \( T_0 \) is defined as specified above. If for \( h = 1, \ldots, r \) and \( \lambda = 0, 1, \ldots, k_m - 1 \) the vector functions \( g_\lambda(t)e^{(k)} \) belong to \( \mathbb{D}^* \), then there exist matrix functions \( \Pi_{\mu}(t) \in \mathcal{L}_r^* [a, b], (\mu = 0, 1, \ldots, m) \), such that

\begin{equation}
L[y](t) = \sum_{\mu=0}^{m} A_\mu[y; II_\mu](t) \text{ for } y \in \mathbb{R}^n[a, b];
\end{equation}

also \( \mathbb{R}^n[a, b] \subset \mathbb{D}^* \) and

\[ (T_0^*z)(t) = L^*[z](t) = \sum_{\mu=0}^{m} A_\mu[z; (-1)^mII_\mu^*](t), \text{ for } z \in \mathbb{R}^n[a, b]. \]

Moreover, \( II_{\mu} \in \mathcal{L}_r^*[a, b], (\mu = 0, 1, \ldots, m) \), if and only if

\[ T_0^*[g_\lambda e^{(k)}] \in \mathcal{L}_r^*[a, b], \text{ (h = 1, \ldots, r; } \lambda = 0, 1, \ldots, k_m - 1 \),

and \( P_\mu \in \mathcal{L}_r^*[a, b], (\mu = 0, 1, \ldots, m - k_m) \).

The result of the above theorem is a direct consequence of Theorem 3.2 of Reid [8] applied to the associated scalar differential operators

\[ L_{bb}[u](t) = \sum_{\mu=0}^{m} [e^{(k)}^* P_\mu(t)e^{(k)}]u^{(\mu)}, \text{ (h, k = 1, \ldots, r),} \]

and expressing in matrix form the scalar results thus obtained.

If for a differential expression (5.1) with \( m = 2n \) we have that \( L[y] \) is given in a corresponding form (5.4) then the differential equation \( L[y](t) = 0 \) is of the form (2.2) with the \((n + 1)r \times (n + 1)r \) matrix function \( F(t) \) expressible in partitioned form \([F_{ij}(t)]\) with \( F_{ij}, (i, j = 0, 1, \ldots, n) \), the \( r \times r \) matrix functions specified for \( i, j = 0, 1, \ldots, n \) as
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\[ F_{ij}(t) = 0, \text{ if } |i - j| > 1; \]
\[ F_{ij}(t) = (-1)^i \Pi_{i+j}(t), \text{ if } |i - j| \leq 1. \]

For such a matrix function \( F(t) \) we have that \( \Re F(t) = G(t) \equiv [G_{jk}(t)], (j, k = 0, 1, \cdots, n) \), where each \( G_{jk} \) is an \( r \times r \) matrix function specified for \( j, k = 0, 1, \cdots, n \) as

\[ G_{jk}(t) = 0, \text{ if } |j - k| > 1; \]
\[ G_{jj}(t) = (-1)^j \Re \Pi_{2j}(t); \]
\[ G_{j,j+1}(t) = \sqrt{-1}(-1)^j \Im \Pi_{2j+1}(t); \]
\[ G_{j,j-1}(t) = \sqrt{-1}(-1)^j \Im \Pi_{2j-1}(t). \]

Correspondingly, \( \Im F(t) = H(t) = [H_{jk}(t)], (j, k = 0, 1, \cdots, n) \), where each \( H_{jk} \) is an \( r \times r \) matrix function specified for \( j, k = 0, 1, \cdots, n \) as

\[ H_{jk}(t) = 0, \text{ if } |j - k| > 1; \]
\[ H_{jj}(t) = (-1)^j \Im \Pi_{2j}(t); \]
\[ H_{j,j+1}(t) = \sqrt{-1}(-1)^{j+1} \Re \Pi_{2j+1}(t); \]
\[ H_{j,j-1}(t) = \sqrt{-1}(-1)^{j+1} \Re \Pi_{2j-1}(t). \]

As an application of the result of Theorem 4.1 with multipliers \( \lambda_0 = 1, \lambda_1 = 0, \) or \( \lambda_0 = -1, \lambda_1 = 0 \), one has the following special criterion for disconjugacy of a differential equation (2.2).

**Theorem 5.2.** Suppose that (5.1) with \( m = 2n \) is expressible in the form (5.4) with coefficient matrices \( \Pi_0(t), \cdots, \Pi_{2n}(t) \) satisfying the conditions given in Theorem 5.1, while \( \Im \Pi_{2j-1}(t) = 0, j = 1, \cdots, n, \) and on a given nondegenerate compact subinterval \([a, b]\) of \( I \) we have either \( \Re \Pi_{2n}(t) > 0 \) or \( \Re \Pi_{2n}(t) < 0. \) If the associated self-adjoint differential system

\[ \mathcal{L}_t[y](t) = \sum_{j=0}^{n} A_{2j}[y; \Re \Pi_{2j}](t) = 0 \]

is disconjugate on \([a, b]\) then the differential equation (5.4) is also disconjugate on this subinterval.

In particular, the functions \( \Im \Pi_{2j-1}(t), (j = 1, \cdots, n) \) are all zero in the scalar case when \( r = 1 \), and the coefficients of (5.1) are real-valued.

**References**

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